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ORDERINGS OF THE RATIONALS AND DYNAMICAL SYSTEMS

BY

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Abstract. This paper is devoted to a systematic study of a class of binary trees encoding the structure of rational numbers both from arithmetic and dynamical point of view. The paper is divided into three parts. The first one is mainly expository and consists in a critical review of rather standard topics such as Stern–Brocot and Farey trees and their connections with continued fraction expansion and the question mark function. In the second part we introduce two classes of (invertible and non-invertible) one-dimensional maps which can be used to generate the binary trees in different ways and study their ergodic properties. This also leads us to study, in the third part, some random processes (Markov chains and martingales) which arise in a natural way from the action of the transfer operators associated to the non-invertible maps.

1. PART ONE: ARITHMETIC

Notational warning: We shall use the following notations:

$$I := [0,1], \qquad J := [0,\infty) \cup \{\infty\},$$
$$\mathbb{Q}_1 := \mathbb{Q} \cap [0,1],$$
$$\mathbb{Q}_p := \{k/p^s : s \in \mathbb{N},\, 0 \le k \le p^s\}, \qquad p \ge 2.$$

1.1. A class of binary trees. We start with the Stern–Brocot (SB) tree $\mathcal{T}$, which is a way to order (and thus to count) the elements of $\mathbb{Q}^+$, the set of positive rational numbers, so that every number appears (and thus is counted) exactly once (see [St], [Br] and, for a modern account, [GKP]). The basic operation needed to construct $\mathcal{T}$ is the *Farey sum*: given $\frac{p}{q}$ and $\frac{p'}{q'}$ in $\mathbb{Q}^+$ set

$$\frac{p}{q} \oplus \frac{p'}{q'} = \frac{p+p'}{q+q'}$$

One notes that the *child* $\frac{p}{q} \oplus \frac{p'}{q'}$ turns out to be in lowest terms whenever the *parents* $\frac{p}{q}$ and $\frac{p'}{q'}$ are. Moreover, the child always lies somewhere in between its parents, e.g., assuming $\frac{p}{q} < \frac{p'}{q'}$, we have $\frac{p}{q} < \frac{p+p'}{q+q'} < \frac{p'}{q'}$.

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Starting from the *ancestors* 0 and ∞ (written "in lowest terms") one then writes genealogically one generation after the other using the above operation:

$$\frac{0}{1} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{1}{0}$$

$$\frac{1}{1}$$

$$\frac{1}{2} \qquad\qquad\qquad\qquad \frac{2}{1}$$

$$\frac{1}{3} \qquad\quad \frac{2}{3} \qquad\quad \frac{3}{2} \qquad\quad \frac{3}{1}$$

$$\frac{1}{4} \quad \frac{2}{5} \quad \frac{3}{5} \quad \frac{3}{4} \quad \frac{4}{3} \quad \frac{5}{3} \quad \frac{5}{2} \quad \frac{4}{1}$$

$$\frac{1}{5}\ \frac{2}{7}\ \frac{3}{8}\ \frac{3}{7}\ \frac{4}{7}\ \frac{5}{8}\ \frac{5}{7}\ \frac{4}{5}\ \frac{5}{4}\ \frac{7}{5}\ \frac{8}{5}\ \frac{7}{4}\ \frac{7}{3}\ \frac{8}{3}\ \frac{7}{2}\ \frac{5}{1}$$

and so on. The easily verified property which makes the above interesting and useful is the following fact: if $\frac{p}{q}$ and $\frac{p'}{q'}$ are consecutive fractions at any stage of the construction then the unimodular relation $qp' - pq' = 1$ is in force.

Finally, the subtree $\mathcal{F}$ of $\mathcal{T}$ having $\frac{1}{2}$ as root node and vertex set $\mathbb{Q}_1$ is called the *Farey tree*. It can be obtained exactly in the same way as $\mathcal{T}$ taking as ancestors $\frac{0}{1}$ and $\frac{1}{1}$ instead of $\frac{0}{1}$ and $\frac{1}{0}$.

LEMMA 1.1. *Let $\phi : J \to I$ be the invertible map defined by $\phi(\infty) = 1$ and*

$$\phi(x) = \frac{x}{x+1}, \qquad x \in \mathbb{R}^+.$$

Then $\phi(\mathcal{T}) = \mathcal{F}$.

Proof. It suffices to notice that $\phi\left(\frac{0}{1}\right) = \frac{0}{1}$, $\phi\left(\frac{1}{0}\right) = \frac{1}{1}$, $\phi\left(\frac{1}{1}\right) = \frac{1}{2}$ and for $x, x' \in \mathbb{Q}^+$ we have $\phi(x) \oplus \phi(x') = \phi(x \oplus x')$. ∎

Another structure we shall deal with is the *dyadic* tree $\mathcal{D}$, whose first two levels are as in $\mathcal{F}$ and which is then constructed from the root node $\frac{1}{2}$ by writing under each vertex $\frac{p}{q}$ the pair $\frac{2p-1}{2q}$ and $\frac{2p+1}{2q}$. The vertex set of $\mathcal{D}$ is $\mathbb{Q}_2$. We shall see later how it is related to $\mathcal{T}$ and $\mathcal{F}$.

1.2. Continued fractions and the $\{L, R\}$ coding. Every $x \in \mathbb{Q}^+$ appears exactly once in the above construction and corresponds to a unique finite path on $\mathcal{T}$ starting at the root node $\frac{1}{1}$ and whose number of vertices equals the *depth* of x, i.e. the level of $\mathcal{T}$ it belongs to. For $x \in \mathbb{Q}_1$ one may just consider the path on the subtree $\mathcal{F}$ which starts at the root node $\frac{1}{2}$ and whose number of vertices is the *rank* of x. For $x \in \mathbb{Q}^+$ we have

$$\mathrm{depth}(x) = [x] + \mathrm{rank}(\{x\}) + 1.$$

In order to properly code these paths we start by recalling that every rational number $x \in \mathbb{Q}^+$ has a unique finite continued fraction expansion [Kh]

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} \equiv [a_0; a_1, \ldots, a_n]$$

with $a_0 \geq 0$, $a_i \geq 1$ for $1 \leq i < n$ and $a_n > 1$.

LEMMA 1.2. *Let $x \in \mathbb{Q}^+$. Then*

$$x = [a_0; a_1, \ldots, a_n] \ \Rightarrow \ \operatorname{depth}(x) = \sum_{i=0}^{n} a_i.$$

Proof. Setting $\operatorname{depth}\left(\frac{0}{1}\right) = \operatorname{depth}\left(\frac{1}{0}\right) = 0$ we have $\operatorname{depth}\left(\frac{1}{1}\right) = 1$. Let now $x = [a_0; a_1, \ldots, a_n]$ be such that $\operatorname{depth}(x) = d > 1$. Then, in order to reach the leaf x from the root $\frac{1}{1}$ one has to first move a_0 steps to the right, thus reaching the node $a_0 + \frac{1}{1}$. Then, moving a_1 steps to the left one reaches $a_0 + \frac{1}{a_1 + \frac{1}{1}}$. After a_2 further steps to the right one reaches the point $a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{1}}}$ and so on. In this way, one sees that the path to reach x makes exactly n turns and the length of the blocks between the $(i-1)$st and the ith turn is given by the partial quotient a_i for $1 \leq i < n$, whereas the last block has length $a_n - 1$. More precisely, the blocks moving to the left are related to partial quotients with odd index, and those moving to the right to those with even index. It then follows at once that $d = \sum_{i=0}^{n} a_i$. ∎

The argument sketched above actually allows us to say more. To this end, we shall first construct a matrix representation of the positive rationals. We start by noting that a given $x \in \mathbb{Q}^+$ can be uniquely decomposed as

$$x = \frac{p}{q} \oplus \frac{p'}{q'} \quad \text{with} \quad qp' - pq' = 1.$$

The neighbours $\frac{p}{q}$ and $\frac{p'}{q'}$ are thus the parents of x as an element of $\mathcal{T}$. We then identify

$$x \ \leftrightarrow \ \begin{pmatrix} p' & p \\ q' & q \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Note that the left column bears on the right parent and vice versa. In this way, the root node yields the identity matrix:

$$\frac{1}{1} = \frac{0}{1} \oplus \frac{1}{0} \ \leftrightarrow \ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover, given $M \in \mathrm{SL}(2,\mathbb{Z})$ which represents the fraction $x \in \mathbb{Q}^+$, the matrix UMU represents the symmetric fraction $1/x$, with

$$U = U^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular,

$$\frac{1}{2} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} =: L \quad \text{and} \quad \frac{2}{1} \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =: R.$$

More generally, for $k \in \mathbb{N}$,

$$L^k = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \leftrightarrow \frac{1}{k+1} \quad \text{and} \quad R^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \leftrightarrow k+1.$$

Now, the point x considered above has in turn a unique pair of (left and right) children, given by

$$\frac{m}{s} \oplus \frac{m+n}{s+t} \quad \text{and} \quad \frac{m+n}{s+t} \oplus \frac{n}{t}$$

respectively. Moreover,

$$\begin{pmatrix} n & m \\ t & s \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} m+n & m \\ s+t & s \end{pmatrix} \leftrightarrow \frac{m}{s} \oplus \frac{m+n}{s+t}$$

and

$$\begin{pmatrix} n & m \\ t & s \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n & m+n \\ t & s+t \end{pmatrix} \leftrightarrow \frac{m+n}{s+t} \oplus \frac{n}{t}.$$

In other words, the matrices L and R, when acting from the right, move to the left and right child in $\mathcal{T}$, respectively. Together with the argument of the proof given above this yields the following result.

