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ON THE AVERAGE OF THE SUM-OF-a-DIVISORS FUNCTION

BY

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Abstract. We prove an Ω result on the average of the sum of the divisors of n which are relatively coprime to any given integer a. This generalizes the earlier result for a prime proved by Adhikari, Coppola and Mukhopadhyay.

1. Introduction. The estimate of the error term in the average of the sum-of-divisors function has attracted attention of many people (see Walfisz [7], Pétermann [5], [6]). Recently, Adhikari, Coppola and Mukhopadhyay [1] proved that

$$\sum_{n\le x} D_p(n) - \frac{\pi^2 x^2}{12}\left(1-\frac{1}{p}\right) = \Omega_{\pm}(x\log\log x),$$

where $D_p(n)$ is defined to be the sum of the divisors of n which are relatively coprime to a prime p, that is,

$$D_p(n) = \sum_{\substack{d\mid n\\ (d,p)=1}} d.$$

Their main ideas come from Erdős and Shapiro [2] and Pétermann [4].

In this paper, we generalize the results of [1] to the general case. We define $D_a(n)$ to be the sum of the divisors of n which are relatively coprime to an integer a, that is,

$$D_a(n) = \sum_{\substack{d\mid n\\ (d,a)=1}} d.$$

We prove the following theorem.

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THEOREM. *For any integer $a > 1$, we have*

$$\sum_{n\le x} D_a(n) - \frac{\pi^2 x^2 \phi(a)}{12a} = \Omega_{\pm}(x \log\log x),$$

where $\phi(n)$ is Euler's function.

2. Lemmas. We will use p, q to denote primes. Let $a > 1$ be an integer with standard factorization $a = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s}$, where $\beta_i > 0$ $(1 \le i \le s)$. It is convenient to define

$$E_a(x) := \sum_{n\le x} D_a(n) - \frac{\pi^2 x^2 \phi(a)}{12a}, \tag{1}$$

$$F_a(x) := \sum_{n\le x} \frac{D_a(n)}{n} - \frac{\pi^2 x \phi(a)}{6a}. \tag{2}$$

LEMMA 1. *For each natural number n we have*

$$\frac{D_a(n)}{n} = \sum_{d\mid n} \frac{1}{d} \prod_{p\mid (a,d)} (1-p).$$

Proof. Since both sides are multiplicative functions of n, it suffices to check the equality for $n = q^b$, where q is a prime and b is a nonnegative integer. In fact, for $n = q^b$, if $q \mid a$, then both sides are $1/q^b$; if $q \nmid a$, then both sides are $(1 + q + \cdots + q^b)/q^b$.

LEMMA 2.

$$\sum_{n\le x} \frac{1}{n} \prod_{p\mid (a,n)} (1-p) = \begin{cases} \log p_1 + O(1/x) & \text{if } s = 1, \\ O(1/x) & \text{if } s > 1. \end{cases}$$

Proof. Since

$$\sum_{n\le y} \frac{1}{n} = \log y + \gamma + O(1/y),$$

we have

$$\begin{aligned} \sum_{n\le x} \frac{1}{n} \prod_{p\mid (a,n)} (1-p) &= \sum_{n\le x} \frac{1}{n} \sum_{d\mid (a,n)} \mu(d) d = \sum_{d\mid a} \mu(d) \sum_{n\le x, d\mid n} \frac{d}{n} \\ &= \sum_{d\mid a} \mu(d) \sum_{n\le x/d} \frac{1}{n} = \sum_{d\mid a} \mu(d) \Big(\log\frac{x}{d} + \gamma + O(d/x)\Big) \\ &= (\log x + \gamma) \sum_{d\mid a} \mu(d) - \sum_{d\mid a} \mu(d) \log d + O(1/x) \end{aligned}$$

$$= (\log x + \gamma) \sum_{d|p_1\cdots p_s} \mu(d) + \sum_{i=1}^{s} \log p_i \sum_{d|(p_1\cdots p_s/p_i)} \mu(d) + O(1/x)$$

$$= \begin{cases} \log p_1 + O(1/x) & \text{if } s = 1, \\ O(1/x) & \text{if } s > 1. \end{cases}$$

