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ON THE ENTROPY OF DARBOUX FUNCTIONS

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Abstract. We prove some results concerning the entropy of Darboux (and almost continuous) functions. We first generalize some theorems valid for continuous functions, and then we study properties which are specific to Darboux functions. Finally, we give theorems on approximating almost continuous functions by functions with infinite entropy.

Introduction. In the classical theory of dynamical systems one usually assumes that all the functions under consideration are continuous. However, some investigations lead to considering Darboux functions or almost continuous functions (cf. e.g. a question raised by W. Transue, cited in [10]); a discussion of this topic can be found in [20]. There are a lot of recent papers dealing with dynamical systems generated by Darboux-like functions (e.g. [7], [10], [16], [18], [21]). The main aim of the current one is to give some results on the entropy of Darboux or almost continuous functions.

Our direct inspiration was M. Čiklová's paper [7]. She generalizes a certain theorem valid for continuous functions to the case of functions whose graph is a connected G_δ set (in particular, such functions have the Darboux property). We mainly aim at the properties specific to dynamical systems generated by Darboux functions. But we also extend several classical results, regarding them as important tools (see Section 2).

In Section 3 we define almost fixed points of a Darboux function. This notion is characteristic for discontinuous Darboux functions. We investigate the fundamental properties of Darboux (or Darboux-like) functions having at least one almost fixed point. In Section 4 we consider approximation of an arbitrary almost continuous function by almost continuous functions having almost fixed points (or having infinite entropy).

The paper is completed by an open problem concerning the relationship between the entropy of a function and the set of periodic points of this function.

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1. Preliminaries. We will use standard definitions and notations (see [1], [2], [4], [5], [9]). The graph of a function f will be denoted by $\Gamma(f)$.

We will consider exclusively Darboux functions (or functions belonging to smaller classes) from the unit interval into itself. We say that $f:[0,1]\to[0,1]$ is a *Darboux function* if whenever $x,y\in[0,1]$ and α is any number between $f(x)$ and $f(y)$, there is a number z between x and y such that $f(z)=\alpha$. Equivalently, f is a Darboux function if $f(C)$ is connected for any connected set $C\subset[0,1]$. It is well known that f is a Darboux function iff every point $x\in[0,1]$ is a Darboux point of f (the definition will be given shortly). We will use some notions and notations pertaining to Darboux points of f ([12], [5]).

Let us recall the definitions of the *left* and *right range* of f at x_0:

$$\begin{aligned}\alpha\in \mathrm{R}^-(f,x_0)) &\quad\text{iff}\quad f^{-1}(\alpha)\cap(x_0-\delta,x_0)\neq\emptyset,\ \text{for any } \delta>0,\\ \alpha\in \mathrm{R}^+(f,x_0)) &\quad\text{iff}\quad f^{-1}(\alpha)\cap(x_0,x_0+\delta)\neq\emptyset,\ \text{for any } \delta>0.\end{aligned}$$

An element β is a *left-hand* (resp. *right-hand*) *cluster value* of f at x_0 iff there exists a sequence $\{x_n\}$ (resp. $\{y_n\}$) such that $x_n\nearrow x_0$ and $f(x_n)\to\beta$ (resp. $y_n\searrow x_0$ and $f(y_n)\to\beta$). Obviously, if $\alpha\in\mathrm{R}^-(f,x_0)$ (resp. $\alpha\in\mathrm{R}^+(f,x_0)$) then α is a left-hand (resp. right-hand) cluster value of f, but the converse is false.

We will say that x_0 is a *left-hand* (resp. *right-hand*) *Darboux point* of f if for each left-hand (resp. right-hand) cluster value β of f at x_0 different from $f(x_0)$ and each γ belonging to the interval with end-points $f(x_0)$ and β we have $\gamma\in\mathrm{R}^-(f,x_0)$ (resp. $\gamma\in\mathrm{R}^+(f,x_0)$) (if $x_0=0$ or $x_0=1$ then we consider only one-sided cluster values).

We shall say that x_0 is a *Darboux point* of a function f if it is simultaneously a right-hand and a left-hand Darboux point of f.

A function $f:[0,1]\to[0,1]$ is *almost continuous* (in the sense of Stallings) if every open set $U\subset[0,1]\times[0,1]$ containing $\Gamma(f)$ contains the graph of a continuous function $g:[0,1]\to[0,1]$. The family of all almost continuous functions is denoted by $\mathcal{A}$.

In the class of all functions from the unit interval into itself the following inclusions hold:

$$\mathcal{C}\subset\mathcal{A}\subset\mathcal{C}onn\subset\mathcal{D},$$

where $\mathcal{C}$ (resp. $\mathcal{D}$) is the family of all continuous (resp. Darboux) functions and $\mathcal{C}onn$ is the class of all functions with connected graph.

In Section 4 we consider the following two topologies on the space $\mathcal{A}$ of almost continuous functions. Let ϱ_u be the metric of uniform convergence (i.e. $\varrho_u(f,g)=\sup_{x\in[0,1]}|f(x)-g(x)|$), and $\mathcal{T}_u$ the topology (of uniform convergence) generated by ϱ_u. Let $\mathcal{T}_\Gamma$ be the topology generated by a neighbourhood system $\{B_\Gamma(t):t\in\mathcal{A}\}$ defined in the following way: if t is a function and U is an open set in $[0,1]\times[0,1]$ containing $\Gamma(t)$ then $U_t = \{\tau : \Gamma(\tau) \subset U\}$ and $B_\Gamma(t) = \{U_t : \Gamma(t) \subset U$ and U is an open set in $[0,1] \times [0,1]\}$. Obviously, $W \in \mathcal{T}_\Gamma$ if and only if W is a union of sets belonging to $\bigcup_{t\in\mathcal{A}} B_\Gamma(t)$.

Let $F_1 \subset F_2$ be two families of functions and let $\mathcal{T}$ be a topology in F_2. We say that a function $f \in F_2$ can be $\mathcal{T}$*-approximated* by functions belonging to F_1 if for each $\mathcal{T}$-neighbourhood U_f of f, there exists $g \in F_1 \cap U_f$.

Let f be a real function. We denote by $B_u(f,\varepsilon)$ the open ball in the metric space $(\mathcal{A}, \varrho_u)$ with centre at f and radius $\varepsilon > 0$.

We denote by $\mathrm{Int}(A)$ the *interior* of a set A (in the space $[0,1]$ with the natural metric). The cardinality of a finite set A will be denoted by $\#(A)$.