PROPOSITION 1.3. *To each entry $x \in \mathcal{T}$ there corresponds a unique element $X \in \mathrm{SL}(2,\mathbb{Z})$ for which we have the following two possibilities*:

- $x = [a_0; a_1, \dots, a_n]$, *$n$ even* $\Rightarrow X = R^{a_0} L^{a_1} \cdots L^{a_{n-1}} R^{a_n - 1}$.
- $x = [a_0; a_1, \dots, a_n]$, *$n$ odd* $\Rightarrow X = R^{a_0} L^{a_1} R^{a_2} \cdots L^{a_n - 1}$.

As an easy consequence we have

COROLLARY 1.4. *Let $x = [a_0; a_1, \dots, a_n]$ with $a_n > 1$ and n even. Then its left and right children in $\mathcal{T}$ are given by $x' = [a_0; a_1, \dots, a_n - 1, 2]$ and $x'' = [a_0; a_1, \dots, a_n + 1]$, respectively. If instead n is odd, the expansions for x' and x'' are interchanged.*

Proof. For n even and larger than one we have $x = [a_0; a_1, \dots, a_n] \Leftrightarrow X = R^{a_0} L^{a_1} R^{a_2} \cdots R^{a_n - 1}$. Therefore $x' \leftrightarrow X' = R^{a_0} L^{a_1} R^{a_2} \cdots R^{a_n - 1} L$ and

$x'' \leftrightarrow X'' = R^{a_0}L^{a_1}R^{a_2}\cdots R^{a_n}$, which yields the claim. A similar reasoning applies for $n = 0$ and for n odd. ∎

1.3. The infinite coding. One can extend the above construction by associating to each $x \in \mathbb{R}^+$ a unique *infinite* path in $\mathcal{T}$, or else a unique semi-infinite word in $\pi(x) \in \{L, R\}^{\mathbb{N}}$, in the natural way. First, to $x \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ with infinite continued fraction expansion $x = [a_0; a_1, a_2, a_3, \ldots]$ there will correspond the (unique) sequence $\pi(x) = R^{a_0}L^{a_1}R^{a_2}L^{a_3}\cdots$, where now R and L are just elements of a binary alphabet. For rational x we can proceed as follows. First we set $\pi\left(\frac{0}{1}\right) = L^\infty$ and $\pi\left(\frac{1}{0}\right) = R^\infty$. Then note that each $x \in \mathbb{Q}^+$ has two infinite paths which agree down to node x: they are those starting with the finite sequence coding the path to reach x from the root node according to Proposition 1.3 and terminating with either RL^∞ or LR^∞. We shall agree that $\pi(x)$ terminates with RL^∞ or LR^∞ according to whether the number of its partial quotients of x is even or odd. Summarizing, we have the following coding:

- $x = [a_0; a_1, \ldots, a_n]$, n even $\Rightarrow \pi(x) = R^{a_0}L^{a_1}\cdots R^{a_n}L^\infty$.
- $x = [a_0; a_1, \ldots, a_n]$, n odd $\Rightarrow \pi(x) = R^{a_0}L^{a_1}\cdots L^{a_n}R^\infty$.
- $x = [a_0; a_1, a_2, a_3, \ldots] \Rightarrow \pi(x) = R^{a_0}L^{a_1}R^{a_2}L^{a_3}\cdots$.

One easily checks that if $\succ$ denotes the lexicographic order on $\{L, R\}^{\mathbb{N}}$ then

$$x > y \;\Rightarrow\; \pi(x) \succ \pi(y).$$

Finally, from the above it follows that for an irrational x the infinite path on $\mathcal{T}$ converging to x coincides with the *slow continued fraction algorithm* (see, e.g., [AO]).

1.4. The (extended) question mark function. Given a number $x \in \mathbb{R}^+$ with continued fraction expansion $x = [a_0; a_1, a_2, \ldots]$, one may ask what is the number obtained by interpreting the sequence $\pi(x)$ defined in Section 1.3 as the binary expansion of a real number in $(0, 1)$. The number so obtained, denoted $\varrho(x)$, reads

$$\varrho(x) = 0\,.\,\underbrace{11\ldots1}_{a_0}\underbrace{00\ldots0}_{a_1}\underbrace{11\ldots1}_{a_2}\;\cdots \tag{1.1}$$

or, what is the same,

$$\varrho(x) = 1 - \sum_{k \geq 0}(-1)^k\, 2^{-(a_0+\cdots+a_k)}.$$

For instance, $\varrho(1/n) = 1/2^n$ and $\varrho(n) = 1 - 1/2^n$ for all $n \geq 1$. Setting $\varrho(0) = 0$ and $\varrho(\infty) = 1$ we see that $\varrho : \mathbb{R}^+ \to I$ satisfies

$$\varrho(x) = ?\circ\phi\,(x) \tag{1.2}$$

where $\phi : J \to I$ is the map defined in Lemma 1.1 and $? : I \to I$ is the *Minkowski question mark function* [M], which for $x = [0; a_1, a_2, \dots]$ is given by

$$?(x) = 0\,.\,\underbrace{00\dots0}_{a_1-1}\underbrace{11\dots1}_{a_2}\underbrace{00\dots0}_{a_3}\ \dots\,. \tag{1.3}$$

In other words, for $x \in (0,1)$ the number $?(x)$ is obtained by interpreting the symbolic sequence corresponding to the path which starts from the root node $\frac{1}{2}$ and approaches x along the Farey tree $\mathcal{F}$ as a binary expansion of a real number in $(0,1)$.

We now need a simple lemma.

Lemma 1.5.

$$x = [0; a_1, a_2, \dots] \ \Leftrightarrow\ 1 - x = \begin{cases} [0; 1 + a_2, a_3, \dots] & \text{if } a_1 = 1, \\ [0; 1, a_1 - 1, a_2, \dots] & \text{if } a_1 > 1. \end{cases}$$

Proof. Apply the identities

$$\frac{1}{a + \frac{1}{b}} + \frac{1}{1 + \frac{1}{a-1+\frac{1}{b}}} = 1 \quad \text{and} \quad \frac{1}{1 + \frac{1}{b+c}} + \frac{1}{1 + b + c} = 1. \ ■$$

Proposition 1.6. *The functions $?$ and ϱ satisfy the functional equations*

$$?(x) + ?(1-x) = 1, \qquad x \in I,$$

and

$$\varrho(x) + \varrho(1/x) = 1, \qquad x \in J.$$

Proof. The equation for $?$ follows at once from Lemma 1.5 and (1.3). That for ϱ then follows from (1.2). ■

Additional properties of ϱ are inherited via (1.2) from the properties of $?$ (see [Sa], [Ki], [VPB], [V]). We only recall that, although singular, $?(x)$ is a Hölder continuous function; moreover, $x \in \mathbb{Q}_1$ iff $?(x) \in \mathbb{Q}_2$, and x is a quadratic irrational iff $?(x)$ is a (non-dyadic) rational.