LEMMA 3.

$$F_a(x) = -\sum_{k=1}^{\infty} \frac{1}{k} \left\{ \frac{x}{k} \right\} \prod_{p|(a,k)} (1-p).$$

Proof. Since

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \prod_{p|(a,k)} (1-p) = \prod_q \left(\sum_{i=0}^{\infty} \frac{1}{q^{2i}} \prod_{p|(a,q^i)} (1-p) \right) \tag{3}$$

$$= \prod_{q|a} \left(1 + \sum_{i=1}^{\infty} \frac{1-q}{q^{2i}} \right) \cdot \prod_{q\nmid a} \sum_{i=0}^{\infty} \frac{1}{q^{2i}}$$

$$= \prod_{q|a} \left(1 - \frac{1}{q} \right) \cdot \prod_{q} \left(1 - \frac{1}{q^2} \right)^{-1} = \frac{\pi^2 \phi(a)}{6a},$$

by (2) and Lemma 1 we have

$$F_a(x) = \sum_{n\le x} \sum_{k|n} \frac{1}{k} \prod_{p|(a,k)} (1-p) - x \sum_{k=1}^{\infty} \frac{1}{k^2} \prod_{p|(a,k)} (1-p)$$

$$= \sum_{k=1}^{\infty} \frac{1}{k} \left[\frac{x}{k} \right] \prod_{p|(a,k)} (1-p) - x \sum_{k=1}^{\infty} \frac{1}{k^2} \prod_{p|(a,k)} (1-p)$$

$$= -\sum_{k=1}^{\infty} \frac{1}{k} \left\{ \frac{x}{k} \right\} \prod_{p|(a,k)} (1-p).$$

LEMMA 4.

$$F_a(x) = -\sum_{k\le y} \frac{1}{k} \left\{ \frac{x}{k} \right\} \prod_{p|(a,k)} (1-p) + O(1)$$

uniformly for $x \ge 2$, $y \ge \frac{1}{2}\sqrt{x}$.

Proof. By Lemma 3 we only need to prove that

$$\sum_{k>y} \frac{1}{k} \left\{ \frac{x}{k} \right\} \prod_{p|(a,k)} (1-p) = O(1).$$

If $y \ge \frac{1}{2}x$, then

$$\sum_{k>y} \frac{1}{k}\left\{\frac{x}{k}\right\} \prod_{p|(a,k)} (1-p) \ll \sum_{k>y} \frac{x}{k^2} \ll \frac{x}{y} \ll 1.$$

Now we assume that $\frac{1}{2}\sqrt{x} \le y < \frac{1}{2}x$. Since

$$\sum_{k>x} \frac{1}{k}\left\{\frac{x}{k}\right\} \prod_{p|(a,k)} (1-p) = O(1),$$

it suffices to prove that

$$\sum_{y<k\le x} \frac{1}{k}\left\{\frac{x}{k}\right\} \prod_{p|(a,k)} (1-p) = O(1).$$

Let M be an integer with $M \le x/y < M+1$. Then, for each integer t with $2 \le t \le M+1$, $\{x/k\}$ is monotone in the range $x/t < k \le x/(t-1)$. Hence, by Lemma 2 and partial summation, we have

$$\sum_{x/t<k\le x/(t-1)} \frac{1}{k}\left\{\frac{x}{k}\right\} \prod_{p|(a,k)} (1-p) = O(t/x)$$

and

$$\sum_{y<k\le x/M} \frac{1}{k}\left\{\frac{x}{k}\right\} \prod_{p|(a,k)} (1-p) = O(M/x).$$

