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WEIGHTED EXTENDED MEAN VALUES

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Abstract. The author generalizes Stolarsky's Extended Mean Values to a four-parameter family of means $F(r,s;a,b;x,y) = E(r,s;ax,by)/E(r,s;a,b)$ and investigates their monotonicity properties.

1. Introduction. The inequalities

$$\sqrt{xy} \le \frac{y-x}{\log y - \log x} \equiv L(x,y) \le \frac{x+y}{2},$$

and the observation that for natural s the inequalities

$$\min(x,y) \le \left(\frac{x^s + x^{s-1}y + \cdots + xy^{s-1} + y^s}{s+1}\right)^{1/s} \le \max(x,y) \tag{1}$$

hold, led Galvani [1] to the investigation of the one-parameter family of means defined as

$$S_p(x,y) = \left(\frac{y^p - x^p}{p(y-x)}\right)^{1/(p-1)},$$

$$S_0(x,y) = L(x,y), \qquad S_1(x,y) = e^{-1}\left(\frac{y^y}{x^x}\right)^{1/(y-x)}.$$

Observe that for $p=-1$ and 2 we obtain the geometric and the arithmetic means. It has been proved that $S_p(x,y) \le S_q(x,y)$ for $p<q$ and that S_p is increasing in both variables. Stolarsky [8] and later Leach and Sholander [2, 3] extended this family to a two-parameter family of extended mean values by

$$E(r,s;x,y) = \begin{cases} \left(\dfrac{r}{s}\,\dfrac{y^s - x^s}{y^r - x^r}\right)^{1/(s-r)}, & sr(s-r)(x-y) \ne 0, \\ \left(\dfrac{1}{r}\,\dfrac{y^r - x^r}{\log y - \log x}\right)^{1/r}, & r(x-y) \ne 0,\ s = 0, \\ e^{-1/r}(y^{y^r}/x^{x^r})^{1/(y^r - x^r)}, & r = s,\ r(x-y) \ne 0, \\ \sqrt{xy}, & r = s = 0,\ x-y \ne 0, \\ x, & x = y. \end{cases} \tag{2}$$

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and proved that E is continuous and increasing in all variables. Other proofs of this fact can be found in [5, 6, 7, 9].

In this paper we extend E to a four-parameter family of means and investigate their monotonicity properties.

The inequalities

$$\min(x,y) \le \left(\frac{(ax)^s+(ax)^{s-1}by+\cdots+ax(by)^{s-1}+(by)^s}{a^s+a^{s-1}b+\cdots+ab^{s-1}+b^s}\right)^{1/s} \le \max(x,y), \tag{3}$$

valid for natural s and positive x, y, a, b, will be the departure point for our investigation.

Following Stolarsky we define

$$F(r,s;a,b;x,y) = \left(\frac{(ax)^s-(by)^s}{a^s-b^s}\Big/\frac{(ax)^r-(by)^r}{a^r-b^r}\right)^{1/(s-r)} \tag{4}$$

for $rs(r-s)(ax-by)(a-b)\neq 0$. Note that (4) can be written as

$$F(r,s;a,b;x,y) = \frac{E(r,s;ax,by)}{E(r,s;a,b)}, \tag{5}$$

thus extending F to a continuous function in $\mathbb{R}^2\times\mathbb{R}^2_+\times\mathbb{R}^2_+$.

In Section 3 we show that F is a mean of x and y and is monotone in all variables though the monotonicity in r, s, a and b may not be the same for different values of other parameters.

2. Tools. Before formulating our main results we define some tools and prove a useful lemma.

For a function $f(x)$ we write $\mathrm{Mon}_x(f) = 1, 0, -1$ if f is increasing, constant or decreasing in x, respectively. Similarly, $\mathrm{Con}_x(f) = 1, 0, -1$ if f is convex, linear or concave in x. We omit the subscript for functions of one variable. It is worth recording some basic properties of the operators Mon and Con:

- $\mathrm{Mon}(f(g)) = \mathrm{Mon}(f)\,\mathrm{Mon}(g)$.
- If $x = f(y)$ then $\mathrm{Mon}_x(g) = \mathrm{Mon}(f)\,\mathrm{Mon}_y(g(f))$.
- $\mathrm{Con}(f) = \mathrm{Mon}(f') = \mathrm{sgn}(f'')$.
- For fixed c and positive f, $\mathrm{Mon}(f^c) = \mathrm{sgn}(c)\,\mathrm{Mon}(f)$.
- $\mathrm{Con}_x(x^c) = \mathrm{sgn}(c(c-1))$.
- $\mathrm{Mon}_x(x^c) = \mathrm{sgn}(c)$.
- $\mathrm{sgn}(f(x)-f(y)) = \mathrm{Mon}(f)\,\mathrm{sgn}(x-y)$ for strictly monotone f.

