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# COMPLETELY MIXING MAPS WITHOUT LIMIT MEASURE

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**Abstract.** We combine some results from the literature to give examples of completely mixing interval maps without limit measure.

Let $X$ be a compact metric space with Borel $\sigma$-algebra $\mathcal{B}$ and equipped with some Borel measure $m$. Consider a transformation $T : X \to X$ that is non-singular with respect to $m$, which means that $m(T^{-1}A) = 0$ whenever $m(A) = 0$ for each Borel set $A$. Let $P : L^1_m \to L^1_m$ be the Frobenius–Perron operator of $T$ so that

$$\int \varphi \cdot P^n f \, dm = \int (\varphi \circ T^n) \cdot f \, dm \quad \forall f \in L^1_m \ \forall \varphi \in L^\infty_m.$$

We adopt the following definitions:

- The system $(X, \mathcal{B}, m, T)$ is *completely mixing* if $\lim_{n\to\infty} \|P^n f\| = 0$ for each $f \in L^1_m$ with $\int f \, dm = 0$.
- A probability measure $\mu$ on $\mathcal{B}$ is a *limit measure* for $(X, \mathcal{B}, m, T)$ if for each probability density $h \in L^1_m$ the measures $P^n h \cdot m$ converge weakly to $\mu$, in other words, if

$$\lim_{n\to\infty} \int \varphi \cdot P^n f \, dm = \int \varphi \, d\mu \cdot \int f \, dm \quad \forall f \in L^1_m \ \forall \varphi \in C(X).$$

- If a system $(X, \mathcal{B}, m, T)$ is completely mixing and has a nontrivial limit measure $\mu$ (i.e. $\mu$ is not a one-point mass), then $\mu$ is called a *stochastic attractor* for the system.
- A probability measure $\mu$ on $\mathcal{B}$ is a *Sinai–Ruelle–Bowen measure* for the system $(X, \mathcal{B}, m, T)$ if for each $\varphi \in C(X)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k x) = \int \varphi \, d\mu \quad m\text{-a.e. } x.$$

**The problems.** Rudnicki [8] posed the following problems:

PROBLEM 1. Does each completely mixing system have a limit measure? If $T$ has an invariant probability density the answer is obviously "yes". In

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general, however, this is not true. In fact, we provide counterexamples in the class of quadratic interval maps.

PROBLEM 2. Is a stochastic attractor necessarily a Bowen–Ruelle–Sinai measure? We give a counterexample in the class of piecewise $C^2$ interval maps with two surjective branches and two neutral fixed points.

### The counterexamples

1. *A completely mixing quadratic interval map without limit measure.* For $0 < a \leq 4$ denote by $T_a : [0,1] \to [0,1]$ the map $T_a(x) = ax(1-x)$. Given a parameter $a$ we denote by $I$ the dynamical interval $[T_a^2(1/2), T_a(1/2)]$ and consider henceforth the restriction of $T_a$ to $I$.

The first ingredient to the construction of our counterexamples is a result by Bruin and Hawkins [2, Theorem 4.2]. It says that if $T_a : I \to I$ is topologically mixing, then it is Lebesgue exact, i.e. the tail $\sigma$-algebra $\mathcal{T} = \bigcap_{n=0}^{\infty} T_a^{-n}\mathcal{B}$ contains only sets of Lebesgue measure zero or full Lebesgue measure (1).

The second ingredient is an old result of Lin [6] (see also [1, Theorem 1.3.3]). It says that a system $(X, \mathcal{B}, m, T)$ is exact if and only if it is completely mixing (2). Hence, if $T_a$ is topologically mixing, then it is completely mixing.

