COLLOQUIUM MATHEMATICUM

VOL. 128 2012 NO. 1

THE POSITIVITY PROBLEM FOR FOURTH ORDER LINEAR RECURRENCE SEQUENCES IS DECIDABLE

BY

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Abstract. The problem whether each element of a sequence satisfying a fourth order linear recurrence with integer coefficients is nonnegative, referred to as the Positivity Problem for fourth order linear recurrence sequence, is shown to be decidable.

1. Introduction. Consider a fourth order linear recurrence of the form

$$u_n = a_1u_{n-1} + a_2u_{n-2} + a_3u_{n-3} + a_4u_{n-4} \quad (n \geq 4), \tag{1.1}$$

where $a_1, a_2, a_3, a_4\ (\neq 0) \in \mathbb{Z}$. The recurrence (1.1) defines a unique sequence of integers provided the initial integers u_0, u_1, u_2, u_3 are given. We are interested in the *Positivity Problem*: Is it possible to decide whether the sequence $(u_n)_{n\geq 0}$ is nonnegative? Equivalently, is it decidable whether $u_n \geq 0$ for all $n \geq 0$? For the significance and applications of the Positivity Problem, we refer to the introduction of [6]. Let us emphasize that the Positivity Problem considered here is to decide whether *all* elements of a sequence are nonnegative. In contrast, the *Eventual Positivity Problem*, which asks whether the terms u_n are nonnegative for all sufficiently large n, is not of interest here.

The Positivity Problem for sequences satisfying a second order linear recurrence has already been shown to be decidable by Halava–Harju–Hirvensalo [5] in 2006; see also [1] and [2]. The Positivity Problem for sequences satisfying a third order linear recurrence has recently been shown to be decidable in [6]. We show here that the same conclusion holds for each sequence satisfying a fourth order linear recurrence with integer coefficients:

MAIN THEOREM 1.1. *The Positivity Problem is decidable for each sequence of integers satisfying a linear fourth order recurrence with integer coefficients.*

There are two main differences between the work here and that in the third order case in [6]:

2010 *Mathematics Subject Classification*: Primary 11B37; Secondary 03D20.
Key words and phrases: positivity problem, recurrence sequences, decidability.

DOI: 10.4064/cm128-1-11

1. In [6], we started by finding shapes of the roots of a third degree polynomial with integer coefficients, and proceeded to determine the positivity by case analysis. Here, we abolish this step and consider all possible shapes of the roots of a fourth degree equation and invoke a result of Bell–Gerhold [1], mentioned below in Proposition 2.1, to get rid of about one half of the possible cases where there are no positive dominating roots.
2. In some cases, the signs of the cosine values (Lemma 2.2 below), in contrast to [6], are needed not only at a few points but for infinitely many points.

The proof of Theorem 1.1 is divided into two cases (Sections 3 and 4) depending on whether or not the characteristic polynomial has only real roots.

2. Preliminaries. We start by briefly recalling some facts about recurrence sequences; for general references, see [8] or [4]. A sequence $(u_n)_{n\geq 0}$ is called a *recurrence sequence* of order $h \in \mathbb{N}$ if it satisfies a recurrence relation of the form

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_h u_{n-h} \quad (n \geq h), \tag{2.1}$$

where $a_1, \ldots, a_h$ ($\neq 0$) are given real numbers. The characteristic polynomial associated with the relation (2.1) is

$$\mathrm{Char}(z) := z^h - a_1 z^{h-1} - \cdots - a_{h-1} z - a_h.$$

Let $\lambda_k \in \mathbb{C} \setminus \{0\}$ $(k = 1, \ldots, m)$ be all the distinct roots with multiplicities $\ell_1, \ldots, \ell_m$, respectively, of $\mathrm{Char}(z)$, so that $\ell_1 + \cdots + \ell_m = h$. Each sequence element satisfying (2.1) can be written as

$$u_n = \sum_{k=1}^{m} P_k(n) \lambda_k^n \quad (n \geq 0),$$

with $P_k(n) \in \mathbb{C}[n] \setminus \{0\}$, $\deg P_k = \ell_k - 1$ $(k = 1, \ldots, m)$. The roots of $\mathrm{Char}(z)$ having the largest absolute value are called *dominating roots*. Such roots play a crucial role in the positivity of the sequence (u_n) as witnessed by the following result of Bell–Gerhold [1, Theorem 2], which helps to reduce considerably the number of cases to consider.