For any pair $\frac{p}{q}$ and $\frac{p'}{q'}$ of consecutive fractions in $\mathcal{T}$ the function ϱ (as $?$ on $\mathcal{F}$) equates their child to the arithmetic average. For instance we have

$$\varrho\left(\frac{p+p'}{q+q'}\right) = \frac{1}{2}\left[\varrho\left(\frac{p}{q}\right) + \varrho\left(\frac{p'}{q'}\right)\right].$$

Therefore the functions ϱ and $?$ map the SB tree $\mathcal{T}$ and the Farey tree $\mathcal{F}$ to the *dyadic tree* $\mathcal{D}$ mentioned above. Note that the set $\mathcal{D}_k$ of dyadic fractions belonging to the first $k+1$ levels of $\mathcal{D}$ is the uniformly spaced sequence $l/2^k$, $l = 0, 1, \dots, 2^k$. Reducing to the lowest terms we get

$$\mathcal{D}_0 = \left(\frac{0}{1}, \frac{1}{1}\right), \quad \mathcal{D}_1 = \left(\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right), \quad \mathcal{D}_2 = \left(\frac{0}{1}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{1}\right)$$

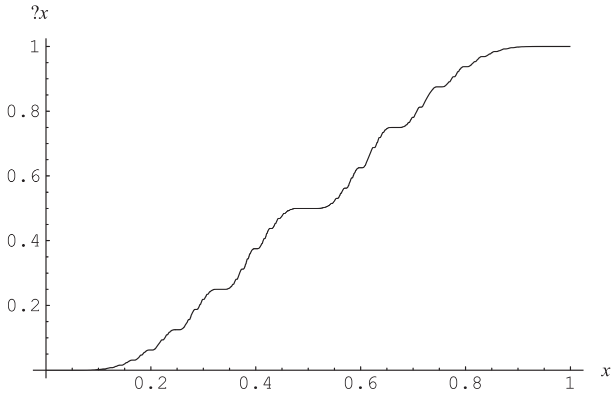

Fig. 1. The function ?

and so on. Hence, an immediate consequence of the fact that $\varrho(\mathcal{T})=?(\mathcal{F})=\mathcal{D}$ is that $\varrho(x)$ and $?(x)$ are the asymptotic distribution functions of the sequences of SB fractions and Farey fractions, respectively.

THEOREM 1.7. *Set* $\mathcal{T}_k := \bigl\{\frac{p}{q} \in \mathcal{T} : \operatorname{depth}(x) \le k\bigr\}$. *Since*

$$\left| x - \frac{\#\bigl\{\frac{p}{q} \in \mathcal{D}_k : \frac{p}{q} \le x\bigr\}}{2^k} \right| \le 2^{-k}$$

we have

$$\left| \varrho(x) - \frac{\#\bigl\{\frac{p}{q} \in \mathcal{T}_k : \frac{p}{q} \le x\bigr\}}{2^k} \right| \le 2^{-k}.$$

The same holds for ? *with* $\mathcal{T}_k$ *replaced by* $\mathcal{F}_k := \{\frac{p}{q} \in \mathcal{F} : \operatorname{rank}(x) \le k\}$.

In particular, the Fourier–Stieltjes coefficients of ϱ and ? are as in

COROLLARY 1.8. *Let*

$$c_n := \int_0^\infty e^{2\pi i n x}\, d\varrho(x).$$

Then

$$c_n = \lim_{k\to\infty} \frac{1}{2^k} \sum_{\frac{p}{q}\in\mathcal{T}_k} e^{2\pi i n \frac{p}{q}}.$$

The same holds for the coefficients of ? *with* $\mathcal{T}_k$ *replaced by* $\mathcal{F}_k$.

1.5. Permuted trees. Let $X \in \mathrm{SL}(2,\mathbb{Z})$ represent a number $x \in \mathbb{Q}^+$ as above and write it as $X = \prod_{i=1}^k M_i$ where $M_i \in \{L, R\}$ and $k = \operatorname{depth}(x)$. Let $\widehat{x}$ denote the positive rational number represented by the reversed matrix product $\widehat{X} = \prod_{i=k}^1 M_i$. Clearly $\operatorname{depth}(x) = \operatorname{depth}(\widehat{x})$, but $x = \widehat{x}$ if and only if the sequence $M_1 \ldots M_k$ is a palindrome. The permutation map $x \mapsto \widehat{x}$

yields the permuted version $\widehat{\mathcal{T}}$ of the SB tree whose first five levels are shown below (the ancestors $\frac{0}{0}$ and $\frac{1}{0}$ are omitted).

$$
\begin{array}{cccccccccccccccc}
 & & & & & & & \frac{1}{1} & & & & & & & & \\
 & & & \frac{1}{2} & & & & & & & & \frac{2}{1} & & & & \\
 & \frac{1}{3} & & & & \frac{3}{2} & & & & \frac{2}{3} & & & & \frac{3}{1} & & \\
\frac{1}{4} & & \frac{4}{3} & & \frac{3}{5} & & \frac{5}{2} & & \frac{2}{5} & & \frac{5}{3} & & \frac{3}{4} & & \frac{4}{1} & \\
\frac{1}{5} & \frac{5}{4} & \frac{4}{7} & \frac{7}{3} & \frac{3}{8} & \frac{8}{5} & \frac{5}{7} & \frac{7}{2} & \frac{2}{7} & \frac{7}{5} & \frac{5}{8} & \frac{8}{3} & \frac{3}{7} & \frac{7}{4} & \frac{4}{5} & \frac{5}{1}
\end{array}
$$

LEMMA 1.9. *Under the ancestors $\frac{0}{1}$ and $\frac{1}{0}$, the permuted SB tree $\widehat{\mathcal{T}}$ can be constructed starting from the root node $\frac{1}{1}$ and writing under each vertex $\frac{p}{q}$ the set of descendants $\left\{\frac{p}{p+q}, \frac{p+q}{q}\right\}$.*

Proof. Note that

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} n & m \\ t & s \end{pmatrix} = \begin{pmatrix} n & m \\ n+s & m+t \end{pmatrix} \leftrightarrow \frac{m+n}{m+n+s+t}$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} n & m \\ t & s \end{pmatrix} = \begin{pmatrix} n+s & m+t \\ s & t \end{pmatrix} \leftrightarrow \frac{m+n+s+t}{s+t}.$$

In other words, the matrices L and R, when acting from the left, give the left and right descendants, respectively.

Also note that if $\frac{p}{q} = [a_0; a_1, \ldots, a_n]$ then $\frac{p}{p+q} = [0; 1, a_0, a_1, \ldots, a_n]$ and $\frac{p+q}{q} = [a_0 + 1; a_1, \ldots, a_n]$. Therefore

$$\text{depth}\left(\frac{q}{p+q}\right) = \text{depth}\left(\frac{p+q}{q}\right) = \text{depth}\left(\frac{p}{q}\right) + 1.$$

This yields the claim. ∎

REMARK 1.10. The tree $\widehat{\mathcal{T}}$ has been considered in [CW] where the authors argued that if we read it row by row, and each row from left to right, then for $i \geq 2$ we can write the ith element in the form $x_i = b(i-2)/b(i-1)$, where $b(n)$ is the number of *hyperbinary* representations of n, that is, the number of ways of writing the integer n as a sum of powers of two, each power being used at most twice. For example, $8 = 2^3 = 2^2+2^2 = 2^2+2+2 = 2^2 + 2 + 1 + 1$ and therefore $b(8) = 4$. This property plainly entails that if we read, from left to right, any sequence of fractions with fixed depth, then the denominator of each fraction is the numerator of its successor.

We finally define the corresponding permutation of both the Farey tree $\mathcal{F}$ and the dyadic tree $\mathcal{D}$, denoted $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{D}}$ respectively. Clearly, we have $\widehat{\mathcal{F}} = \phi(\widehat{\mathcal{T}})$ (see Lemma 1.1). Reasoning as above one easily obtains the following simple genealogical rules:

Lemma 1.11. *Under the ancestors $\frac{0}{1}$ and $\frac{1}{1}$, the permuted trees $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{D}}$ can be constructed starting from the root node $\frac{1}{2}$ and writing under each vertex $\frac{p}{q}$ the sets of descendants $\left\{\frac{p}{p+q}, \frac{q}{2q-p}\right\}$ and $\left\{\frac{p}{2q}, \frac{p+q}{2q}\right\}$, respectively.*

The first five levels of $\widehat{\mathcal{F}}$ are

$$\frac{1}{2}$$

$$\frac{1}{3} \qquad \frac{2}{3}$$

$$\frac{1}{4} \quad \frac{3}{5} \quad \frac{2}{5} \quad \frac{3}{4}$$

$$\frac{1}{5} \quad \frac{4}{7} \quad \frac{3}{8} \quad \frac{5}{7} \quad \frac{2}{7} \quad \frac{5}{8} \quad \frac{3}{7} \quad \frac{4}{5}$$

$$\frac{1}{6} \; \frac{5}{9} \; \frac{4}{11} \; \frac{7}{10} \; \frac{3}{11} \; \frac{8}{13} \; \frac{5}{12} \; \frac{7}{9} \; \frac{2}{9} \; \frac{7}{12} \; \frac{5}{13} \; \frac{8}{11} \; \frac{3}{10} \; \frac{7}{11} \; \frac{4}{9} \; \frac{5}{6}$$

and the corresponding levels of $\widehat{\mathcal{D}}$ are

$$\frac{1}{2}$$

$$\frac{1}{4} \qquad \frac{3}{4}$$

$$\frac{1}{8} \quad \frac{5}{8} \quad \frac{3}{8} \quad \frac{7}{8}$$

$$\frac{1}{16} \quad \frac{9}{16} \quad \frac{5}{16} \quad \frac{13}{16} \quad \frac{3}{16} \quad \frac{11}{16} \quad \frac{7}{16} \quad \frac{15}{16}$$

$$\frac{1}{32} \; \frac{17}{32} \; \frac{9}{32} \; \frac{25}{32} \; \frac{5}{32} \; \frac{21}{32} \; \frac{13}{32} \; \frac{29}{32} \; \frac{3}{32} \; \frac{19}{32} \; \frac{11}{32} \; \frac{27}{32} \; \frac{7}{32} \; \frac{23}{32} \; \frac{15}{32} \; \frac{31}{32}$$