If A, B are subsets of the domain of f, then $f{\restriction}A$ denotes the *restriction* of f to A. We say that a set A f*-covers* a set B (denoted by $A \underset{f}{\to} B$) if $B \subset f(A)$.

Let $f : [0,1] \to [0,1]$. Then an m*-horseshoe* for f $(2 \le m < \infty)$ is an ordered pair (J, D), where $J \subset [0,1]$ is an interval and D is a family of pairwise disjoint closed intervals $I_1, \dots, I_m \subset J$ such that each element of D f-covers J.

A function $f : [0,1] \to [0,1]$ is *turbulent* if there are compact subintervals $I_1, I_2 \subset [0,1]$ with at most one point in common such that

$$I_1 \cup I_2 \subset f(I_1) \cap f(I_2).$$

Let $f : [0,1] \to [0,1]$. Then we define $f^0(x) = x$, and $f^n(x) = f(f^{n-1}(x))$ if $n > 0$. A point x such that $f^M(x) = x$ but $f^n(x) \ne x$ for $n \in \{1, \dots, M-1\}$ is called a *periodic point* of f of *prime period* M. The set of all periodic points of f of prime period M is denoted by $\mathrm{Per}_M(f)$.

The set of all fixed points of f is denoted by $\mathrm{Fix}(f)$.

Denote by $\mathcal{DB}_1$ the class of all Darboux functions of Baire class one from the unit interval into itself ([4]).

We now proceed to the definitions and notations connected with the Sharkovskiĭ property. Consider the following *Sharkovskiĭ ordering* of the set of all positive integers:

$$3 \prec 5 \prec 7 \prec \cdots \prec 2\cdot 3 \prec 2\cdot 5 \prec 2\cdot 7 \prec \cdots \prec 2^2\cdot 3 \prec 2^2\cdot 5 \prec \cdots$$
$$\cdots \prec 2^3 \prec 2^2 \prec 2 \prec 2^0 = 1.$$

We shall say that f is a *Sharkovskiĭ function* (or f has the *Sharkovskiĭ property*) provided that if $\mathrm{Per}_M(f) \ne \emptyset$ and $M \prec K$, then $\mathrm{Per}_K(f) \ne \emptyset$.

In the next definitions the subscripts are taken modulo M (i.e. $M+1 \equiv 1$). Let $(I_1, \dots, I_M)$ be a finite sequence of subcontinua of $[0,1]$ and let $f_1, \dots, f_M : [0,1] \to [0,1]$. We say that $(I_1, \dots, I_M)$ is an $(f_1, \dots, f_M)$*-cycle* if

$$I_1 \underset{f_1}{\to} I_2 \underset{f_2}{\to} \cdots \underset{f_{M-1}}{\longrightarrow} I_M \underset{f_M}{\to} I_{M+1} = I_1.$$

If $x_0 \in I_1$ is such that

$$(f_i \circ \cdots \circ f_1)(x_0) \in I_{i+1} \quad \text{for } i \in \{1, \ldots, M\},$$

then we say that x_0 *is associated* with the $(f_1, \ldots, f_M)$-cycle $(I_1, \ldots, I_M)$.

We say ([21]) that a family of functions $\mathcal{F}$ has the *property* $\mathcal{J}$ if for any $(f_1, \ldots, f_M)$-cycle $(I_1, \ldots, I_M)$ with $f_1, \ldots, f_M \in \mathcal{F}$, there exists a point x_0 associated with this cycle and such that $(f_M \circ \cdots \circ f_1)(x_0) = x_0$.

Let $\mathcal{K} = \{I_1, \ldots, I_n\}$ be a finite set of closed intervals contained in $[0,1]$ and $f : [0,1] \to [0,1]$. Consider the oriented graph $G_f(\mathcal{K}) = (\mathcal{K}, \Phi)$, where $\Phi = \{(I_i, I_k) : I_i \underset{f}{\to} I_k\} \subset \mathcal{K} \times \mathcal{K}$, and the matrix $M_f(\mathcal{K}) = M(G_f(\mathcal{K})) = [a_{ik}]$ defined by

$$a_{ik} = \begin{cases} 1 & \text{if } I_i \underset{f}{\to} I_k, \\ 0 & \text{if } \neg(I_i \underset{f}{\to} I_k). \end{cases}$$

We will write $\sigma(M)$ and $\operatorname{tr}(M)$ for the *maximal absolute value of an eigenvalue* (in other words, the spectral radius [1]) and the *trace* of the matrix M.

Let $f : [0,1] \to [0,1]$, $\varepsilon > 0$ and n be a positive integer. A set $M \subset [0,1]$ is (n, ε)-*separated* if for each $x, y \in M$, $x \neq y$ there is $0 \leq i < n$ such that $|f^i(x) - f^i(y)| > \varepsilon$. Let $S(f, n, \varepsilon)$ denote an (n, ε)-separated set with the maximal possible number of points and $s_n(\varepsilon)$ its cardinality.

The *topological entropy* of the function f is the number

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon).$$

The above definition is the Bowen and Dinaburg version of the topological entropy ([3], [8]). Note that this is in agreement with other definitions of topological entropy ([7], [1]).

To avoid unnecessary repetitions let us make a standing assumption that **all functions considered are Darboux functions**.

2. Generalizations of classical results

THEOREM 2.1. *Let $f : [0,1] \to [0,1]$ and let $\{I_1, \ldots, I_m\}$ $(m \geq 2)$ be a set of pairwise disjoint closed intervals. Then*

$$h(f) \geq \log \sigma(M_f(I_1, \ldots, I_m)).$$

Proof. Let $M = M_f(I_1, \ldots, I_m)$ and $\sigma_M = \sigma(M)$. It is sufficient to consider the case $\log \sigma_M > 0$.

Set

$$\varepsilon_0 := \frac{1}{2} \min\{\operatorname{dist}(I_i, I_j) : i, j \in \{1, \ldots, m\} \text{ and } i \neq j\}.$$

It is well known (e.g. [1]) that

$$\sigma_M = \limsup_{n\to\infty} \sqrt[n]{\operatorname{tr}(M^n)}.$$

Fix $\beta \in (0, \sigma_M)$. Then there exists a sequence $\{n_k\}$ of positive integers such that

$$\sqrt[n_k]{\operatorname{tr}(M^{n_k})} > \beta > 0 \quad \text{for } k = 1, 2, \ldots. \tag{1}$$

For each $k = 1, 2, \ldots$ let $a_1^{n_k}, \ldots, a_m^{n_k}$ be the diagonal entries of the matrix M^{n_k}. By (1), $a_i^{n_k} \neq 0$ for at least one $i \in \{1, \ldots, m\}$. Note that

> for any $i \in \{1, \ldots, m\}$ there exist $a_i^{n_k}$ pairwise different paths $P_i^1, \ldots, P_i^{a_i^{n_k}}$ of length n_k beginning and terminating at I_i.