LEMMA 2.1. *Let (u_n) be a nonzero recurrence sequence with no positive dominating characteristic root. Then the sets $\{n \in \mathbb{N} : u_n > 0\}$ and $\{n \in \mathbb{N} : u_n < 0\}$ each have positive density, and so both contain infinitely many elements.*

Following [1, p. 334], the *density* of a set $A \subseteq \mathbb{N}$ is defined as

$$\lim_{x\to\infty} \frac{1}{x} \sharp\{n \leq x \,;\, n \in A\}.$$

Some auxiliary results about oscillating behavior of the cosine function are also needed:

Lemma 2.2. *Let $\varphi, \theta \in [-\pi, \pi)$ with $\theta \notin \{-\pi, 0\}$.*

I. *If $\theta = s\pi/t$ is a rational multiple of π where $s, t\ (>0) \in \mathbb{Z} \setminus \{0\}$, $\gcd(s,t) = 1$, then as n varies over $\mathbb{N} \cup \{0\}$, the function $\cos(\varphi + n\theta)$ is periodic and takes at most $2t$ explicitly computable distinct values corresponding to $n = 0, 1, \ldots, 2t-1$.*
II. *If θ is not a rational multiple of π, then as n varies over $\mathbb{N} \cup \{0\}$ the range of values of $\cos(\varphi + n\theta)$ is dense in $[-1, 1]$.*
III. *The function $\cos(\varphi + n\theta)$ takes both positive and negative values for infinitely many $n \in \mathbb{N} \cup \{0\}$.*

Proof. Part I is contained in the statement and proof of [6, Lemma 1.3(a)], part II is [6, Lemma 1.3(b)], and part III is [7, Propositions 2.1 and 2.2]. ∎

The results in the next lemma are shapes of sequence elements that have already been shown to be decidable; those proved for second order recurrence sequences from [5] are in Part I, and those for third order recurrence sequences from [6] are in Part II.

Lemma 2.3. *The Positivity Problem for the following forms of sequence elements is decidable:*

Part I:

(HHH1) $u_n = A\lambda_1^n + B\lambda_2^n \quad (A, B \in \mathbb{R};\ \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\})$,
(HHH2) $u_n = (A + Bn)\lambda^n \quad (A, B \in \mathbb{R};\ \lambda \in \mathbb{R} \setminus \{0\})$,
(HHH3) $u_n = A\lambda^n + \bar{A}\bar{\lambda}^n \quad (A \in \mathbb{C};\ \lambda \in \mathbb{C} \setminus \mathbb{R})$.

Part II:

(LT1) $u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n \quad (A, B, C \in \mathbb{R};\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \setminus \{0\})$,
(LT2) $u_n = (A + Bn + Cn^2)\lambda^n \quad (A, B, C \in \mathbb{R};\ \lambda \in \mathbb{R} \setminus \{0\})$,
(LT3) $u_n = A\lambda_1^n + (B + Cn)\lambda_2^n \quad (A, B, C \in \mathbb{R};\ \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\})$,
(LT4) $u_n = A\lambda_1^n + B\lambda_2^n + \bar{B}\bar{\lambda}_2^n \quad (A \in \mathbb{R}, \lambda_1 \in \mathbb{R} \setminus \{0\}, B \in \mathbb{C}, \lambda_2 \in \mathbb{C} \setminus \mathbb{R})$.

3. Proof of Theorem 1.1 when Char(z) has only real roots. In this case, the general term of the sequence is

$$u_n = P_1(n)\lambda_1^n + P_2(n)\lambda_2^n + \cdots + P_m(n)\lambda_m^n \quad (n \geq 0,\ m \leq 4),$$

where $\lambda_1, \ldots, \lambda_m$ are distinct nonzero real numbers and

$$P_i(n) = A_{i,1} + A_{i,2}n + \cdots + A_{i,\ell_i}n^{\ell_i - 1} \quad (\ell_i \in \mathbb{N};\ i = 1, \ldots, m;\ A_{i,\ell_i} \neq 0),$$

with $\ell_1 + \cdots + \ell_m = 4$. We have two possibilities to consider.