1.6. Random walks on the permuted trees. We now construct a sequence of random variables $Z_1, Z_2, \ldots$ on $\mathbb{Q}^+$ defined recursively in the following way: set $Z_1 = \frac{1}{1}$ and if $Z_k = \frac{p}{q}$ then either $Z_{k+1} = \frac{p}{p+q}$ or $Z_{k+1} = \frac{p+q}{q}$, both with probability $\frac{1}{2}$. The sequence $(Z_k)_{k\geq 1}$ can be regarded as a (symmetric) random walk on $\widehat{\mathcal{T}}$ ([1]).

Theorem 1.12. *The random walk $(Z_k)_{k\geq 1}$ enters any non-empty open interval $(a,b) \subset \mathbb{R}^+$ almost surely.*

([1]) After this paper was submitted we became aware of the work [AS], where a result akin to Theorem 1.12 is proved. See also [CLM] for a different approach.

Proof. Pick up an irrational number $x = [a_0; a_1, \dots] \in (a, b)$. Then for n large enough we can find a closed subinterval $A \subset (a, b)$ such that the c.f. expansion of each element of A starts as $[a_0; a_1, \dots, a_n, \dots]$. To fix ideas and with no loss, let n be odd. Then, according to the above and Proposition 1.3 the path on $\widehat{\mathcal{T}}$ starting from the root $\frac{1}{1}$ and entering A (for the first time) will eventually end with the word $W = L^{a_n - 1} \cdots R^{a_2} L^{a_1} R^{a_0}$. Hence it has the form UW with prefix $U \in \{L, R\}^*$ such that UW does not contain subwords equal to W but W itself.

We now proceed by induction on the length ℓ of W, writing W_ℓ for words of length ℓ. If $\ell = 1$ then there is exactly one prefix U of length k for each $k \geq 1$ (e.g. if $W_1 = L$ then $U = R^k$ is the only possible prefix) occurring with probability 2^{-k}. Summing over the prefixes we get $\sum_{k \geq 1} 2^{-k} = 1$. Therefore the claim is true for $\ell = 1$.

Now suppose it is true for $\ell = m$. When passing to $\ell = m + 1$, we have either $W_{m+1} = W_m L$ or $W_{m+1} = W_m R$, hence we have two families of paths $UW_m L$ and $UW_m R$, one of which being UW_{m+1} and thus, by the induction hypothesis, having probability $\frac{1}{2}$. We are now left with all paths starting with the "bad" ones and eventually ending with W_{m+1}. But then we can use the self-similarity of the tree and iterate the above construction. Suppose for instance that the "bad" set was $UW_m R$, that is, $W_{m+1} = W_m L$. Then at some point we will end up with the alternative $UW_m RU' W_m L$ and $UW_m RU' W_m R$ for some $U' \in \{L, R\}^*$, and the "good" set $UW_m RU' W_m L$ has probability $\frac{1}{2} \cdot \frac{1}{2}$. Iteration of this argument yields the probability $\frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{1}{2}$, which has to be added to the probability $\frac{1}{2}$ of the initial "good" set $UW_m L$. ∎

Remark 1.13. The above result can be easily extended to both $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{D}}$. However, it seems to be specific to the particular permutation which defines these trees, in particular it is plainly false for the original Stern–Brocot tree $\mathcal{T}$ (as well as for $\mathcal{F}$ and $\mathcal{D}$). We shall see later a further improvement (see Corollary 3.7).

2. PART TWO: DYNAMICS

We shall now be dealing with a class of transformations which generate the permuted trees $\widehat{\mathcal{T}}$, $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{D}}$, either one generation after the other or in genealogical way, i.e. producing elements with increasing depth.

2.1. Rank one ergodic transformations with dense orbits of rationals. It was noticed in [N] that the sequence x_i of elements of $\widehat{\mathcal{T}}$ satisfies the iteration

$$x_{i+1} = \frac{1}{1 - \{x_i\} + [x_i]}, \qquad i \geq 0. \tag{2.1}$$

We are thus led to study the map ([2]) $R : J \to J$ given by $R(\infty) := 0$ and

$$R(x) := \frac{1}{1 - \{x\} + [x]}, \qquad x \in \mathbb{R}^+.$$

PROPOSITION 2.1.

(i) *R is a Borel bijection of $\mathbb{R}^+$ onto itself.*
(ii) *For any $x \in \mathbb{R}^+$, $R(x) \in \mathbb{Q}^+$ if and only if $x \in \mathbb{Q}^+$.*
(iii) *R enumerates the set $\mathbb{Q}^+ \cup \{0\} \cup \{\infty\}$ in the following sense: let x_i be the sequence obtained by reading $\widehat{\mathcal{T}}$ row by row and each row from left to right (except the zero-th one). Then $x_i = R^i(\frac{1}{0})$.*

Proof. One easily checks that R is one-to-one and onto, with inverse

$$R^{-1}(x) = 2n + 1 - \frac{1}{x}, \qquad \frac{1}{n+1} \le x < \frac{1}{n}, \qquad n \ge 0.$$

This proves (i). Statement (ii) is immediate. Moreover, if $x \in \mathbb{N}$ then $R(x) = 1/(x+1)$ so that $\mathrm{depth}(R(x)) = \mathrm{depth}(x) + 1$. If instead $x = [a_0; a_1, \ldots, a_n]$ with $n \ge 1$ then $R(x) = 1/(a_0 + 1 - \{x\})$ so that $\mathrm{depth}(R(x)) = \mathrm{depth}(x)$ since $\mathrm{depth}(1/x) = \mathrm{depth}(x)$ and $\mathrm{rank}(\{x\}) = \mathrm{rank}(1 - \{x\})$. This yields the first part of statement (iii).

To see the second part we start by observing that if we write x in the form $x = (kq+r)/q$ with $k \ge 0$ and $0 \le r < q$ we have $R(x) = q/(kq+q-r)$. Now if $k = 0$ then $x = r/q$ and $R(x) = q/(q-r)$, namely x and $R(x)$ are left and right descendants of the fraction $r/(q-r)$. If instead $k > 0$ then x is the right descendant of $x' = ((k-1)q + r)/q$ whereas $R(x)$ is the left descendant of $x'' = q/(kq - r)$, and $x'' = R(x')$. ∎

REMARK 2.2. Note that, although the sequence $(x_i)_{i \ge 0}$ defined in (2.1) is dense in $\mathbb{R}^+$, it "diffuses" only logarithmically. Indeed, we have $x_i = n$ for $i = 2^n$ and therefore $\sup_{0<i\le n} x_i = \mathcal{O}(\log n)$. In fact, from what is proved below it follows that all orbits $\{R^i(x) : i \ge 0\}$, $x \in \mathbb{R}^+$, are dense and have this property. An automorphism of the unit circle with similar properties has been constructed in [Bo].

We now restrict to the unit interval and consider two automorphisms on it. The first one is the map $S : I \to I$ defined by (see Lemma 1.1)

$$S(x) := \phi \circ R \circ \phi^{-1}(x) \tag{2.2}$$

or else by $S(1) = 0$ and

$$S(x) = \frac{1}{2 - \left\{\dfrac{x}{1-x}\right\} + \left[\dfrac{x}{1-x}\right]}, \qquad x \in [0, 1).$$

([2]) The study of this map was suggested to one of us (C.B.) by Don Zagier.

Its inverse is

$$S^{-1}(x) = \frac{2\,n\,x-1}{(2n+1)x-1}, \qquad \frac{1}{n+1} \le x < \frac{1}{n}, \qquad n \ge 1.$$

The second is the classical *von Neumann–Kakutani transformation* $T : I \to I$ given by $T(1) := 0$ and

$$T(x) := x + \frac{3}{2^n} - 1, \qquad 1 - \frac{1}{2^{n-1}} \le x < 1 - \frac{1}{2^n}, \qquad n \ge 1.$$

It was defined in [VN] and is also called *van der Corput's transformation* or else *dyadic rotation*.