Set $T_{n_k} = \{i \in \{1, \ldots, m\} : a_i^{n_k} \neq 0\}$ and let $i \in T_{n_k}$. Then we can write P_i^j $(j = 1, \ldots, a_i^{n_k})$ in the form

$$I_i = I_{i,j}^1 \underset{f}{\to} I_{i,j}^2 \underset{f}{\to} \cdots \underset{f}{\to} I_{i,j}^{n_k} \underset{f}{\to} I_{i,j}^{n_k+1} = I_i,$$

where $I_{i,j}^s \in \{I_1, \ldots, I_m\}$ for $s \in \{1, \ldots, n_k\}$. Choose $x_{i,j}^{n_k} \in I_i$ such that $f^s(x_{i,j}^{n_k}) \in I_{i,j}^{s+1}$ (for $s \in \{1, \ldots, n_k\}$). We shall show that

$$x_{i,j_1}^{n_k} \neq x_{i,j_2}^{n_k} \quad \text{for } j_1, j_2 \in \{1, \ldots, a_i^{n_k}\}.$$

Indeed, suppose, contrary to our claim, that $x_{i,j_1}^{n_k} = x_{i,j_2}^{n_k}$ for some distinct $j_1, j_2 \in \{1, \ldots, a_i^{n_k}\}$. Since $P_i^{j_1} \neq P_i^{j_2}$ there exists $t \in \{2, \ldots, n_k\}$ such that $I_{i,j_1}^t \cap I_{i,j_2}^t = \emptyset$ and so $f^{t-1}(x_{i,j_1}^{n_k}) \in I_{i,j_1}^t$ and $f^{t-1}(x_{i,j_2}^{n_k}) \notin I_{i,j_1}^t$, contrary to $x_{i,j_1}^{n_k} = x_{i,j_2}^{n_k}$.

Let $A_{n_k} := \{x_{i,j}^{n_k} : i \in T_{n_k},\ j \in \{1, \ldots, a_i^{n_k}\}\}$. Note that

$$\#(A_{n_k}) = a_1^{n_k} + \cdots + a_m^{n_k}. \tag{2}$$

Now, consider any two distinct $x_{i_0,j_0}^{n_k}, x_{i_1,j_1}^{n_k} \in A_{n_k}$. If $i_0 \neq i_1$ then

$$|x_{i_0,j_0}^{n_k} - x_{i_1,j_1}^{n_k}| \geq \operatorname{dist}(I_{i_0}, I_{i_1}) > \varepsilon_0.$$

In the case $i_0 = i_1$ we have $j_0 \neq j_1$ $(j_0, j_1 \leq a_{i_0}^{n_k})$. By the definition of $x_{i_0,j}^{n_k}$ $(j = 1, \ldots, a_{i_0}^{n_k})$ there exists a positive integer $t < n_k$ such that $f^t(x_{i_0,j_0}^{n_k}), f^t(x_{i_0,j_1}^{n_k})$ belong to different intervals of the family $\{I_1, \ldots, I_m\}$. This means that

$$|f^t(x_{i_0,j_0}^{n_k}) - f^t(x_{i_0,j_1}^{n_k})| > \varepsilon_0.$$

Therefore

$$s_{n_k}(\varepsilon_0) \geq a_1^{n_k} + \cdots + a_m^{n_k}.$$

From the above considerations and (1) we have

$$\overline{s}(\varepsilon_0) := \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon_0) \geq \limsup_{k\to\infty} \frac{1}{n_k} \log(a_1^{n_k} + \cdots + a_m^{n_k}) \geq \log \beta.$$

Since ([7, Lemma 3.2]) $\overline{s}(\varepsilon) := \limsup_{n\to\infty} \frac{1}{n}\log s_n(\varepsilon) \geq \overline{s}(\varepsilon_0)$ for any $\varepsilon \in (0, \varepsilon_0)$, we have

$$h(f) = \lim_{\varepsilon\to 0} \overline{s}(\varepsilon) \geq \log \beta.$$

As $\beta \in (0, \sigma_M)$ can be arbitrarily close to σ_M, the conclusion follows. ∎

The above theorem is similar to a well-known theorem on continuous functions. But in the case of continuous functions we can use the following statement: *if f is a continuous function and $I_1 \underset{f}{\rightarrow} I_2$ then there exists an interval $J \subset I_1$ such that $f(J) = I_2$*. Note that for f discontinuous this need not be true: set $f(0) = 0$ and $f(x) = \max\left(x^2, \sin\frac{1}{x}\right)$ and consider $I_1 = [0, 1/2]$ and $I_2 = [0, 3/4]$.

Corollary 2.2. *If $f : [0,1] \to [0,1]$ has an m-horseshoe, then $h(f) \geq \log m$.*

Proof. Let (J, D) be an m-horseshoe for f and let $D = \{I_1, \ldots, I_m\}$ ($I_i \cap I_j = \emptyset$ for $i \neq j$). Then $M = M_f(I_1, \ldots, I_m)$ is an $m \times m$ matrix and

$$M^n = \begin{bmatrix} m^{n-1} & m^{n-1} & \cdots & m^{n-1} \\ m^{n-1} & m^{n-1} & \cdots & m^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ m^{n-1} & m^{n-1} & \cdots & m^{n-1} \end{bmatrix},$$

which gives $\mathrm{tr}(M^n) = m^n$, and consequently $\sigma(M) = m$.

From Theorem 2.1 it follows that $h(f) \geq \log m$. ∎

Lemma 2.3 ([7]). *$h(f) = 0$ iff $h(f^k) = 0$ for every positive integer k.*

The above results yield a formally stronger version of [7, Proposition 4.2].

Proposition 2.4. *If $f : [0,1] \to [0,1]$ is a turbulent function then $h(f) > 0$.*

Proof. Let $I_1 = [\alpha_1, \beta_1]$ and $I_2 = [\alpha_2, \beta_2]$ have at most one point in common and

$$I_1 \cup I_2 \subset f(I_1) \cap f(I_2).$$

For simplicity, we may assume that $\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2$. Set $J = [\alpha_1, \beta_2]$.

First, suppose $I_1 \cap I_2 = \emptyset$. Set $D = \{I_1, I_2\}$. Then (J, D) is a 2-horseshoe for f. From Corollary 2.2 we infer $h(f) \geq \log 2 > 0$.