3.1. There are two roots λ_i with the same absolute value. Without loss of generality, let the two roots be $\lambda_1 > 0$ and $\lambda_2 = -\lambda_1$. Here,

$$u_n = \{P_1(n) + (-1)^n P_2(n)\}\lambda_1^n + P_3(n)\lambda_3^n + \cdots + P_m(n)\lambda_m^n \quad (n \geq 0),$$

i.e.,

$$\begin{aligned} u_{2k} &= (P_1(2k) + P_2(2k))\lambda_1^{2k} + P_3(2k)\lambda_3^{2k} + \cdots + P_m(2k)\lambda_m^{2k}, \\ u_{2k+1} &= (P_1(2k+1) - P_2(2k+1))\lambda_1^{2k+1} \\ &\quad + P_3(2k+1)\lambda_3^{2k+1} + \cdots + P_m(2k+1)\lambda_m^{2k+1} \quad (k \geq 0). \end{aligned}$$

The sequence (u_n) is nonnegative if and only if both the sequences (u_{2k}) and (u_{2k+1}) are nonnegative. The two sequences (u_{2k}) and (u_{2k+1}) are decidable as they satisfy recurrence relations of lower order, i.e., they must be one of the forms stated in Lemma 2.3.

3.2. All roots have different absolute values. By Proposition 2.1, we need only treat the case where there is a positive dominating root, say $\lambda_1 > 0$. Without loss of generality, assume $\lambda_1 > |\lambda_2| > \cdots > |\lambda_m|$. Here,

$$u_n = \lambda_1^n \{P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n\} \quad (n \geq 0),$$

and so (u_n) is nonnegative if and only if

$$P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n \geq 0 \quad \text{for all } n \geq 0.$$

- If $A_{1,\ell_1} < 0$, then

$$P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n < 0$$

for all sufficiently large n, and so this case is untenable.

- If $A_{1,\ell_1} > 0$, since

$$P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n \to \infty \quad (n \to \infty),$$

there is an explicitly computable least $M_0 \in \mathbb{N} \cup \{0\}$ such that

$$P_1(n) + P_2(n)(\lambda_2/\lambda_1)^n + \cdots + P_m(n)(\lambda_m/\lambda_1)^n \geq 0 \quad \text{for all } n \geq M_0,$$

and so the sequence (u_n) is nonnegative if and only if $M_0 = 0$.

4. Proof of Theorem 1.1 when Char(z) has non-real roots. The possible shapes of the four roots are:

1. two complex conjugate pairs, either distinct or identical, denoted by $C(z_1\bar{z}_1z_2\bar{z}_2)$;
2. two identical real numbers and one complex conjugate pair, denoted by $C(r_1^2z\bar{z})$;
3. two distinct real numbers and one complex conjugate pair, denoted by $C(r_1r_2z\bar{z})$.

The possibility $C(z_1\bar{z}_1z_2\bar{z}_2)$ is decidable by Proposition 2.1 as Char(z) has no positive dominating roots. The remaining two possible cases are now treated.

4.1. Case $C(r_1^2z\bar{z})$. In this case, the general term of the sequence is

$$u_n = (A + Bn)\lambda_1^n + C\lambda_2^n + \bar{C}\bar{\lambda}_2^n \quad (n \geq 0),$$

where $A, B, \lambda_1\ (\neq 0) \in \mathbb{R}$, $C \in \mathbb{C}$, $\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$ and the bar denotes complex conjugate. Let $\lambda_2 = |\lambda_2|e^{i\theta}$, $C = |C|e^{i\varphi}$ where $\theta, \varphi \in [-\pi, \pi)$, $\theta \notin \{-\pi, 0\}$ so that

$$u_n = (A + Bn)\lambda_1^n + 2|C|\,|\lambda_2|^n \cos(\varphi + n\theta).$$

By Proposition 2.1, we need only treat the case where there is a positive dominating root, say $\lambda_1 > 0$. There are two possibilities.