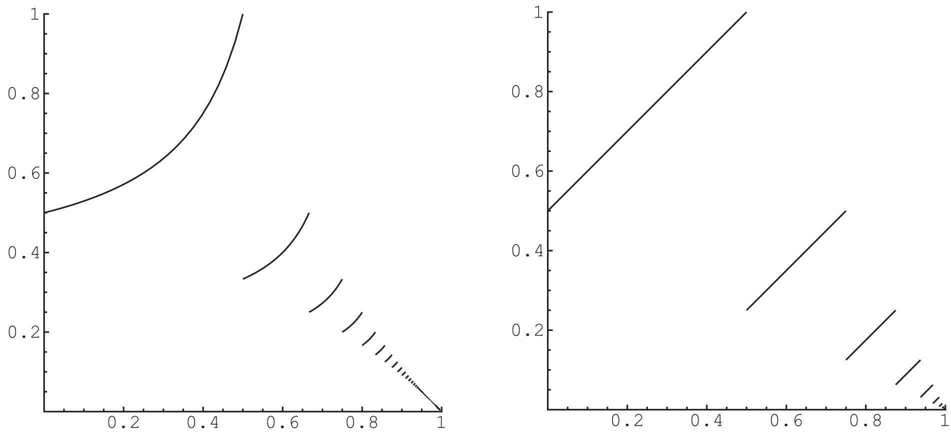

Fig. 2. The maps S (left) and T (right)

THEOREM 2.3. *We have the commutative diagram*

$$\begin{array}{ccc} J & \xrightarrow{R} & J \\ \phi\big\downarrow & & \big\downarrow\phi \\ I & \xrightarrow{S} & I \\ ?\big\downarrow & & \big\downarrow ? \\ I & \xrightarrow{T} & I \end{array}$$

Proof. The upper square follows immediately from (2.2). To check the lower square, let $x \in (0,1)$ be given by $x = [0; a_1, a_2, \dots]$. We have $\frac{x}{1-x} = [0; a_1 - 1, a_2, \dots]$ so that

$$\left[\frac{x}{1-x}\right] = \begin{cases} a_2 & \text{if } a_1 = 1, \\ 0 & \text{if } a_1 > 1, \end{cases} \qquad \left\{\frac{x}{1-x}\right\} = \begin{cases} [0; a_3, a_4, \dots] & \text{if } a_1 = 1, \\ [0; a_1 - 1, a_2, \dots] & \text{if } a_1 > 1, \end{cases}$$

and therefore

$$S(x)=\begin{cases}(2+a_2-[0;a_3,a_4,\dots])^{-1} & \text{if } a_1=1,\\ (2-[0;a_1-1,a_2,\dots])^{-1} & \text{if } a_1>1.\end{cases}$$

By Lemma 1.5 this becomes

$$S(x)=\begin{cases}[0;a_2+1,1,a_3-1,a_4,\dots] & \text{if } a_1=1,\\ [0;1,1,a_1-2,a_2,\dots] & \text{if } a_1>1,\end{cases} \tag{2.3}$$

where for $a_k \ge \ell \ge 1$ for some $k \ge 1$ we set

$$[a_0;a_1,a_2,\dots,a_{k-1},a_k-\ell,a_{k+1},\dots]=[a_0;a_1,a_2,\dots,a_{k-1}+a_{k+1},\dots]$$

if $a_k=\ell$. On the other hand, the map $T(x)$ is also named the dyadic rotation because of the following fact (see, e.g., [PF, p. 120]): if we expand $x\in[0,1]$ in base two, i.e. we write $x=\sum_{k=0}^{\infty}\omega_k 2^{-k-1}=0.\omega_1\omega_2\dots$ with $\omega_k\in\{0,1\}$, it acts as $T(0.111\dots)=0.000\dots$, and for $n\ge 1$,

$$T(0.\underbrace{11\dots1}_{n-1}0\,\omega_{n+1}\omega_{n+2}\dots)=0.\underbrace{00\dots0}_{n-1}1\,\omega_{n+1}\omega_{n+2}\dots$$

Therefore if $x=[0;a_1,a_2,\dots]$ then using (1.3) we find

$$T(?(x))=\begin{cases}0.\underbrace{00\dots0}_{a_2}1\underbrace{00\dots0}_{a_3-1}\underbrace{11\dots1}_{a_4}\dots & \text{if } a_1=1,\\ 0.1\underbrace{00\dots0}_{a_1-2}\,\underbrace{11\dots1}_{a_2}\dots & \text{if } a_1>1,\end{cases}$$

which is identical to what we obtain by applying ? to (2.3). ∎

We now derive some consequences of the above theorem.

COROLLARY 2.4. *The maps S and T enumerate the sets $\mathbb{Q}_1$ and $\mathbb{Q}_2$, respectively. More specifically, let for instance y_i be the sequence obtained by reading $\widehat{\mathcal{F}}$ row by row and each row from left to right (except the zero-th one). Then $y_i=S^i\big(\frac{1}{1}\big)$. A similar statement holds for $\widehat{\mathcal{D}}$ and T.*

Proof. Follows at once from Proposition 2.1 and Theorem 2.3. ∎

COROLLARY 2.5. *The systems (J,R) and (I,S) are uniquely ergodic and hence $(J,R,d\varrho)$ and $(I,S,d?)$ are ergodic.*

Proof. The first statement follows from the above as topological conjugacy preserves unique ergodicity and the system (I,T) has this property. Moreover, the Lebesgue measure dx is T-invariant so that by the above and (1.2) the maps R and S preserve the measures $d\varrho$ and $d?$ respectively. ∎

Corollary 2.6. *The systems (J,R) and (I,S) are of rank one. Moreover, they have the same spectrum which is discrete with eigenvalues $e^{2\pi i\alpha}$ for any dyadic rational α.*

Proof. The system (I,T) has this property. Let us briefly recall how this is obtained. One starts by setting $A(1,n) = [0, 2^{-n})$ for $n \geq 0$ and noticing that T maps in an affine way $A(i,n) = T^{i-1}A(1,n)$ onto $A(i+1,n)$ for $i = 1,\dots,2^n$. Clearly, these intervals are not ordered lexicographically but in the way induced by T. For example, for $n=3$ we have $000 \mapsto 100 \mapsto 010 \mapsto 110 \mapsto 001 \mapsto 101 \mapsto 011 \mapsto 111$. One may then write the intervals so ordered one above the other, thus making a stack which partitions the whole space. The action of T is then that of climbing up one level in the n-stack but is not defined on the top level. At step $n+1$, i.e. looking at the action of the iterates of T on $A(1,n+1)$, the stack is cut into two equal halves and the right half is stacked on top the left half. This defines the action of T on a finer partition of the space. This procedure eventually leads to determining T on the whole space. Finally, to get the same property for (I,S) it will suffice to follow the above procedure with the family of intervals $B(i,n) = ?^{-1}(A(i,n))$ (stacked in the same order). Clearly, although all the intervals $A(i,n)$, $i=1,\dots,2^n$, have the same length 2^{-n}, the corresponding $B(i,n)$ do not. A similar construction can be done for (J,G) with the intervals $C(i,n) = \phi^{-1}(B(i,n))$. The last assertion follows again from the same property for (I,T) along with topological conjugacy (see, e.g., [PF, p. 23]). ∎

2.2. Non-invertible Markov maps. We now introduce three non-invertible maps which generate the trees $\widehat{\mathcal{T}}$, $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{D}}$ genealogically, i.e. via descendants. With the notations of Theorem 2.3, the first one is the map $G : J \to J$ given by

$$G(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } 0 \leq x < 1, \\ x - 1 & \text{if } x \geq 1. \end{cases}$$

The second is the *modified Farey map* $F : I \to I$ given by

$$F(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } 0 \leq x < 1/2, \\ 2 - 1/x & \text{if } 1/2 \leq x \leq 1, \end{cases}$$

and the third is the *doubling map* $D : I \to I$ given by

$$D(x) = 2x \,(\mathrm{mod}\, 1).$$

They are expansive orientation preserving piecewise analytic endomorphisms such that the sets $G^{-1}(x)$, $F^{-1}(x)$ and $D^{-1}(x)$ consist of exactly two points for each x. More specifically,

$$G^{-1}(x) = \left\{ \frac{x}{1+x}, x+1 \right\}, \qquad x \in J,$$
$$F^{-1}(x) = \left\{ \frac{x}{1+x}, \frac{1}{2-x} \right\}, \qquad D^{-1}(x) = \left\{ \frac{x}{2}, \frac{x}{2} + \frac{1}{2} \right\}, \qquad x \in I.$$

Both F and D fix the boundary points 0 and 1, but for F these are *indifferent fixed points*, i.e. $F'(0) = F'(1) = 1$. More specifically, 0 is a *weakly repelling* fixed point whereas 1 is *weakly attracting*. On the other hand, we can say that G has two indifferent fixed points at 0 and ∞.