Now, suppose $I_1 \cap I_2 \neq \emptyset$. Then $\beta_1 = \alpha_2$. If $f(\alpha_2) \neq \beta_2$ then we can choose $p, q \in (\alpha_2, \beta_2]$ such that $f(p) \in (\alpha_1, \beta_1)$ and $f(q) = \beta_2$ (assume that $p < q$). Let $I_1' = I_1$ and $I_2' = [p, q] \subset I_2$.

If $f(\alpha_2) = \beta_2$, then pick $a, b \in [\alpha_1, \beta_1)$ such that $f(a) = \alpha_1$ and $f(b) \in (\alpha_2, \beta_2]$. We can assume that $a < b$. Let $I_1' = [a, b] \subset I_1$ and $I_2' = I_2$.

In both cases (J, D_1), where $D_1 = \{I_1', I_2'\}$, is a 2-horseshoe for f^2. By Corollary 2.2 we have $h(f^2) > 0$, and Lemma 2.3 yields $h(f) > 0$. ∎

3. Almost fixed points. We now introduce the concept of an almost fixed point.

DEFINITION 1. Let $f : [0,1] \to [0,1]$ be a Darboux function. We will say that a point x_0 is an *almost fixed point* of f (written $x_0 \in \mathrm{aFix}(f)$) if

$$x_0 \in \mathrm{Int}(\mathrm{R}^-(f, x_0)) \cup \mathrm{Int}(\mathrm{R}^+(f, x_0)).$$

If $x_0 = 0$ or $x_0 = 1$, then we only consider $\mathrm{R}^+(f, x_0)$ or $\mathrm{R}^-(f, x_0)$, respectively. Note that if $x_0 \in \mathrm{aFix}(f)$ then x_0 is a discontinuity point of f. What is more, if f is a Darboux function and x_0 is a discontinuity point of f, then $\mathrm{Int}(\mathrm{R}^-(f, x_0)) \cup \mathrm{Int}(\mathrm{R}^+(f, x_0)) \neq \emptyset$.

The following theorem seems to be interesting from the point of view of combinatorial dynamical systems.

THEOREM 3.1. *If $f, g : [0,1] \to [0,1]$ are topologically conjugate via a homeomorphism φ (i.e. $\varphi \circ f = g \circ \varphi$), and $x_0 \in \mathrm{aFix}(f)$, then $\varphi(x_0) \in \mathrm{aFix}(g)$.*

Proof. By our assumptions we have

$$x_0 \in \mathrm{Int}(\mathrm{R}^-(f, x_0)) \cup \mathrm{Int}(\mathrm{R}^+(f, x_0)).$$

Let, for instance, φ be a decreasing function and $x_0 \in \mathrm{Int}(\mathrm{R}^+(f, x_0))$ (hence $x_0 < 1$).

If $x_0 = 0$ then $\varphi(x_0) = 1$. Pick $\lambda \in (0,1)$ with $[0, \lambda] \subset \mathrm{Int}(\mathrm{R}^+(f, x_0))$.

If $x_0 \in (0,1)$, then there exist $\alpha, \beta \in (0,1)$ such that

$$x_0 \in (\alpha, \beta) \subset \mathrm{Int}(\mathrm{R}^+(f, x_0)).$$

Set $Z = [0, \lambda)$ if $x_0 = 0$, and (α, β) if $x_0 \in (0,1)$. Note that $\varphi(x_0) \in \mathrm{Int}(\varphi(Z))$. The proof will be completed by showing that

$$\varphi(Z) \subset \mathrm{R}^-(g, \varphi(x_0)). \tag{3}$$

So, pick $t \in Z$. To prove that $\varphi(t) \in \mathrm{R}^-(g, \varphi(x_0))$, we fix $\delta > 0$ and show that

(4) there exists a point $y \in (\varphi(x_0) - \delta, \varphi(x_0))$ such that $g(y) = \varphi(t)$.

Indeed, since φ is a continuous function, there exists $\delta_0 > 0$ such that

$$\varphi((x_0 - \delta_0, x_0 + \delta_0) \cap [0,1]) \subset (\varphi(x_0) - \delta, \varphi(x_0) + \delta). \tag{5}$$

Since $t \in Z$, we have $t \in \mathrm{R}^+(f, x_0)$, and consequently there exists a point $z \in (x_0, x_0 + \delta_0)$ such that $f(z) = t$. Set $y = \varphi(z)$. We have

$$g(y) = \varphi(f(z)) = \varphi(t).$$

Moreover, (5) shows that $y \in (\varphi(x_0) - \delta, \varphi(x_0) + \delta)$ and $\varphi(z) < \varphi(x_0)$. Thus $y \in (\varphi(x_0) - \delta, \varphi(x_0))$, proving (4). ∎

In the next theorem (under a slightly stronger assumption than the Darboux property) we establish a relation between having almost fixed points and fixed points. Moreover, this theorem justifies the name of "almost fixed point".

THEOREM 3.2. *Let* $f \in \mathcal{DB}_1$ *and let* $x_0 \in \mathrm{aFix}(f)$. *Then*

$$(x_0 - \varepsilon, x_0 + \varepsilon) \cap \mathrm{Fix}(f) \neq \emptyset \quad \textit{for each } \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$. We can assume that $x_0 \in \mathrm{Int}(\mathrm{R}^+(f, x_0))$ and choose real numbers β_1, β_2 such that $x_0+\varepsilon > \beta_2 > \beta_1 > x_0$ and $[x_0, \beta_2] \subset \mathrm{R}^+(f, x_0)$. Set

$$K = [(x_0, x_0), (\beta_1, \beta_1)] \subset [0,1] \times [0,1].$$

Since the values x_0 and β_2 are attained by f in (x_0, β_1), there exists $c \in (x_0, \beta_1)$ such that $(c, f(c)) \in K$.

From the definition of K we have $c \in \mathrm{Fix}(f)$. ∎

THEOREM 3.3. *Let* $f : [0,1] \to [0,1]$ *be such that* $\mathrm{aFix}(f) \neq \emptyset$. *Then* f *has an* m*-horseshoe for any* $m \geq 2$.

Proof. Fix $m \geq 2$ and $x_0 \in \mathrm{aFix}(f)$. We can assume that $x_0 \in \mathrm{Int}(\mathrm{R}^+(f, x_0))$.

If $f(x_0) = x_0$ then choose $\alpha \in \mathrm{R}^+(f, x_0)$ such that $x_0 < \alpha$. If $f(x_0) \neq x_0$, say $x_0 < f(x_0)$, then pick $\alpha \in (x_0, f(x_0))$.