Subcase 1: $\lambda_1 = |\lambda_2|$. Write

$$u_n = \lambda_1^n\{A + Bn + 2|C|\cos(\varphi + n\theta)\} \quad (n \geq 0).$$

The sequence (u_n) is nonnegative if and only if

$$A \geq -Bn - 2|C|\cos(\varphi + n\theta) \quad \text{for all } n \geq 0. \tag{4.1}$$

- If $B < 0$, then (4.1) cannot be fulfilled.
- If $B = 0$, then $u_n = A\lambda_1^n + C\lambda_2^n + \bar{C}\bar{\lambda}_2^n$, which is of the form (LT4).
- If $B > 0$, then $-Bn - 2|C|\cos(\varphi + n\theta) \to -\infty$ as $n \to \infty$. Thus, there exists an explicitly computable $N_0 \in \mathbb{N}$ depending on B, C, φ, θ such that

$$\max_{n\in\mathbb{N}\cup\{0\}}\{-Bn - 2|C|\cos(\varphi + n\theta)\} = -BN_0 - 2|C|\cos(\varphi + N_0\theta).$$

Consequently, the sequence (u_n) is nonnegative if and only if

$$A \geq -BN_0 - 2|C|\cos(\varphi + N_0\theta).$$

Subcase 2: $\lambda_1 > |\lambda_2|$. Rewrite the general term of the sequence as

$$u_n = \lambda_1^n\{A + Bn + 2|C|(|\lambda_2|/\lambda_1)^n\cos(\varphi + n\theta)\} \quad (n \geq 0).$$

Since $\operatorname{sign}(A+Bn+2|C|(|\lambda_2|/\lambda_1)^n\cos(\varphi+n\theta)) = \operatorname{sign}(B)$ when n sufficiently large provided $B \neq 0$, for the sequence (u_n) to be nonnegative we must have $B \geq 0$ and

$$A \geq -Bn - 2|C|(|\lambda_2|/\lambda_1)^n\cos(\varphi + n\theta) \quad (n \geq 0).$$

By Lemma 2.2, there is a least $N_L \in \mathbb{N} \cup \{0\}$ such that $\cos(\varphi + N_L\theta) < 0$. Since $(|\lambda_2|/\lambda_1)^n \to 0$ as $n \to \infty$, there is $N_M > N_L$ such that for all $n \geq N_M$ we have

$$-2|C|(|\lambda_2|/\lambda_1)^n\cos(\varphi + n\theta) < -2|C|(|\lambda_2|/\lambda_1)^{N_L}\cos(\varphi + N_L\theta).$$

Consequently, the sequence (u_n) is nonnegative if and only if

$$B \geq 0 \quad \text{and} \quad A \geq \max\{-Bn-2|C|(|\lambda_2|/\lambda_1)^n\cos(\varphi+n\theta)\,;\, N_L \leq n \leq N_M\}.$$

4.2. Case $C(r_1r_2z\bar{z})$. In this case, the general term of the sequence is

$$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n \quad (n \geq 0),$$

where $A, B \in \mathbb{R}$, $C \in \mathbb{C}$, $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ and $\lambda_3 \in \mathbb{C} \setminus \mathbb{R}$. Let $\lambda_3 = |\lambda_3|e^{i\theta}$, $C = |C|e^{i\varphi}$ where $\theta, \varphi \in [-\pi, \pi)$, $\theta \notin \{-\pi, 0\}$ so that

$$u_n = A\lambda_1^n + B\lambda_2^n + 2|C|\,|\lambda_3|^n \cos(\varphi + n\theta).$$

By Proposition 2.1, we need only decide the situation where $\mathrm{Char}(z)$ has a positive dominating root, say $\lambda_1 > 0$, and so $\lambda_1 \geq \max\{|\lambda_2|, |\lambda_3|\}$. There are three possibilities:

1. The three λ's have the same absolute values, i.e., $\lambda_1 = |\lambda_2| = |\lambda_3|$.
2. All three roots λ_1, λ_2 and λ_3 have different absolute values.
3. There are exactly two λ_i's having the same absolute value, i.e., $\lambda_1 = |\lambda_2|$ or $\lambda_1 = |\lambda_3|$ or $|\lambda_2| = |\lambda_3|$.