Let us now recall two properties of convex functions that will be extremely useful [4].

Property 1. *f is convex (resp. concave) if and only if the difference quotient function $\frac{f(x)-f(y)}{x-y}$, $x \neq y$, is increasing (resp. decreasing) in both x and y.*

Property 2. *If f is convex and $z > 0$ (resp. $z < 0$), then the function $g(x) = f(x+z)-f(x)$ is increasing (resp. decreasing). For concave functions, the monotonicities reverse.*

The above properties can be written as

$$\operatorname{Con}(f) = \operatorname{Mon}_x\left(\frac{f(x)-f(y)}{x-y}\right) = \operatorname{Mon}_y\left(\frac{f(x)-f(y)}{x-y}\right), \tag{6}$$

$$\operatorname{Con}(f) = \operatorname{sgn}(z)\operatorname{Mon}_x(f(x+z)-f(x)). \tag{7}$$

Lemma 1. *If $A, B > 0$, $A, B \neq 1$, $A \neq B$, $A \neq B^{-1}$, then the function*

$$H(t) = \log\left|\frac{1-A^t}{1-B^t}\right|, \qquad H(0) = \log\left|\frac{\log A}{\log B}\right|,$$

is strictly convex or concave and

$$\operatorname{Con}(H) = \operatorname{sgn}(\log^2 A - \log^2 B). \tag{8}$$

Proof. We have

$$\begin{aligned} H''(t) &= \frac{B^t \log^2 B}{(1-B^t)^2} - \frac{A^t \log^2 A}{(1-A^t)^2} \\ &= C(t)\left(\frac{A^t - 2 + A^{-t}}{\log^2 A} - \frac{B^t - 2 + B^{-t}}{\log^2 B}\right) \\ &= 2C(t)\sum_{k=2}^{\infty}\frac{(\log^2 A)^{k-1} - (\log^2 B)^{k-1}}{(2k)!}t^{2k} \\ &= 2C(t)(\log^2 A - \log^2 B)\sum_{k=2}^{\infty}\frac{\sum_{j=0}^{k-2}(\log^2 A)^j(\log^2 B)^{k-2-j}}{(2k)!}t^{2k}, \end{aligned} \tag{9}$$

where $C(t) = \frac{B^t \log^2 B}{(1-B^t)^2}\frac{A^t \log^2 A}{(1-A^t)^2}$ is positive. ∎

3. Monotonicity of $F(r, s; a, b; x, y)$

Theorem 1 (Monotonicity in x and y).

$$\operatorname{Mon}_x(F) = \operatorname{Mon}_y(F) = 1.$$

Proof. The result follows immediately from (5) and monotonicity of E, but we will give an independent proof.

Suppose first that $rs(r-s)(a-b)(x-y) \neq 0$ and write F as

$$\left(\frac{((ax)^r)^{s/r} - ((by)^r)^{s/r}}{(ax)^r - (by)^r}\frac{a^r - b^r}{a^s - b^s}\right)^{1/(s-r)}.$$

One can see immediately that F as a function of x is a composition of four monotone functions: $f_1(t) = (at)^r$, f_2 is the difference quotient function obtained from $t^{s/r}$ (see Property 1), $f_3(t) = \frac{a^r-b^r}{a^s-b^s}t$, and $f_4(t) = t^{1/(s-r)}$. So F is monotone and

$$\begin{aligned}\operatorname{Mon}_x(F) &= \operatorname{sgn}(r)\operatorname{Con}(t^{s/r})\operatorname{sgn}\frac{a^r-b^r}{a^s-b^s}\operatorname{sgn}\frac{1}{s-r}\\ &= \operatorname{sgn}\left(r\,\frac{s}{r}\left(\frac{s}{r}-1\right)\frac{r}{s}\,\frac{1}{s-r}\right) = 1.\end{aligned}$$