THEOREM 2.7. *The permuted tree* $\widehat{\mathcal{T}}$ *can be constructed genealogically from its root* $\frac{1}{1}$ *by writing under each leaf* x *the set of descendants* $G^{-1}(x)$. *The same can be done for* $\widehat{\mathcal{F}}$ *and* $\widehat{\mathcal{D}}$ *starting from their root* $\frac{1}{2}$ *with the sets of descendants* $F^{-1}(x)$ *and* $D^{-1}(x)$, *respectively. Furthermore*, *we have the commutative diagram*

$$\begin{array}{ccc} J & \xrightarrow{G} & J \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ I & \xrightarrow{F} & I \\ {\scriptstyle ?}\big\downarrow & & \big\downarrow{\scriptstyle ?} \\ I & \xrightarrow{D} & I \end{array}$$

Proof. The first assertion follows from Lemma 1.9, equation (3.4) and Lemma 1.11. The proof of the conjugation between G and F is immediate. That for F and D can be obtained by reasoning along the same lines as in the proof of Theorem 2.3, starting from the observation that D acts as the shift on binary expansions whereas the action of F is the Farey shift $[0; a_1, a_2, \ldots] \mapsto [0; a_1 - 1, a_2, \ldots]$ on the interval $[0, 1/2]$ and $[0; 1, a_2, \ldots] \mapsto 1 - [0; a_2, \ldots]$ on $(1/2, 1]$. Then use Lemma 1.5. We leave the details to the interested reader. ∎

REMARK 2.8. Conversely, using the maps G, F and D one can retrace the path from a leaf x in any of the trees $\mathcal{T}$, $\mathcal{F}$ or $\mathcal{D}$ back to the root. For instance, for $x \in \mathcal{T}$ let $X = \prod_{i=1}^{k} M_i$ be the element which uniquely represents x in $\mathrm{SL}(2, \mathbb{Z})$ with $k = \mathrm{depth}(x)$, according to Proposition 1.3. One then sees that the following rule is in force: if $G^{(i-1)}(x) < 1$ then $M_i = L$, and if $G^{(i-1)}(x) > 1$ then $M_i = R$ for $i = 1, \ldots, k$ with $k = \mathrm{depth}(x)$ such that $G^k(x) = 1$.

REMARK 2.9. The map D preserves the Lebesgue measure dx on I, whereas the map F preserves the a.c. infinite measure $\mu(dx) = dx/x(1-x) = \big(\frac{d}{dx} \log \phi^{-1}(x)\big) dx$ on I, as one easily checks. This entails that G preserves the (infinite) measure $\nu(dx) = \mu\circ\phi(dx) = dx/x$ on J. Each of these invariant measures is ergodic and exact [Th1], [Z]. On the other hand, from the above theorem it follows that also the measure $d?$ is invariant under F (just as $d\varrho$ for G) and the entropy of $(I, F, d?)$ is $\log 2$. Therefore $d?$ is the *measure of maximal entropy* for (I, F) (as is $d\varrho$ for (J, G)).

3. TRANSFER OPERATORS, MARKOV CHAINS AND MARTINGALES

To the map G we associate a *generalised transfer operator* L_q acting on $f : J \to \mathbb{C}$ as

$$(L_q f)(x) = \sum_{y\in G^{-1}(x)} \frac{f(y)}{|G'(y)|^q}$$

or else

$$(L_q f)(x) = \frac{1}{(1+x)^{2q}}\, f\left(\frac{x}{1+x}\right) + f(x+1) \tag{3.1}$$

where q is a real or complex parameter. We point out that a continuous fixed function for L_q satisfies the functional equation

$$f(x) = f(x+1) + \frac{1}{(1+x)^{2q}}\, f\left(\frac{x}{1+x}\right)$$

which is called the *Lewis–Zagier three-term functional equation* and is related to the spectral theory of the hyperbolic laplacian on the modular surface (see [LeZa] and references therein).

In the same way, the operators associated to D and F act on $f : I \to \mathbb{C}$ as

$$f(x) \mapsto \frac{1}{2^q}\, f\left(\frac{x}{2}\right) + \frac{1}{2^q}\, f\left(\frac{x}{2}+\frac{1}{2}\right) \tag{3.2}$$

and

$$f(x) \mapsto \frac{1}{(1+x)^{2q}}\, f\left(\frac{x}{1+x}\right) + \frac{1}{(2-x)^{2q}}\, f\left(\frac{1}{2-x}\right) \tag{3.3}$$

respectively. For the spectral theory of an operator closely related to (3.3) see [I] and [BGI].

Let $\varPhi_s$, $s \in \{0,1\}$, be the inverse branches of G, i.e.

$$\varPhi_0(x) = \frac{x}{1+x}, \qquad \varPhi_1(x) = x+1. \tag{3.4}$$

They satisfy

$$\varPhi_s(1/x) = \frac{1}{\varPhi_{1-s}(x)}, \qquad s \in \{0,1\}. \tag{3.5}$$

Let moreover $p_s(\cdot)$, $s \in \{0,1\}$, be a pair of positive Borel functions such that $p_0(x) + p_1(x) = 1$ for all $x \in J$. We now want to study the

Markov chain with state space J where at each step, starting from a state $x \in J$, two transitions are possible towards the states $\Phi_0(x)$ and $\Phi_1(x)$, with probabilities $p_0(x)$ and $p_1(x)$ respectively. Note that for $x = \frac{1}{1}$ and $p_i(x) = \frac{1}{2}$, $i = 0, 1$, this Markov chain reduces to the random walk on $\widehat{\mathcal{T}}$ discussed in Theorem 1.12.

We now briefly adapt to our context some basic facts about canonical Markov chains associated to Markov transfer operators (see [CoRa1], also [CoRa2] for an application to the dyadic transfer operator (3.2) and [Ra]). Let $P : L^\infty(J) \to L^\infty(J)$ be the Markov operator acting as

$$(Pf)(x) = p_0(x) f(\Phi_0(x)) + p_1(x) f(\Phi_1(x)).$$

A measurable function $h : J \to \mathbb{C}$ satisfying $Ph = h$ is called *P-harmonic*. We shall make the further assumption that the transition probabilities satisfy

$$p_s(1/x) = p_{1-s}(x), \quad s \in \{0, 1\}, \quad \forall x \in J. \tag{3.6}$$

The symmetries (3.5) and (3.6) yield at once the following

LEMMA 3.1. *The averaging operator $A : L^\infty(J) \to L^\infty(J)$ acting as*

$$(Af)(x) = \frac{f(x) + f(1/x)}{2}$$

commutes with P. In particular, if $h : J \to \mathbb{C}$ is a bounded P-harmonic function then Ah has the same property.

A positive measure ν is called *P-invariant* if $\nu P = \nu$, i.e. $\int_J Pf \, d\nu = \int_J f \, d\nu$ for all measurable $f : J \to \mathbb{C}$. In turn, one readily realizes that this condition is equivalent to

$$\frac{d(\nu \circ \Phi_s)}{d\nu} = p_s(\cdot), \quad s \in \{0, 1\}.$$

Now, for $\Omega := \{0, 1\}^{\mathbb{N}}$, an n-dimensional *cylinder* of Ω is a subset of the type $C(i_1, \ldots, i_n) = \{\omega \in \Omega : \omega_1 = i_1, \ldots, \omega_n = i_n\}$. The cylinder sets generate the topology of Ω and its Borel σ-algebra $\mathcal{F}$.

Given $x \in J$ let $U(x)$ be the closure of the set of all possible paths starting at x, i.e.

$$U(x) = \overline{\bigcup_{\omega \in \Omega} \{\Phi_{\omega_n} \circ \cdots \circ \Phi_{\omega_1}(x) : n \geq 1\}}.$$

This is clearly a compact invariant set, in the sense that if $y \in U(x)$ then $\Phi_i(y) \in U(x)$, $i \in \{0, 1\}$. More generally, a compact subset V of J is called *invariant* if for all $x \in V$ and all $i \in \{0, 1\}$ such that $p_i(x) > 0$ we have $\Phi_i(x) \in V$.