Subcase 1: $\lambda_1 = |\lambda_2| = |\lambda_3|$. Here, $0 < \lambda_1 = -\lambda_2 = |\lambda_3|$. The sequence term is of the form

$$u_n = \{A + (-1)^n B\}\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n \quad (n \geq 0),$$

i.e.,

$$u_{2k} = \{A + B\}\lambda_1^{2k} + C\lambda_3^{2k} + \bar{C}\bar{\lambda}_3^{2k},$$
$$u_{2k+1} = \{A - B\}\lambda_1^{2k+1} + C\lambda_3^{2k+1} + \bar{C}\bar{\lambda}_3^{2k+1} \quad (k \geq 0).$$

The two sequences (u_{2k}) and (u_{2k+1}) are decidable because they are of the form (LT4).

Subcase 2: *All three roots λ_1, λ_2 and λ_3 have different absolute values.* Rewrite the general term of the sequence as

$$u_n = \lambda_1^n\{A + B(\lambda_2/\lambda_1)^n + 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta)\} \quad (n \geq 0).$$

The sequence (u_n) is nonnegative if and only if

$$A + B(\lambda_2/\lambda_1)^n + 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta) \geq 0 \quad (n \geq 0). \tag{4.2}$$

If $A = 0$, then $u_n = B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$, which is of the form (LT4). If $A < 0$, then

$$A + B(\lambda_2/\lambda_1)^n + 2|C|(|\lambda_3|/\lambda_1)^n \cos(\varphi + n\theta) \to A < 0 \quad (n \to \infty),$$

showing that (4.2) cannot be fulfilled. If $A > 0$, then there is an explicitly computable least integer $N_0 \in \mathbb{N} \cup \{0\}$ such that (4.2) holds for all $n \geq N_0$. Consequently, in this case the sequence (u_n) is nonnegative if and only if $N_0 = 0$.

Subcase 3: $\lambda_1 = |\lambda_2|$ *or* $\lambda_1 = |\lambda_3|$ *or* $\lambda_2 = |\lambda_3|$.

3.1: $\lambda_1 = |\lambda_2| = -\lambda_2 > |\lambda_3|$. Here,

$$u_n = \{A + (-1)^n B\}\lambda_1^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n \quad (n \geq 0),$$

i.e.,

$$u_{2k} = \{A+B\}\lambda_1^{2k} + C\lambda_3^{2k} + \bar{C}\bar{\lambda}_3^{2k},$$
$$u_{2k+1} = \{A-B\}\lambda_1^{2k+1} + C\lambda_3^{2k+1} + \bar{C}\bar{\lambda}_3^{2k+1} \quad (k \geq 0).$$

The two sequences (u_{2k}) and (u_{2k+1}) are both decidable because they are of the form (LT4).

3.2: $\lambda_1 > |\lambda_2| = |\lambda_3| > 0$. If $\lambda_2 > 0$, then $|\lambda_3| = |\lambda_2| = \lambda_2$, and so

$$\begin{aligned} u_n &= A\lambda_1^n + \{B + 2|C|\cos(\varphi + n\theta)\}\lambda_2^n \\ &= \lambda_1^n\{A + (B + 2|C|\cos(\varphi + n\theta))(\lambda_2/\lambda_1)^n\}. \end{aligned} \tag{4.3}$$

Since $(B + 2|C|\cos(\varphi + n\theta))(\lambda_2/\lambda_1)^n \to 0$ as $n \to \infty$, either

$$-(B + 2|C|\cos(\varphi + n\theta))(\lambda_2/\lambda_1)^n < 0 \quad \text{for all } n \geq 0,$$

or there is an explicit $N_1 \in \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} &-(B + 2|C|\cos(\varphi + N_1\theta))(\lambda_2/\lambda_1)^{N_1} \\ &\qquad = \max_{n\geq 0}\{-(B + 2|C|\cos(\varphi + n\theta))(\lambda_2/\lambda_1)^n\} \geq 0. \end{aligned}$$

The sequence (u_n) is thus nonnegative if and only if

$$A \geq \max\{0, -(B + 2|C|\cos(\varphi + N_1\theta))(\lambda_2/\lambda_1)^{N_1}\}.$$

If $\lambda_2 < 0$, then $|\lambda_3| = |\lambda_2| = -\lambda_2$, and so

$$u_n = A\lambda_1^n + \{(-1)^n B + 2|C|\cos(\varphi + n\theta)\}|\lambda_3|^n.$$

The two subsequences corresponding to even and odd subscripts, i.e,

$$u_{2k} = A\lambda_1^{2k} + \{B + 2|C|\cos(\varphi + 2k\theta)\}|\lambda_3|^{2k} \quad (k \geq 0),$$
$$u_{2k+1} = A\lambda_1^{2k+1} + \{-B + 2|C|\cos(\varphi + (2k+1)\theta)\}|\lambda_3|^{2k+1} \quad (k \geq 0),$$

are of the form (4.3) and so similar arguments show that both are decidable.