Hence, because $y \ge \frac{1}{2}\sqrt{x}$, we have

$$\sum_{y<k\le x} \frac{1}{k}\left\{\frac{x}{k}\right\} \prod_{p|(a,k)} (1-p) = O\left(\frac{1}{x} \sum_{1\le t\le x/y} t\right) = O\left(\frac{1}{x} \cdot \frac{x^2}{y^2}\right) = O(1).$$

Lemma 5.

$$\frac{E_a(x)}{x} - F_a(x) = O(1).$$

Proof. By (1) we have

$$\frac{E_a(x)}{x} = \frac{1}{x} \sum_{n\le x} D_a(n) - \frac{\pi^2 x\phi(a)}{12a}. \tag{4}$$

By Lemmas 1, 2 and 4, we have

$$\sum_{n\le x} D_a(n) = \sum_{n\le x} n \sum_{d|n} \frac{1}{d} \prod_{p|(a,d)} (1-p) = \sum_{d\le x} \prod_{p|(a,d)} (1-p) \sum_{n\le x/d} n \tag{5}$$

$$
\begin{aligned}
&= \sum_{d\le x}\left(\frac{1}{2}\left[\frac{x}{d}\right]^2+\frac{1}{2}\left[\frac{x}{d}\right]\right)\prod_{p|(a,d)}(1-p)\\
&= \sum_{d\le x}\left(\frac{x^2}{2d^2}+\frac{x}{2d}-\frac{x}{d}\left\{\frac{x}{d}\right\}\right)\prod_{p|(a,d)}(1-p)+O(x)\\
&= \frac{1}{2}x^2\sum_{d\le x}\frac{1}{d^2}\prod_{p|(a,d)}(1-p)-x\sum_{d\le x}\frac{1}{d}\left\{\frac{x}{d}\right\}\prod_{p|(a,d)}(1-p)+O(x)\\
&= \frac{1}{2}x^2\sum_{d\le x}\frac{1}{d^2}\prod_{p|(a,d)}(1-p)+xF_a(x)+O(x).
\end{aligned}
$$

By (3)–(5), we finally obtain

$$
\begin{aligned}
\frac{E_a(x)}{x}-F_a(x) &= -\frac{1}{2}x\sum_{d>x}\frac{1}{d^2}\prod_{p|(a,d)}(1-p)+O(1)\\
&= O\left(x\sum_{d>x}\frac{1}{d^2}\right)+O(1)=O(1).
\end{aligned}
$$

LEMMA 6 (Montgomery [3]). *If b, r are positive integers such that $(b,r) = 1$ and β is a real number, then for any positive integer N, we have*

$$
\sum_{n=1}^{N}\left\{\frac{nb}{r}+\beta\right\}=\frac{N}{r}\{r\beta\}+\frac{N}{r}\left(\frac{r-1}{2}\right)+O(r).
$$

LEMMA 7. *Let $A = m!/(p_1^{e_1}\cdots p_s^{e_s})$ be an integer with $(A,a) = 1$ and $A \ge ma$. Then*

$$
\sum_{k\le A}\frac{(k,A)}{k^2}\prod_{p|(a,k)}(1-p)\ge\frac{\phi(a)}{a}\log m+O(1).
$$

Proof. We have

$$
\begin{aligned}
\sum_{k\le A}\frac{(k,A)}{k^2}\prod_{p|(a,k)}(1-p) &= \sum_{k\le A}\frac{(k,A)}{k^2}\sum_{d|a,d|k}\mu(d)d\\
&= \sum_{d|a}\mu(d)d\sum_{k\le A,d|k}\frac{(k,A)}{k^2}=\sum_{d|a}\mu(d)\frac{1}{d}\sum_{k\le A/d}\frac{(k,A)}{k^2}\\
&= \sum_{d|a}\frac{\mu(d)}{d}\sum_{k\le A/a}\frac{(k,A)}{k^2}+\sum_{d|a}\frac{\mu(d)}{d}\sum_{A/a<k\le A/d}\frac{(k,A)}{k^2}\\
&= \frac{\phi(a)}{a}\sum_{k\le A/a}\frac{(k,A)}{k^2}+\sum_{d|a}\mu(d)\frac{1}{d}\sum_{A/a<k\le A/d}\frac{(k,A)}{k^2}.
\end{aligned}
$$