Certainly, x_0 is a right-hand Darboux point of f. Thus

$$f^{-1}(\alpha) \cap (x_0, x_0 + \eta) \neq \emptyset \neq f^{-1}(x_0) \cap (x_0, x_0 + \eta) \quad \text{for any } \eta > 0.$$

So, we can find two sequences $\{a_n\}$, $\{b_n\}$ such that

$$\alpha > a_1 > b_1 > a_2 > b_2 > \cdots > x_0$$

and

$$f(a_n) = \alpha \quad \text{and} \quad f(b_n) = x_0 \quad \text{for any } n = 1, 2, \ldots.$$

Set $J = [x_0, a_1] \subset [x_0, \alpha]$ and $D = \{I_i = [a_i, b_i] : i = 1, \ldots, m\}$. Note that $I_i \subset J$ for any $i = 1, \ldots, m$, and (f is a Darboux function)

$$f(I_i) \supset J \quad \text{for any } i = 1, \ldots, m.$$

This shows that (J, D) is an m-horseshoe for f. ∎

Corollary 2.2 and the above theorem may be summarized as follows:

COROLLARY 3.4. *Let* $f : [0,1] \to [0,1]$ *be such that* $\mathrm{aFix}(f) \neq \emptyset$. *Then* $h(f)$
$= \infty$. ∎

The next corollary is a simple consequence of Theorem 3.3.

COROLLARY 3.5. *Let* $f : [0,1] \to [0,1]$ *be a Darboux function such that* $\mathrm{aFix}(f) \neq \emptyset$. *Then* f *is turbulent.* ∎

The following statement completes the discussion of basic properties of Darboux functions having at least one almost fixed point.

THEOREM 3.6. *Let $f \in \mathcal{DB}_1$ with* $\mathrm{aFix}(f) \neq \emptyset$. *Then* $\mathrm{Per}_n(f) \neq \emptyset$ *for any* $n = 1, 2, \ldots$.

Proof. Using the sequences $\{a_n\}$, $\{b_n\}$ from the proof of Theorem 3.3, set $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$. Then $I_1 \cap I_2 = \emptyset$ and $I_1, I_2 \subset f(I_1) \cap f(I_2)$, and consequently

$$I_1 \underset{f}{\to} I_2 \underset{f}{\to} I_2 \underset{f}{\to} I_1. \tag{6}$$

Since every $\mathcal{DB}_1$ function has the property $\mathcal{J}$ ([20], [21]), there exists a point y_0 associated with the cycle (6) such that $f^3(y_0) = y_0$. Since the intervals I_1 and I_2 are disjoint, we infer that $f^i(y_0) \neq y_0$ for $i \in \{1, 2\}$. This gives $\mathrm{Per}_3(f) \neq \emptyset$. Hence f is a Sharkovskiĭ function ([17], [20]), and consequently $\mathrm{Per}_n(f) \neq \emptyset$ for any $n = 1, 2, \ldots$. ∎

4. Approximation in spaces of almost continuous functions. Almost continuity was defined by Stallings ([19]) in order to generalize Brouwer's fixed point theorem and very soon became intensively studied by many mathematicians. Note that each almost continuous function from a compact space into itself is a Darboux function and has a fixed point, and moreover a lot of interesting classes of functions from the unit interval into itself are subsets of the family of all almost continuous functions (for example, Darboux Baire one functions, and consequently: all derivatives, all approximately continuous functions, etc.).

The choice of the family of all almost continuous functions as an object of study in this part of the paper is not accidental. Notice that a function f is almost continuous iff it can be $\mathcal{T}_\Gamma$-approximated by continuous functions. We, however, begin by considering the $\mathcal{T}_u$-topology.

LEMMA 4.1. ([19], [15]) *If F is a closed set and $f \in \mathcal{A}$, then $f\restriction F \in \mathcal{A}$.* ∎

LEMMA 4.2 ([15]). *Let $[0,1]$ be a union of countably many closed intervals I_n such that $\mathrm{Int}(I_i) \cap \mathrm{Int}(I_j) = \emptyset$ for $i \neq j$ and $I_i \cap I_{i+1} \neq \emptyset$ for each i. Then for any function $f : [0,1] \to [0,1]$, $f \in \mathcal{A}$ iff $f\restriction I_i \in \mathcal{A}$ for each i.* ∎

LEMMA 4.3 ([15]). *If $J = [a, b]$, $f : [a, b] \to [0, 1]$ is an almost continuous function and $U \subset J \times [0, 1]$ is an open neighbourhood of $\Gamma(f)$, then there exists a continuous function $g : J \to [0, 1]$ such that $\Gamma(g) \subset U$, $g(a) = f(a)$ and $g(b) = f(b)$.* ∎

Let $\mathcal{F}$ be a family of functions from the unit interval into itself. We will write $\mathcal{F}_a = \{f \in \mathcal{F} : \mathrm{aFix}(f) \neq \emptyset\}$ and $\mathcal{F}_\infty = \{f \in \mathcal{F} : h(f) = \infty\}$.

THEOREM 4.4.

(1) *Any almost continuous function can be $\mathcal{T}_u$-approximated by functions belonging to $\mathcal{A}_a$.*

(2) *Any almost continuous function can be $\mathcal{T}_u$-approximated by functions belonging to $(\mathcal{A} \setminus \mathcal{C})_\infty$.*

(3) *Any continuous function can be $\mathcal{T}_u$-approximated by functions belonging to $\mathcal{C}_\infty$.*

Proof. (3) is well known (e.g. [2, Proposition 31, p. 216]) ([1]).

For (1) and (2), according to Corollary 3.4, it is sufficient to show that any almost continuous function can be $\mathcal{T}_u$-approximated by functions in $(\mathcal{A}\setminus\mathcal{C})_a$.

Fix $f \in \mathcal{A}$ and $\varepsilon > 0$. Then there exists $x_0 \in \mathrm{Fix}(f)$. The proof falls naturally into two cases: $x_0 \neq 0$ and $x_0 \neq 1$. In both cases the proofs are similar, so we will only consider the case $x_0 < 1$.