3.3: $\lambda_1 = |\lambda_3| > |\lambda_2| > 0$. Write

$$u_n = \lambda_1^n\{A + 2|C|\cos(\varphi + n\theta) + B(\lambda_2/\lambda_1)^n\}.$$

The sequence (u_n) is nonnegative if and only if

$$A \geq -2|C|\cos(\varphi + n\theta) - B(\lambda_2/\lambda_1)^n \quad (n \geq 0). \tag{4.4}$$

• If θ is a rational multiple of π, say $\theta = s\pi/t$ where $s, t\ (>0) \in \mathbb{Z} \setminus \{0\}$ and $\gcd(s,t) = 1$, then by Lemma 2.2, $\cos(\varphi + n\theta)$ is periodic and takes at most $2t$ distinct explicit (positive and negative) values at $n \in \{0, 1, \ldots, 2t-1\}$ (mod $2t$); among these values, let c_t be the least (negative). Since $-B(\lambda_2/\lambda_1)^n \to 0$ as $n \to \infty$, (4.4) holds if and only if

$$A \geq \begin{cases} -2c_t|C| & \text{if "}B = 0\text{" or "}B > 0,\ \lambda_2 > 0\text{"}, \\ \max_{n=0,1,\ldots,2t-1}\{-2|C|\cos(\varphi + n\theta) - B(\lambda_2/\lambda_1)^n\} & \text{otherwise.} \end{cases}$$

• If θ is not a rational multiple of π, rewrite the terms of the sequence as

$$u_n = |\lambda_2|^n\{(A + 2|C|\cos(\varphi + n\theta))(\lambda_1/|\lambda_2|)^n + B(\lambda_2/|\lambda_2|)^n\}.$$

The sequence (u_n) is nonnegative if and only if

$$\{A + 2|C|\cos(\varphi + n\theta)\}(\lambda_1/|\lambda_2|)^n + B(\lambda_2/|\lambda_2|)^n \geq 0 \quad (n \geq 0). \tag{4.5}$$

If $C = 0$, then $u_n = A\lambda_1^n + B\lambda_2^n$, which is of the form (HHH1). Assume henceforth that $C > 0$.

(a) If $A \leq 0$, since the values of $\cos(\varphi + n\theta)$ are dense in the closed interval $[-1, 1]$ ([3, Chapters 3–4]), and $(\lambda_1/|\lambda_2|)^n \to \infty$ as $n \to \infty$, there is an explicitly computable $N_L \in \mathbb{N} \cup \{0\}$ such that

$$\{A + 2|C|\cos(\varphi + N_L\theta)\}(\lambda_1/|\lambda_2|)^{N_L} + B(\lambda_2/|\lambda_2|)^{N_L} < 0,$$

and so (4.5) cannot be fulfilled.

(b) If $0 < A < 2|C|$, let $2|C| - A = \Delta > 0$ so that

$$\begin{aligned}\{A + 2|C|\cos(\varphi + n\theta)\}&(\lambda_1/|\lambda_2|)^n + B(\lambda_2/|\lambda_2|)^n \\ &= \{2|C|(1 + \cos(\varphi + n\theta)) - \Delta\}(\lambda_1/|\lambda_2|)^n + B(\lambda_2/|\lambda_2|)^n.\end{aligned} \tag{4.6}$$

Taking a subsequence (n_k) for which $\cos(\varphi + n_k\theta) \to -1$ as $k \to \infty$, the left-hand expression in (4.6) tends to $-\infty$ as $k \to \infty$, showing that (4.5) cannot be fulfilled.