If $r = 0$ then

$$F(s,0) = F(0,s) = \left(\frac{\exp(s\log(ax)) - \exp(s\log(by))}{\log(ax)-\log(by)}\,\frac{\log a - \log b}{a^s-b^s}\right)^{1/s}$$

and we have a similar situation with $f_1(t) = \log(at)$ and f_2 coming from e^{st}. So

$$\operatorname{Mon}_x(F) = \operatorname{Mon}(f_1)\operatorname{Con}_t(e^{st})\operatorname{sgn}\left(\frac{\log a-\log b}{a^s-b^s}\right)\operatorname{sgn}\frac{1}{s} = 1.$$

In the case $r = s$,

$$\log F = -\frac{1}{s} + \frac{1}{s}\,\frac{(ax)^s\log(ax)^s - (by)^s\log(by)^s}{(ax)^s-(by)^s} - \log E(s,s;a,b)$$

is monotone in x for the same reason as above, and

$$\operatorname{Mon}_x(F) = \operatorname{Mon}_x(\log F) = \operatorname{Mon}_t(t^s)\operatorname{Con}_t(s^{-1}t\log t) = 1.$$

We leave the case $a = b$ to the reader.

The proof of the monotonicity in y is exactly the same. ■

THEOREM 2 (Monotonicity in r and s).

$$\operatorname{Mon}_r(F) = \operatorname{Mon}_s(F) = \operatorname{sgn}(x-y)\operatorname{sgn}(a^2x-b^2y). \tag{10}$$

Proof. We consider four cases:

CASE 1: $x = y$ or $a^2x = b^2y$. In this case the right hand side of (10) equals 0. An easy calculation shows that

$$F(r,s;a,b;x,y) = \begin{cases} x & \text{if } x=y,\\ \sqrt{xy} & \text{if } a^2x=b^2y,\end{cases} \tag{11}$$

is constant in r and s, so our theorem holds.

CASE 2: $a = b$. The right hand side of (10) equals 1 and from (2) and (5) we obtain

$$\begin{aligned}\log F(r,s;a,a;x,y) &= \log E(r,s;x,y)\\ &= \frac{\log\left|\frac{y^s-x^s}{s}\right| - \log\left|\frac{y^r-x^r}{r}\right|}{s-r}.\end{aligned} \tag{12}$$

As the function $f(s)=\log\left|\frac{y^s-x^s}{s}\right|$ is convex (the proof is almost the same as the proof of Lemma 1), it follows from (12) and Property 1 that $\log F$ and $\log E$, hence F and E are increasing in r and s.

CASE 3: $ax=by$. Then

$$\operatorname{sgn}((x-y)(a^2x-b^2y))=\operatorname{sgn}((x-y)(ax)^2(x^{-1}-y^{-1}))=-1.$$

By (12) and (5),

$$F(r,s;a,b;x,y)=\frac{\sqrt{abxy}}{E(r,s;a,b)},$$

hence from $\operatorname{Mon}_{r,s}(E)=1$ it follows that $\operatorname{Mon}_{r,s}(F)=-1$.

CASE 4: all other cases. We have

$$\begin{aligned}
\operatorname{sgn}(x-y)\operatorname{sgn}(a^2x-b^2y) &= \operatorname{sgn}\left(\log\frac{x}{y}\right)\operatorname{sgn}\left(\log\frac{a^2x}{b^2y}\right)\\
&= \operatorname{sgn}\left(\log\frac{x}{y}\right)\operatorname{sgn}\left(2\log\frac{a}{b}+\log\frac{x}{y}\right)\\
&= \operatorname{sgn}\left(\log^2\frac{ax}{by}-\log^2\frac{a}{b}\right)\\
&= \operatorname{Con}_t\left(\log\left|\frac{1-\left(\frac{ax}{by}\right)^t}{1-\left(\frac{a}{b}\right)^t}\right|\right) \qquad \text{(by Lemma 1)}\\
&= \operatorname{Mon}_{r,s}\left(\frac{1}{s-r}\left(\log\left|\frac{1-\left(\frac{ax}{by}\right)^s}{1-\left(\frac{a}{b}\right)^s}\right|-\log\left|\frac{1-\left(\frac{ax}{by}\right)^r}{1-\left(\frac{a}{b}\right)^r}\right|\right)\right) \qquad \text{(by (6))}\\
&= \operatorname{Mon}_{r,s}(-\log y+\log F)=\operatorname{Mon}_{r,s}(F). \ \blacksquare
\end{aligned}$$