The third ingredient are real quadratic maps without asymptotic measure constructed by Hofbauer and Keller [3]. Denote by $\bar{\omega}_a(m)$ the set of all weak accumulation points of the sequence of probability measures $(n^{-1}\sum_{k=0}^{n-1} m \circ T_a^{-k})_{n>0}$, where $m$ denotes the normalized Lebesgue measure on $I$. Theorem 1 of [3] provides an uncountable family of parameters $a$ for which the set of ergodic measures in $\bar{\omega}_a(m)$ is infinite (3). Such maps do not, in particular, have a limit measure, because $m \circ T_a^{-k} = P^k 1 \cdot m$ so that the existence of a limit measure for $T_a$ (in the sense of the above definition) would imply $\bar{\omega}_a(m) = \{\mu\}$.

The missing link that combines these results to produce examples of completely mixing maps without limit measure is the observation that the maps constructed in [3, 4] are topologically mixing. For a unimodal interval map topological mixing is equivalent to the nondecomposability of its kneading

(1) More precisely, Bruin and Hawkins assume that the map $T_a$ has no Cantor attractor in the sense of Milnor. But Lyubich [7] showed that a topologically mixing quadratic map $T_a$ never has such an attractor. For our construction, however, this deep result need not be invoked, because the denseness of the critical orbit in the examples below excludes the existence of a Cantor attractor.

(2) The reader should be warned that Lin [6] uses a different terminology concerning the notion of complete mixing. The terminology used in this note is adopted from [8].

(3) In [4] this construction is modified in such a way that, for uncountably many parameters $a$, $\bar{\omega}_a(m)$ is even the set of *all* invariant probability measures of $T_a$.

sequence (equivalently to the nonrenormalizability of the map) (4). But this follows readily from equations (3.6) and (3.7) in [3].

Finally, we remark that we have obtained a bit more than only a negative answer to the above problem. We showed:

THEOREM. *There are uncountably many maps* $T_a$ *in the quadratic family which are completely mixing with respect to Lebesgue measure, but for which* $\bar{\omega}_a(m)$ *is the set of all* $T_a$*-invariant probability measures. In particular, the sequence of measures* $(n^{-1}\sum_{k=0}^{n-1} m\circ T_a^{-k})_{n>0}$ *does not converge weakly for these parameters.*

2. *A stochastic attractor which is not a Sinai–Ruelle–Bowen measure.* The second example uses interval maps with two indifferent fixed points where the order of contact of the graph of the map to the diagonal is higher than two. To be definite we consider the map $T:[0,1]\to[0,1]$,

$$T(x)=\begin{cases} x+4x^3 & \text{for } x\in[0,1/2),\\ x-4(1-x)^3 & \text{for } x\in[1/2,1].\end{cases}$$

The map $T$ has a smooth $\sigma$-finite invariant density with nonintegrable singularities at $x=0$ and $x=1$. Thaler [9, Theorem 1] proved that such maps are Lebesgue exact, so by the result of Lin again, they are completely mixing. Since $\lim_{n\to\infty}\int_\delta^{1-\delta} P^n 1\,dm=0$ for all $\delta>0$, the set of weak accumulation points of the measures $P^n 1\cdot m$ is contained in $\{a\delta_0+(1-a)\delta_1 : 0\le a\le 1\}$. Since $T$ has the symmetry $T(x)=1-T(1-x)$, it maps symmetric densities $h$ (i.e. $h(x)=h(1-x)$) to symmetric ones. In particular, all $P^n 1$ are symmetric. Hence $P^n 1\cdot m\to\mu:=\frac{1}{2}(\delta_0+\delta_1)$ weakly so that $\mu$ is a stochastic attractor for $(X,\mathcal{B},m,T)$. On the other hand, a result of Inoue [5, Corollary 2.2] shows that $\mu$ is not a Sinai–Ruelle–Bowen measure for the system. Indeed, he proves

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} 1_A(T^k x)=1 \qquad m\text{-a.e. } x$$

for all intervals $A=(0,\delta)$ and $A=(1-\delta,1)$ and all $\delta>0$.

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(4) Recall that we restricted $T_a$ to its dynamical interval.

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