The proof will be divided into two cases:

Case I: x_0 is a continuity point. Then there exists $\delta \in (0, \varepsilon)$ such that $f([x_0 - \delta, x_0 + \delta] \cap [0,1]) \subset (x_0 - \varepsilon/2, x_0 + \varepsilon/2)$. Define

$$g(x) = \begin{cases} f(x) & \text{if } x \notin (x_0, x_0 + \delta), \\ \max\left(0, x_0 + \dfrac{\varepsilon}{2} \sin \dfrac{1}{x - x_0}\right) & \text{if } x \in (x_0, x_0 + \delta/2), \\ \text{linear in } [x_0 + \delta/2, x_0 + \delta]. \end{cases}$$

It is easy to see that $x_0 \in \mathrm{aFix}(g)$. According to Lemmas 4.1 and 4.2 we obtain $g \in \mathcal{A} \setminus \mathcal{C}$. Moreover,

$$\varrho_u(f, g) = \varrho_u(f\restriction[x_0, x_0 + \delta], g\restriction[x_0, x_0 + \delta]) < \varepsilon.$$

Case II: x_0 is a discontinuity point of f, say a right-hand discontinuity point. Then there exists a sequence $\{x_n\}$ such that $x_n \searrow x_0$ and $f(x_n) \to \alpha \neq x_0$. For simplicity assume that $x_0 < \alpha$. Set

$$\sigma = \min\left(\frac{\varepsilon}{2}, \frac{\alpha - x_0}{2}\right)$$

and define $k : [0,1] \to [0,1]$ in the following way:

$$k(x) = \begin{cases} f(x) & \text{if } x \in [0, x_0], \\ \max(0, f(x) - \sigma) & \text{if } x \in (x_0, 1]. \end{cases}$$

To show that $k \in \mathcal{A}$, let W be an open set in $[0,1] \times [0,1]$ containing $\mathit{\Gamma}(k)$ and let $\lambda > 0$ be such that

$$([x_0 - \lambda, x_0 + \lambda] \cap [0,1]) \times ([f(x_0) - \lambda, f(x_0) + \lambda] \cap [0,1]) \subset W.$$

As f is a Darboux function, there exists $x_1 \in (x_0, x_0 + \lambda)$ such that $f(x_1) = x_0 + \sigma$.

([1]) In (3) we cannot replace "continuous" with "almost continuous". Later we prove a similar theorem for the $\mathcal{T}_\Gamma$-topology and in this case it will be possible to $\mathcal{T}_\Gamma$-approximate almost continuous functions by continuous functions.

By Lemma 4.1, $k\restriction[0,x_0] \in \mathcal{A}$. Thus (Lemma 4.3) there exists a continuous function $\xi_1 : [0,x_0] \to [0,1]$ such that $\xi_1(x_0) = k(x_0) = x_0$ and

$$\Gamma(\xi_1) \subset W \cap ([0,1] \times [0,1]).$$

Note that $k\restriction[x_1,1] \in \mathcal{A}$ and $k(x_1) = x_0$. Just as above, there exists a continuous function $\xi_2 : [x_1,1] \to [0,1]$ such that $\xi_2(x_1) = x_0$ and

$$\Gamma(\xi_2) \subset W \cap ([0,1] \times [0,1]).$$

We now define $\xi : [0,1] \to [0,1]$ by

$$\xi(x) = \begin{cases} \xi_1(x) & \text{if } x \in [0,x_0], \\ x_0 & \text{if } x \in [x_0,x_1], \\ \xi_2(x) & \text{if } x \in [x_1,1]. \end{cases}$$

Then ξ is a continuous function and $\Gamma(\xi) \subset W$, proving that $k \in \mathcal{A}$.

We can now prove that $x_0 \in \mathrm{aFix}(k)$. More precisely, we will show that

$$x_0 \in \mathrm{Int}(\mathrm{R}^+(k,x_0)).$$

Let $\{p_n\}$ be a sequence such that $p_n \searrow x_0$ and $f(p_n) \searrow x_0$. Then $k(p_n) \to \beta = \max(0, x_0 - \sigma)$ and $k(x_n) \to \alpha - \sigma$. Moreover, $\beta \le x_0 < \alpha - \sigma$. Let $U = [0, \alpha - \sigma)$ if $x_0 = 0$, and $U = (\beta, \alpha - \sigma)$ if $x_0 > 0$. Then U is an open set in $[0,1]$ containing x_0 such that $U \subset \mathrm{R}^+(k,x_0)$.

Finally, note that $\varrho_u(f,k) = \sigma < \varepsilon$. ■

The above theorem shows that $\mathcal{A}_a$ is dense in the space $(\mathcal{A}, \mathcal{T}_u)$. The next theorem gives a more precise description of the topological properties of $\mathcal{A}_a$. To state it, we set

$$\mathcal{A}_a^* = \{f \in \mathcal{A} : \mathrm{aFix}(f) \cap (0,1) \neq \emptyset\}.$$

THEOREM 4.5.

(a) *If $f \in \mathcal{A}_a^*$, then there exists $\varepsilon > 0$ such that $B_u(f,\varepsilon) \subset \mathcal{A}_a^*$.*

(b) *If $f \in \mathcal{A}_a \setminus \mathcal{A}_a^*$, then $B_u(f,\varepsilon) \setminus \mathcal{A}_a \neq \emptyset$ for any $\varepsilon > 0$.*

Proof. Let $f \in \mathcal{A}_a^*$ and fix $x_0 \in \mathrm{aFix}(f) \cap (0,1)$. Suppose $x_0 \in \mathrm{Int}(\mathrm{R}^+(f,x_0))$ and let $\varepsilon > 0$ be such that $x_0 + \varepsilon < 1$ and

$$(7) \qquad [x_0 - 3\varepsilon, x_0 + 3\varepsilon] \subset \mathrm{R}^+(f,x_0) \cap (0,1).$$

Pick $\xi \in B_u(f,\varepsilon)$. By (7) there exists a sequence $\{x_n^+\}$ such that $x_n^+ \searrow x_0$ and $f(x_n^+) \in (x_0 + 2\varepsilon, x_0 + 3\varepsilon)$. Thus $\xi(x_n^+) > x_0 + \varepsilon$. Analogously, there exists a sequence $\{y_n^+\}$ such that $y_n^+ \searrow x_0$ and $\xi(y_n^+) < x_0 - \varepsilon$.

Let $\delta > 0$. Then there are n_1, n_2 such that $x_{n_1}^+, y_{n_2}^+ \in (x_0, x_0 + \delta)$. Since $\xi \in \mathcal{A}$, for every $\gamma \in (x_0 - \varepsilon, x_0 + \varepsilon)$, in the open interval I^0 with end-points $x_{n_1}^+, y_{n_2}^+$ there exists a number z_γ such that $\xi(z_\gamma) = \gamma$. Since δ was chosen arbitrarily, $(x_0 - \varepsilon, x_0 + \varepsilon) \subset \mathrm{R}^+(\xi, x_0)$. Hence $x_0 \in \mathrm{aFix}(\xi)$ and $\xi \in \mathcal{A}_a^*$, proving (a).