THEOREM 3 (Monotonicity in a and b).

$$\operatorname{Mon}_a(F)=-\operatorname{Mon}_b(F)=\operatorname{sgn}(x-y)\operatorname{sgn}(r+s).$$

Proof. First observe that $F(r,-r;a,b;x,y)=\sqrt{xy}$, so the theorem holds if the right hand side equals 0.

For $r\neq s$ we have

$$\begin{aligned}
\operatorname{sgn}(x-y)\operatorname{sgn}(r+s) &= \operatorname{sgn}(x-y)\operatorname{sgn}(s-r)\operatorname{sgn}(s^2-r^2)\\
&= \operatorname{sgn}(s-r)\operatorname{sgn}\left(\log\frac{x}{y}\right)\operatorname{sgn}(\log^2 e^s-\log^2 e^r)
\end{aligned}$$

$$= \operatorname{sgn}(s-r)\operatorname{sgn}\left(\log\frac{x}{y}\right)\operatorname{Con}_t\left(\log\left|\frac{1-e^{st}}{1-e^{rt}}\right|\right) \tag{13}$$

$$= \operatorname{sgn}(s-r)\operatorname{sgn}\left(\log\frac{x}{y}\right)\operatorname{sgn}(z) \tag{14}$$

$$\times \operatorname{Mon}_t\left(\log\left|\frac{1-e^{s(t+z)}}{1-e^{r(t+z)}}\right| - \log\left|\frac{1-e^{st}}{1-e^{rt}}\right|\right),$$

where (13) and (14) follow from Lemma 1 and Property 2.

Let $z = \log(x/y)$ and $t = \log(a/b)$. Note that $\operatorname{Mon}_t(a) = -\operatorname{Mon}_t(b) = 1$, and (14) transforms into

$$\begin{aligned}
&\operatorname{sgn}(x-y)\operatorname{sgn}(r+s)\\
&\quad= \operatorname{sgn}(s-r)\operatorname{Mon}_t(a)\operatorname{Mon}_a\left(\log\left|\frac{1-\left(\frac{ax}{by}\right)^s}{1-\left(\frac{ax}{by}\right)^r}\right| - \log\left|\frac{1-\left(\frac{a}{b}\right)^s}{1-\left(\frac{a}{b}\right)^r}\right|\right)\\
&\quad= \operatorname{sgn}(s-r)\operatorname{Mon}_a(\log y^{r-s} + \log F^{s-r}) = \operatorname{Mon}_a(F)
\end{aligned}$$

and also

$$\begin{aligned}
&\operatorname{sgn}(x-y)\operatorname{sgn}(r+s)\\
&\quad= \operatorname{sgn}(s-r)\operatorname{Mon}_t(b)\operatorname{Mon}_b\left(\log\left|\frac{1-\left(\frac{ax}{by}\right)^s}{1-\left(\frac{ax}{by}\right)^r}\right| - \log\left|\frac{1-\left(\frac{a}{b}\right)^s}{1-\left(\frac{a}{b}\right)^r}\right|\right)\\
&\quad= -\operatorname{sgn}(s-r)\operatorname{Mon}_b(\log y^{r-s} + \log F^{s-r}) = -\operatorname{Mon}_b(F).
\end{aligned}$$

The case $s = r$ follows from continuity of F. ∎

THEOREM 4.

$$\min(x,y) \le F(r,s;a,b;x,y) \le \max(x,y).$$

Proof. As F is monotone in a it is enough to show that $\lim_{a\to 0} F$ and $\lim_{a\to\infty} F$ satisfy the same inequalities. But

$$\lim_{a\to 0} F = \sqrt{xy}\left(\sqrt{\frac{y}{x}}\right)^{\frac{r+s}{|r|+|s|}}, \qquad \lim_{a\to\infty} F = \sqrt{xy}\left(\sqrt{\frac{x}{y}}\right)^{\frac{r+s}{|r|+|s|}},$$

which completes the proof. ∎

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