A first basic fact (see [CoRa1, Sec. 3.4]; or else [Jo, Chap. 2.4]) is that for each $x \in J$ there is a unique probability measure $\mathbb{P}_x$ on Ω such that

$$\mathbb{P}_x(C(i_1,\ldots,i_n)) = \prod_{k=1}^{n} p_{i_k}(\Phi_{i_1}^{i_k}(x))$$

where we have used the notation

$$\Phi_{i_1}^{i_k} := \Phi_{i_k} \circ \cdots \circ \Phi_{i_1}.$$

The symmetries (3.5) and (3.6) entail the following

Lemma 3.2. *For each $x \in J$ we have*

$$\mathbb{P}_x(C(i_1,\ldots,i_n)) = \mathbb{P}_{1/x}(C(1-i_1,\ldots,1-i_n)).$$

For $\omega \in \Omega$ let $X_k(\omega) = \omega_k$ be the kth coordinate function on Ω and $\mathcal{Z}_n$ the subalgebra of $C(\Omega)$ generated by the first n coordinates $\{X_k : 1 \le k \le n\}$. The $\mathcal{Z}_n$ form a filtration in that $\mathcal{Z}_n \subset \mathcal{Z}_{n+1}$. Note that if $X \in \mathcal{Z}_n$ then

$$\mathbb{E}_x[X] = \sum_{(\omega_1,\ldots,\omega_n)\in\{0,1\}^n} \prod_{k=1}^{n} p_{\omega_k}(\Phi_{\omega_1}^{\omega_k}(x)) X(\omega_1,\ldots,\omega_n).$$

In particular, if there is $h : J \to \mathbb{C}$ such that

$$X(\omega_1,\ldots,\omega_n) = h(\Phi_{\omega_1}^{\omega_n}(x))$$

then

$$\mathbb{E}_x[X] = (P^n h)(x) \tag{3.7}$$

since

$$(P^n f)(x) = \sum_{(\omega_1,\ldots,\omega_n)\in\{0,1\}^n} \Big[\prod_{j=1}^{n} p_{\omega_j}(\Phi_{\omega_1}^{\omega_{j-1}}(x))\Big] f(\Phi_{\omega_1}^{\omega_n}(x)).$$

Now, having fixed $x \in J$, define

$$W_0(x,\omega) := x \quad \text{and} \quad W_n(x,\omega) := \Phi_{X_1(\omega)}^{X_n(\omega)}(x), \quad n \ge 1. \tag{3.8}$$

The process $\{W_n(x,\,\cdot\,) : n \ge 0\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P}_x)$ is a Markov chain on J with initial state x and for any measurable function $f : J \to \mathbb{C}$ we have

$$\lim_{n\to\infty} (P^n f)(x) = \lim_{n\to\infty} \mathbb{E}_x[f(W_n(x,\,\cdot\,))].$$

Moreover, if $h : J \to \mathbb{C}$ is a measurable bounded P-harmonic function then

$$\begin{aligned}\mathbb{E}_x[h(W_{n+1}(x,\,\cdot\,)) \,|\, \mathcal{Z}_n] &= \sum_{\omega_{n+1}\in\{0,1\}} p_{\omega_{n+1}}(x) h(\Phi_{\omega_1}^{\omega_{n+1}}(x)) \\ &= (Ph)(\Phi_{\omega_1}^{\omega_n}(x)) = h(\Phi_{\omega_1}^{\omega_n}(x)) = h(W_n(x,\,\cdot\,)).\end{aligned}$$

In other words, the sequence of random variables $\{h(W_n(x,\,\cdot\,)) : n \ge 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P}_x)$ is a bounded martingale (relative to the filtration $\{\mathcal{Z}_n : n \ge 1\}$)

and therefore it converges pointwise $\mathbb{P}_x$-a.e. The limit random variable $H(x,\,\cdot\,) = \lim_{n\to\infty} h(W_n(x,\,\cdot\,))$ satisfies

$$H(x,\omega) = H(\Phi_{\omega_1}(x), \sigma\omega) \tag{3.9}$$

where $\sigma : \Omega \to \Omega$ is the left shift acting as $(\sigma\omega)_i = \omega_{i+1}$. A bounded measurable function $H : J \times \Omega \to \mathbb{C}$ satisfying (3.9) is said to be a *cocycle*. Conversely, by (3.7), h may be recovered from the cocycle H as

$$h(x) = \mathbb{E}_x[H(x,\,\cdot\,)]. \tag{3.10}$$

REMARK 3.3. Note that $P : L^\infty(J) \to L^\infty(J)$ has norm one. Therefore if $h \in L^\infty(J)$ is an eigenfuction of P corresponding to a real and positive eigenvalue then the sequence $\{h(W_n(x,\,\cdot\,)) : n \ge 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P}_x)$ is a supermartingale, which again converges $\mathbb{P}_x$-a.e. to a limit cocycle H.

REMARK 3.4. As pointed out in [Jo, p. 50], (3.10) can be thought of as an analogue of the classical result about the existence of boundary functions for bounded harmonic functions via Poisson integral.

We now discuss two specific Markov chains of the above type, denoted MC_0 and MC_1, corresponding to the choices $q = 0$ and $q = 1$ in (3.1).

3.1. The Markov chain MC_0. Setting $q=0$ in (3.1) we have $\frac{1}{2}L_0 1 = 1$. One can then consider the Markov (i.e. normalised) operator P_0 acting as $P_0 f = \frac{1}{2}L_0 f$. More explicitly,

$$(P_0 f)(x) = \frac{1}{2}\, f\left(\frac{x}{1+x}\right) + \frac{1}{2}\, f(x+1).$$

LEMMA 3.5. *Let ϱ be as in* (1.1). *The probability measure $d\varrho$ on J is P_0-invariant.*

Proof. From the fact that the function ϱ is the distribution function of the (permuted) Stern–Brocot fractions (cf. Theorem 1.7) and Lemma 1.9 one readily finds that ϱ satisfies the functional equation

$$2\varrho(x) = \begin{cases} \varrho\left(\dfrac{x}{1-x}\right) & \text{if } 0 < x < 1, \\ \varrho(x-1) + 1 & \text{if } x \ge 1. \end{cases}$$

The claim now follows directly. ∎

Setting $p_0(x) = p_1(x) = 1/2$ we see that there is only one compact invariant set, the set J, and according to [CoRa2, Sec. IV] the only bounded continuous P_0-harmonic functions are the constants. Moreover, the unique probability measure $\mathbb{P}_x^{(0)}$ on Ω such that

$$\mathbb{P}_x^{(0)}(C(i_1, \ldots, i_n)) = 2^{-n}$$

is atomless for each $x \in J$. The Markov chain MC_0 is then defined as in (3.8) on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_x^{(0)})$. We summarise the above in the following

THEOREM 3.6. *For $f \in L^1(\mathbb{R}^+, d\varrho)$ we have $\mathbb{P}_x^{(0)}$-a.s.*

$$\lim_{n\to\infty} (P_0{}^n f)(x) = \lim_{n\to\infty} \mathbb{E}_x^{(0)}[f(W_n(x,\,\cdot\,))] = \int_0^\infty f\, d\varrho.$$

Taking $f = 1_{(a,b)}$, $(a,b) \subset \mathbb{R}^+$, one gets the following improvement of Theorem 1.12.

COROLLARY 3.7. *The random walk $(Z_k)_{k\ge1}$ visits any non-empty open interval $I = (a,b) \subset \mathbb{R}^+$ a.s. with asymptotic frequency $\varrho(I)$.*

3.2. The Markov chain MC_1. Setting $q = 1$ in (3.1) we have $L_1 g = g$ where $g(x) = 1/x$ is the G-invariant density. We then consider the Markov operator P_1 acting as $P_1 f = g^{-1} L_1(f \cdot g)$, or

$$(P_1 f)(x) = \frac{1}{x+1}\, f\left(\frac{x}{1+x}\right) + \frac{x}{1+x}\, f(x+1).$$

The validity of $L_1 g = g$ is equivalent to the fact that the infinite measure $\nu(dx) = dx/x$ on J is P_1-invariant.

Set

$$p_0(\infty) = p_1(0) = 0, \qquad p_0(0) = p_1(\infty) = 1 \tag{3.11}$$

and

$$p_0(x) = \frac{1}{x+1}, \qquad p_1(x) = \frac{x}{x+1}, \qquad x \in (0,\infty). \tag{3.12}$$

These transition probabilities plainly satisfy the symmetry (3.6) although, unlike the previous case, they are not strictly positive.