For (b) choose $f \in \mathcal{A}_a \setminus \mathcal{A}_a^*$. Then $0 \in \mathrm{aFix}(f)$ or $1 \in \mathrm{aFix}(f)$. Let $\varepsilon \in (0, 1/2)$.

If $0 \notin \mathrm{aFix}(f)$ then let $f_1 = f$. Otherwise one can find $x_1, x_2 \in (0, \varepsilon/2)$ such that $x_1 < x_2$ and $f(x_2) = 0$, and define $\xi_1 : [0,1] \to [0,1]$ by

$$\xi_1(x) = \begin{cases} x & \text{if } x \in [0, x_1], \\ \dfrac{x_1 \cdot x - x_1 \cdot x_2}{x_1 - x_2} & \text{if } x \in [x_1, x_2], \\ 0 & \text{if } x \in [x_2, 1]. \end{cases}$$

Then ξ_1 is a continuous function. Set $f_1 = \max(f, \xi_1) \in \mathcal{A}$.

If $1 \notin \mathrm{aFix}(f)$ then we put $f_0 = f_1$. Otherwise there are $y_1, y_2 \in (1 - \varepsilon/2, 1)$ such that $y_2 < y_1$ and $f(y_2) = 1$. Define $\xi_2 : [0,1] \to [0,1]$ by

$$\xi_2(x) = \begin{cases} 1 & \text{if } x \in [0, y_2], \\ \dfrac{y_1 - 1}{y_1 - y_2} x + 1 - \dfrac{y_1 - 1}{y_1 - y_2} y_2 & \text{if } x \in [y_2, y_1], \\ x & \text{if } x \in [y_1, 1]. \end{cases}$$

Then ξ_2 is a continuous function. We set $f_0 = \min(f_1, \xi_2)$. It is obvious that $f_0 \in \mathcal{A}$.

Since $\varrho_u(f, f_0) < \varepsilon$, it remains to prove that

$$\mathrm{aFix}(f_0) = \emptyset. \tag{8}$$

It is obvious that $0, 1 \notin \mathrm{aFix}(f_0)$. Moreover, if $0 \notin \mathrm{aFix}(f)$ then $[0, y_2) \cap \mathrm{aFix}(f_0) = \emptyset$ and $y_2 \notin \mathrm{Int}(\mathrm{R}^-(f_0, y_2))$ (if $1 \notin \mathrm{aFix}(f)$ then $(x_2, 1] \cap \mathrm{aFix}(f_0) = \emptyset$ and $x_2 \notin \mathrm{Int}(\mathrm{R}^+(f_0, x_2))$). If $0, 1 \in \mathrm{aFix}(f)$ we have $(x_2, y_2) \cap \mathrm{aFix}(f_0) = \emptyset$, $x_2 \notin \mathrm{Int}(\mathrm{R}^+(f_0, x_2))$ and $y_2 \notin \mathrm{Int}(\mathrm{R}^-(f_0, y_2))$.

Consequently, it remains to consider an interval $(0, x_2]$ if $0 \in \mathrm{aFix}(f)$, and $[y_2, 1)$ if $1 \in \mathrm{aFix}(f)$.

In the case $0 \in \mathrm{aFix}(f)$, we shall show that

$$(0, x_2] \cap \mathrm{aFix}(f_0) = \emptyset. \tag{9}$$

It is easy to see that $x_2 \notin \mathrm{aFix}(f_0)$, so let $x \in (0, x_2)$ and suppose that $x \in \mathrm{aFix}(f_0)$. Consider the following cases:

Case 1: $x \in (0, x_1]$. Then there exists $t_1 \in (0, x)$ such that $[t_1, x] \subset \mathrm{R}^-(f_0, x)$ or $[t_1, x] \subset \mathrm{R}^+(f_0, x)$. If $x < x_1$ then let $t_2 \in (x, x_1)$. If $x = x_1$, let $t_2 \in (x_1, x_2)$ be such that $\xi_1(t_2) = t_1$. Then

$$f_0^{-1}(t_1) \cap (t_1, t_2) = \emptyset.$$

Hence $t_1 \notin \mathrm{R}^-(f_0, x) \cup \mathrm{R}^+(f_0, x)$, which is impossible.

Case 2: $x \in (x_1, x_2)$. Then there exist $t_1 \in (x_1, x)$ and $t_2 \in (x, x_2)$ such that $(t_1, t_2) \subset \mathrm{R}^-(f_0, x)$ or $(t_1, t_2) \subset \mathrm{R}^+(f_0, x)$. Let, for instance, $(t_1, t_2) \subset \mathrm{R}^-(f_0, x)$. Thus for each $t \in (t_1, t_2)$ and $\delta \in (0, x - x_1)$ there exists $z_t^\delta \in$

$(x-\delta, x)$ such that $f_0(z_t^\delta) = t$. Note that $f_0(z_t^\delta) = f(z_t^\delta)$, and consequently $(t_1, t_2) \subset \mathrm{R}^-(f, x)$, which is impossible.

The proof of (9) is finished.

If $1 \in \mathrm{aFix}(f)$, one can prove similarly that $[y_2, 1] \cap \mathrm{aFix}(f_0) = \emptyset$. ∎

COROLLARY 4.6. *If $f \in \mathcal{A}_a^*$, then f is a continuity point of the entropy function $h : (\mathcal{A}, \mathcal{T}_u) \to [0, \infty]$.* ∎

Let us now turn to the $\mathcal{T}_\Gamma$ topology.

THEOREM 4.7.

(i) *Any almost continuous function can be $\mathcal{T}_\Gamma$-approximated by functions in $\mathcal{A}_a$.*
(ii) *Any almost continuous function can be $\mathcal{T}_\Gamma$-approximated by functions in $(\mathcal{A} \setminus \mathcal{C})_\infty$.*
(iii) *Any almost continuous function can be $\mathcal{T}_\Gamma$-approximated by functions in $\mathcal{C}_\infty$.*

Proof. Similarly to the proof of Theorem 4.4, for (i) and (ii), it is sufficient to show that any almost continuous function can be $\mathcal{T}_\Gamma$-approximated by functions in $(\mathcal{A} \setminus \mathcal{C})_a$.

Fix $f \in \mathcal{A}$, $\varepsilon > 0$ and $x_0 \in \mathrm{Fix}(f)$. The proof will be divided naturally into two cases: $x_0 > 0$ and $x_0 < 1$. As the arguments are similar, we will consider only the case $x_0 < 1$.