(c) If $A > 2|C| > 0$, let $\delta = A - 2|C| > 0$. Since

$$\begin{aligned}\{A + 2|C|\cos(\varphi + n\theta)\}&(\lambda_1/|\lambda_2|)^n + B(\lambda_2/|\lambda_2|)^n \\ &\geq \delta(\lambda_1/|\lambda_2|)^n + B(\lambda_2/|\lambda_2|)^n \to \infty \quad (n \to \infty),\end{aligned}$$

there is an explicitly computable least $N_0 \in \mathbb{N} \cup \{0\}$, depending on $A, B, C, \varphi, \theta, \lambda_1, \lambda_2$, such that

$$\{A + 2|C|\cos(\varphi + n\theta)\}(\lambda_1/|\lambda_2|)^n + B(\lambda_2/|\lambda_2|)^n \geq 0$$

for all $n \geq N_0$. Consequently, (4.5) holds if and only if $N_0 = 0$.

(d) If $A = 2|C|$, then (4.5) becomes

$$2|C|\{1 + \cos(\varphi + n\theta)\}(\lambda_1/|\lambda_2|)^n + B(\lambda_2/|\lambda_2|)^n \geq 0 \quad (n \geq 0). \tag{4.7}$$

We pause to prove two important claims.

Claim 1. *There is at most one integer $N_0 \in \mathbb{N} \cup \{0\}$ such that*

$$1 + \cos(\varphi + N_0\theta) = 0. \tag{4.8}$$

Proof of Claim 1. If there were two distinct such integers N_0 and N_0', then there would exist two integers k_0, k_0' such that

$$\varphi + N_0\theta = (2k_0 + 1)\pi, \qquad \varphi + N_0'\theta = (2k_0' + 1)\pi.$$

Subtracting the two equations, we find that θ is a multiple of π, which is a contradiction.

Claim 2. *If* $(n_k) \subset \mathbb{N} \cup \{0\}$ *is an increasing sequence of positive integers such that*

$$1 + \cos(\varphi + n_k\theta) \to 0 \quad (k \to \infty),$$

then

$$2|C|\{1 + \cos(\varphi + n_k\theta)\}(\lambda_1/|\lambda_2|)^{n_k} + B(\lambda_2/|\lambda_2|)^{n_k} \to \infty \quad (k \to \infty).$$

Proof of Claim 2. By Claim 1, we know that from certain k onward all the values of $1 + \cos(\varphi + n_k\theta)$ are nonzero. Since the function $2|C|\{1 + \cos(\varphi + x\theta)\}(\lambda_1/|\lambda_2|)^x \pm B(\lambda_2/|\lambda_2|)^x$ is infinitely differentiable, the desired result follows from a number of well-known theorems in real analysis, such as L'Hôpital's rule.

Returning to (4.5), if there is $N_0 \in \mathbb{N} \cup \{0\}$ such that (4.8) holds, which must be unique by Claim 1, then for (4.5) to hold we must have $B(\lambda_2/|\lambda_2|)^{N_0} \geq 0$. Moreover, using Claim 2 and the fact that for n large enough the values of $1 + \cos(\varphi + n\theta)$ are positive and bounded by 1, we deduce that

$$2|C|\{1 + \cos(\varphi + n\theta)\}(\lambda_1/|\lambda_2|)^n + B(\lambda_2/|\lambda_2|)^n \to \infty \quad (n \to \infty).$$

Thus, there is an explicitly computable least integer $N_1 \in \mathbb{N} \cup \{0\}$, depending on $B, C, \varphi, \theta, \lambda_1, \lambda_2$, such that

$$2|C|\{1 + \cos(\varphi + n\theta)\}(\lambda_1/|\lambda_2|)^n + B(\lambda_2/|\lambda_2|)^n \geq 0 \quad \text{for all } n \geq N_1.$$

Using all the information gathered, we conclude that (4.5) holds if and only if $N_1 = 0$.

Final remarks. It seems natural to ask whether the preceding proof is applicable to the general case. Should it be so, the general positivity problem would be solved, which in turns implies that a long unsolved conjecture of Skolem would be settled. Though a good deal of the above analysis, such as the case where all roots of $\mathrm{Char}(z)$ are real, does indeed work in the general situation, in the case of fifth order recurrence, there are instances in which we are not yet able to settle decidability.

Acknowledgements. The first two authors are supported by the Thailand Research Fund through the Royal Golden Jubilee Ph.D. Program (Grant No. PHD/0022/2552), and the third author is supported by the Centre of Excellence in Mathematics, CHE, Bangkok 10400, Thailand.

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Received 27 April 2012;
revised 3 September 2012 (5675)