The Markov chain MC_1 is now defined as in (3.8) on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_x^{(1)})$, where $\mathbb{P}_x^{(1)}$ is the transition measure on Ω arising from the probabilities (3.11) and (3.12). It satisfies the following

LEMMA 3.8. *For $x \in (0,\infty)$ the measures $\mathbb{P}_x^{(1)}$ have no atoms. On the other hand, both $\mathbb{P}_0^{(1)}$ and $\mathbb{P}_\infty^{(1)}$ are purely atomic with $\mathbb{P}_0^{(1)} = \delta_{0^\infty}$ and $\mathbb{P}_\infty^{(1)} = \delta_{1^\infty}$.*

Proof. From (3.4) and (3.12) it follows that the path of length n starting at $x \in [1,\infty)$ and having largest probability is that corresponding to the word $\omega = 1\cdots1$. If instead $0 < x < 1$, it corresponds to $\omega = 0\cdots0$. On the other hand, we have

$$\mathbb{P}_x^{(1)}(C(1,\dots,1)) = \mathbb{P}_{1/x}^{(1)}(C(0,\dots,0)) = \prod_{k=0}^{n-1} \frac{x+k}{x+k+1} = \frac{x}{x+n} \to 0$$

as $n \to \infty$, proving the first assertion. The other one is straightforward. ■

Now, a path starting somewhere in J and converging to 0 corresponds to a sequence of the form $(\omega_1, \dots, \omega_n, 0, 0, \dots)$ for some $n \geq 1$. The symmetric sequence $(1-\omega_1, \dots, 1-\omega_n, 1, 1, \dots)$ yields a corresponding path which converges to ∞. By Lemma 3.2, if we let the first path start at x and the second one at $1/x$, all finite equal portions of them have the same probability. We can thus concentrate on the paths starting at x and converging to 0. In turn, these can be put in a one-to-one correspondence with $\mathbb{Q}_2$ via the mapping

$$\mathbb{Q}_2 \ni a = \sum_{i=1}^{n} \omega_i 2^{-i} \mapsto \omega(a) = (\omega_1, \dots, \omega_n, 0, 0, \dots).$$

Another copy of $\mathbb{Q}_2$ is obtained via the mapping

$$1 - \mathbb{Q}_2 \ni 1 - a = \sum_{i=1}^{n} (1-\omega_i) 2^{-i} + 2^{-n}$$
$$\mapsto \omega(1-a) = (1-\omega_1, \dots, 1-\omega_n, 1, 1, \dots).$$

With the identification $a \leftrightarrow \omega(a)$ we set

$$\mathbb{P}_x^{(1)}(\mathbb{Q}_2) = \sum_{a \in \mathbb{Q}_2} \mathbb{P}_x^{(1)}(\omega(a))$$

so that Lemma 3.8 can be rephrased in the form

$$\mathbb{P}_x^{(1)}(\mathbb{Q}_2) + \mathbb{P}_x^{(1)}(1 - \mathbb{Q}_2) = \delta_x^0 + \delta_x^\infty. \tag{3.13}$$

Remark 3.9. Having fixed $x, y \in J$ let $B(x,y)$ be the tail event

$$B(x,y) := \{\omega \in \Omega : \lim_{n \to \infty} W_n(x,\omega) = y\}.$$

According to (3.10), the cocycle H_1 associated to h_1 is

$$H_1(x,\omega) = 1_{B(x,0) \cup B(x,\infty)}(\omega).$$

Note that (3.13) can be further rephrased as

$$\mathbb{P}_x^{(1)}[B(x,0)] + \mathbb{P}_x^{(1)}[B(x,\infty)] = \delta_x^0 + \delta_x^\infty.$$

The following is now a consequence of previous results and of [CoRa2, Sec. IV]).

Theorem 3.10. *The only bounded continuous P_1-harmonic functions are the constants.*

Proof. This follows by repeating the same argument as in Proposition 4.1 and Theorem 4.2 in [CoRa2]. In particular, there exists a set $\Omega_0 \subset \Omega$ such that $\mathbb{P}_x^{(1)}(\Omega_0) = 1$ for all $x \in J$, and for each $\omega \in \Omega_0$ the sequence $h(W_n(1,\omega))$ is convergent for any bounded continuous P_1-harmonic function h. This implies that there exists a compact invariant set C which contains the accumulation points $A(\omega)$ of the sequence $\{W_n(1,\omega)\}$ for all $\omega \in \Omega_0$, that is,

$$\bigcup_{\omega\in\Omega_0} \Big(\bigcap_{N\geq 1} \overline{\{W_n(1,\omega) : n \geq N\}}\Big) \subset C,$$

and such that any bounded continuous harmonic function is completely determined by its values on C. Moreover, the set C can be written as a disjoint union of compact invariant sets on which all bounded continuous harmonic functions are constant, each of the sets $A(\omega)$ being contained in one of these components. On the other hand, from (3.11) it follows that the singletons $\{0\}$ and $\{\infty\}$ are two disjoint compact invariant sets and from (3.12) one sees that they are the only invariant sets of this type (alternatively one can use Theorem 1.12 along with (3.11)). Therefore the only possible decomposition of C into disjoint compact invariant sets is $C = J$. It follows that the only bounded continuous P_1-harmonic functions are the constants. ∎

Finally, the limit properties of the chain MC_1 are stated in the following

THEOREM 3.11. *For any $f \in L^1(\mathbb{R}^+, d\nu)$ with $\nu(dx) = dx/x$, there is a sequence $a_n \sim n/\log n$ such that*

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=0}^{n-1} (P_1^n f)(x) = \int_0^\infty f\, d\nu$$

uniformly on compact subsets of $(0,\infty)$.

Proof. One can first prove the corresponding statement for the dynamical system (I, F, μ) defined above and then the result can be carried over to (J, G, ν) and to the associated Markov chain MC_1 via the diffeomorphism ϕ. On the other hand, for a large class of infinite measure-preserving interval maps this result has been obtained by several authors, notably by Thaler in [Th2] (see also [ADU], [Z]). So we only have to check that the conditions used there are satisfied for (I, F, μ).

The Perron–Frobenius operator M associated to the interval map F acts on a function $f : I \to \mathbb{C}$ as (cf. (3.3) with $q = 1$)

$$\begin{aligned}(Mf)(x) &= \frac{1}{(1+x)^2} f\Big(\frac{x}{1+x}\Big) + \frac{1}{(2-x)^2} f\Big(\frac{1}{2-x}\Big)\\ &=: u(0,x) f(\Psi_0(x)) + u(1,x) f(\Psi_1(x))\end{aligned}$$

where we have set $F^{-1}(x) = \{\Psi_0(x), \Psi_1(x)\}$.

Denoting $x_0 = 0$ and $x_1 = 1$ the two indifferent fixed points of the map F we have

$$F(x) = x + (-1)^s (x - x_s)^2 + o(|x - x_s|^2), \qquad x \to x_s,\ s = 0, 1. \tag{3.14}$$

We recall that from a general result proved in [Th1] it follows that the infinite measure $\mu(dx) = h(x)dx$ with $h(x) := 1/x(1-x)$ is the *unique* a.c. measure invariant for (I, F). Now set

$$u(s_1 \cdots s_n, x) := \prod_{k=1}^{n} u(s_k, \Psi_{s_k} \circ \cdots \circ \Psi_{s_1}(x)), \qquad s_k \in \{0,1\},\ k = 1, \dots, n,$$

and $u_n(s, x) := u(s \cdots s, x)$, $s \in \{0,1\}$. Given $\varepsilon > 0$ set moreover

$$A_\varepsilon := I \setminus \bigcup_{s \in \{0,1\}} (x_s - \varepsilon, x_s + \varepsilon).$$

Lemma 3.12. *For each $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that for all $n \geq 1$ and all $s_1, \dots, s_n \in \{0,1\}^n$ we have*

$$\sum_{j=1}^{n} |u'(s_1 \cdots s_n, x)| \leq C, \qquad x \in A_\varepsilon.$$

Proof. Since the second derivative F'' is strictly monotone on each interval of monotonicity of F, it will suffice to check the property for $u_n(s, x)$, $s = 0, 1$. We have

$$u_n(0, x) = \prod_{k=0}^{n-1} \left(1 + \frac{x}{1 + kx}\right)^{-1} = (1 + nx)^{-2}$$

so that $u_n'(0, x) = 2n/(1 + nx)^3$. On the other hand,

$$u_n(1, x) = \prod_{k=0}^{n-1} \left(\frac{k + 1 - kx}{k + 2 - (k+1)x}\right) = (n + 1 - nx)^{-2}$$

so that $|u_n'(1, x)| = 2n/(n + 1 - nx)^3$. ∎

From now on the proof can follow word for word the path traced in [Th2]. In particular, the asymptotic identification of the normalising sequence follows from (3.14). ∎

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