Since

$$\begin{aligned}\sum_{k\le A/a}\frac{(k,A)}{k^2} &\ge \sum_{k\le m,\,(k,a)=1}\frac{(k,A)}{k^2}=\sum_{k\le m,\,(k,a)=1}\frac{1}{k}\\ &=\sum_{k\le m}\frac{1}{k}\sum_{d|k,\,d|a}\mu(d)=\sum_{d|a}\mu(d)\sum_{k\le m,\,d|k}\frac{1}{k}\\ &=\sum_{d|a}\frac{\mu(d)}{d}\sum_{k\le m/d}\frac{1}{k}=\sum_{d|a}\frac{\mu(d)}{d}\Big(\log\frac{m}{d}+O(1)\Big)\\ &=\frac{\phi(a)}{a}\log m+O(1)\end{aligned}$$

and

$$\sum_{d|a}\mu(d)\,\frac{1}{d}\sum_{A/a<k\le A/d}\frac{(k,A)}{k^2}\ll\sum_{A/a<k\le A}\frac{(k,A)}{k^2}\ll\sum_{A/a<k\le A}\frac{1}{k}\ll 1,$$

we obtain the assertion.

3. Proof of the Theorem. Let A be as in Lemma 7 and let B be an integer with $0\le B<A$. By Lemmas 4, 6 and 2 we have

$$\begin{aligned}\frac{1}{A}\sum_{n=1}^{A}F_a(nA+B)&=-\frac{1}{A}\sum_{n=1}^{A}\sum_{k\le A}\frac{1}{k}\Big\{\frac{nA+B}{k}\Big\}\prod_{p|(a,k)}(1-p)+O(1)\\ &=-\frac{1}{A}\sum_{k\le A}\frac{1}{k}\Big(\prod_{p|(a,k)}(1-p)\Big)\sum_{n=1}^{A}\Big\{\frac{nA+B}{k}\Big\}+O(1)\\ &=-\sum_{k\le A}\frac{(k,A)}{k^2}\Big(\prod_{p|(a,k)}(1-p)\Big)\Big(\Big\{\frac{B}{(k,A)}\Big\}+\frac{k}{2(k,A)}-\frac{1}{2}\Big)+O(1)\\ &=-\sum_{k\le A}\frac{(k,A)}{k^2}\Big(\prod_{p|(a,k)}(1-p)\Big)\Big(\Big\{\frac{B}{(k,A)}\Big\}-\frac{1}{2}\Big)+O(1).\end{aligned}$$

Hence

$$\frac{1}{A}\sum_{n=1}^{A}F_a(nA)=\frac{1}{2}\sum_{k\le A}\frac{(k,A)}{k^2}\prod_{p|(a,k)}(1-p)+O(1)\tag{6}$$

and

$$\begin{aligned}&\frac{1}{A}\sum_{n=1}^{A}F_a(nA+A-1)\\ &\qquad=-\sum_{k\le A}\frac{(k,A)}{k^2}\Big(\prod_{p|(a,k)}(1-p)\Big)\Big(\frac{1}{2}-\frac{1}{(k,A)}\Big)+O(1)\end{aligned}\tag{7}$$

$$= -\frac{1}{2}\sum_{k\le A}\frac{(k,A)}{k^2}\prod_{p|(a,k)}(1-p)+O(1).$$

Noting that

$$\log\log(A^2+B) \ll \log\log A \ll \log(m\log m) \ll \log m,$$

by (6), (7) and Lemma 7 we obtain $F_a(x) = \Omega_\pm(\log\log x)$. By Lemma 5 we conclude that $E_a(x) = \Omega_\pm(x\log\log x)$. This completes the proof.

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