Let $U_f = \{\tau \in \mathcal{A} : \Gamma(\tau) \subset U\} \in B_\Gamma(f)$. Thus U is an open set containing $\Gamma(f)$. Then there exists $\varepsilon_0 > 0$ such that $x_0 + \varepsilon_0 < 1$ and

$$([x_0 - \varepsilon_0, x_0 + \varepsilon_0] \cap [0,1]) \times ([x_0 - \varepsilon_0, x_0 + \varepsilon_0] \cap [0,1]) \subset U.$$

We define

$$g(x) = \begin{cases} f(x) & \text{if } x \notin (x_0, x_0 + \varepsilon_0), \\ \max\left(0, x_0 + \dfrac{\varepsilon_0}{2} \sin \dfrac{1}{x - x_0}\right) & \text{if } x \in (x_0, x_0 + \varepsilon_0/2], \\ \text{linear in } [x_0 + \varepsilon_0/2, x_0 + \varepsilon_0]. \end{cases}$$

From Lemmas 4.1 and 4.2 we conclude that $g \in \mathcal{A}$ (obviously, g is discontinuous).

Now, it is sufficient to note that $\Gamma(g) \subset U$ and $x_0 \in \mathrm{aFix}(g)$.

To prove (iii), for every closed interval $[a, b] \subset [0, 1]$ and a fixed positive integer n let $\{a_i\}_{i=0}^n$, $\{b_i\}_{i=1}^{n+1}$ be such that

$$a = a_0 < b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n < b_{n+1} = b.$$

Then let $h_{a,b}^n : [a, b] \to [a, b]$ be linear in all segments $[a_i, b_{i+1}]$ $(i = 0, 1, \ldots, n)$, $[b_i, a_i]$ $(i = 1, \ldots, n)$ and such that $h_{a,b}^n(a_i) = a$ $(i = 0, 1, \ldots, n)$, $h_{a,b}^n(b_i) = b$ $(i = 1, \ldots, n+1)$.

Now, set $V_f = \{\tau \in \mathcal{A} : \Gamma(\tau) \subset V\} \in B_\Gamma(f)$, where V is an open set containing $\Gamma(f)$. Since $f \in \mathcal{A}$, there exists a continuous function ξ such that

$\Gamma(\xi) \subset V$. Let $y_0 \in \mathrm{Fix}(\xi)$. We may assume that $y_0 \neq 1$ (if $y_0 = 1$ the proof is similar). Then there exists $\delta_0 > 0$ such that $y_0 + \delta_0 < 1$ and

$$([y_0 - \delta_0, y_0 + \delta_0] \cap [0,1]) \times ([y_0 - \delta_0, y_0 + \delta_0] \cap [0,1]) \subset V.$$

So, we can define

$$k(x) = \begin{cases} \xi(x) & \text{if } x \notin (y_0, y_0 + \delta_0), \\ h^n_{y_0+\delta_0/2^{n+1}, y_0+\delta_0/2^n} & \text{if } x \in [y_0+\delta_0/2^{n+1}, y_0+\delta_0/2^n],\ n = 1,2,\ldots, \\ \text{linear in } [y_0 + \delta_0/2, y_0 + \delta_0]. \end{cases}$$

It is easy to see that k is continuous and $\Gamma(k) \subset V$. Moreover, k has an $(n+1)$-horseshoe for $n = 1,2,\ldots$. From Corollary 2.2 we infer that $h(k) = \infty$. ∎

5. An open problem. It is well known that

If f is a continuous function then $h(f) > 0$ iff f has a periodic point of period different from 2^n for $n = 1,2,\ldots$.

M. Čiklová [7] proved in 2005 that we can replace the continuity by the assumption that f has connected and G_δ graph:

If f has a connected and G_δ graph then $h(f) > 0$ iff f has a periodic point of period different from 2^n for $n = 1,2,\ldots$.

An interesting question is:

PROBLEM. *Is an analogous theorem true for f almost continuous?*

It should be mentioned here that every almost continuous function mapping the unit interval into itself has a connected graph, but this graph need not be a G_δ set. Moreover, in 1989 Kellum [10] proved that almost continuous functions need not have the Sharkovskiĭ property.

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REFERENCES

[1] L. Alsedà, J. Llibre and M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, World Sci., 1993.

[2] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, Springer, 1992.

[3] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 153 (1971), 401–414; Errata, ibid. 181 (1973), 509–510.

[4] A. M. Bruckner, *Differentiation of Real Functions*, Springer, 1978.

[5] A. M. Bruckner and J. G. Ceder, *Darboux continuity*, Jahresber. Deutsch. Math.-Verein. 67 (1965), 93–117.

[6] J. G. Ceder, *On Darboux points of real functions*, Period. Math. Hungar. 11 (1980), 69–80.

[7] M. Čiklová, *Dynamical systems generated by functions with connected G_δ graphs*, Real Anal. Exchange 30 (2004/2005), 617–638.
[8] E. I. Dinaburg, *The relation between topological entropy and metric entropy*, Soviet Math. Dokl. 11 (1970), 13–16.
[9] R. Engelking, *General Topology*, PWN–Polish Sci. Publ., 1977.
[10] K. L. Kellum, *Iterates of almost continuous functions and Sharkovskii's theorem*, Real Anal. Exchange 14 (1988-89), 420–422.
[11] E. Korczak, *Rings of H-connected functions*, doctoral thesis (in Polish, in preparation).
[12] J. S. Lipiński, *On Darboux points*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 26 (1978), 689–873.
[13] A. Maliszewski, *Darboux property and quasi-continuity. Uniform approach*, habilitation thesis, 1996.
[14] T. Natkaniec, *On compositions and products of almost continuous functions*, Fund. Math. 139 (1991), 71–78.
[15] —, *Almost continuity*, habilitation thesis, Bydgoszcz, 1992.
[16] H. Pawlak and R. Pawlak, *Transitivity, dense orbits and some topologies finer than the natural topology of the unit interval*, to appear.
[17] R. J. Pawlak, *On the Sharkovskiĭ property of Darboux functions*, to appear.
[18] A. Peris, *Transitivity, dense orbits and discontinuous functions*, Bull. Belg. Math. Soc. 6 (1999), 391–394.
[19] J. Stallings, *Fixed point theorem for connectivity maps*, Fund. Math. 47 (1959), 249–263.
[20] P. Szuca, *Fixed points of Darboux maps*, doctoral thesis, Gdańsk, 2003 (in Polish).
[21] —, *Sharkovskiĭ's theorem holds for some discontinuous functions*, Fund. Math. 179 (2003), 27–41.
[22] P. Walters, *An Introduction to Ergodic Theory*, Grad. Texts in Math. 79, Springer, 1982.

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