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ZEROS OF QUADRATIC FUNCTIONALS ON NON-SEPARABLE SPACES

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Abstract. We construct non-separable subspaces in the kernel of every quadratic functional on some classes of complex and real Banach spaces.

1. Introduction. Investigation of quadratic functionals is an old story [12], [6], [7]. According to [11], for any polynomial functional p with $p(0) = 0$ defined on an infinite-dimensional complex linear space X there is an infinite-dimensional subspace X_0 in the kernel $\ker(p) = p^{-1}(0)$ of p. Quantitative finite-dimensional versions of this fact (estimations of $\dim X_0$ depending on $\dim X$ and the degree of the polynomial) are contained in [1], [4], [5], [14].

The paper [2] started the consideration of subspaces in kernels of polynomials on non-separable spaces. In particular, the authors of [2] proved that if a real Banach space X admits no positive quadratic continuous functional, then every quadratic continuous functional on X vanishes on some infinite-dimensional subspace. They pose the problem of whether in this statement one can replace "infinite-dimensional" by "non-separable" (see also [1, Question 4.8]). Our note continues the investigations of [2]. In particular, we shall construct a non-separable subspace in the kernel of every quadratic functional on a complex Banach space having weak* non-separable dual and on a real Banach space which has controlled separable projection property and admits no positive quadratic continuous functional. On the other hand, we construct a CH-example of a quadratic functional on the normed space $l_1^{\mathrm{f}}(\omega_1)$ whose kernel contains no non-separable linear subspace.

We use the standard notation; in particular dens X stands for the density of a Banach space X, $F^\perp = \{x \in X : \forall f \in F \ \ f(x) = 0\}$ is the annihilator of a subspace $F \subset X^*$ in X, $S(X)$ is the unit sphere of X, and $[M]$ denotes the closed linear span of a subset $M \subset X$. We shall identify cardinals with initial ordinals and will denote by $\overline{\alpha}$ the cardinality of an ordinal α. Elements $x_\alpha \in X$ form a *transfinite basic sequence* if there is a constant $c > 0$ such

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that $\|\sum_{i=1}^{m} a_i x_{\alpha_i}\| \le c\|\sum_{i=1}^{n} a_i x_{\alpha_i}\|$ for any $\alpha_1 < \cdots < \alpha_m < \cdots < \alpha_n$ and any numbers (a_i). *A homogeneous quadratic functional* is a functional $q(x) = B(x,x)$, where $B(x,y)$ is a symmetric bilinear form defined on a linear space X.

2. Complex case. In this section we consider Banach spaces with the following property:

$$\text{dens}\, X/F^{\perp} \le \text{card}\, F \quad \text{for any infinite subset } F \subset X^*. \tag{1}$$

In particular, all WCG spaces have this property (1) (see e.g. [10]).

PROPOSITION 1. *Let q be a continuous homogeneous quadratic functional defined on a non-separable* (*real or complex*) *Banach space X with property* (1). *Then there exists a transfinite basic sequence $x_\alpha \in S(X)$, $\omega_0 \le \alpha < \text{dens}\, X$, such that for every finite collection of scalars* (a_α),

$$q\Big(\sum a_\alpha x_\alpha\Big) = \sum a_\alpha^2 q(x_\alpha). \tag{2}$$

Proof. We construct the x_α by transfinite induction. Take an arbitrary $x_{\omega_0} \in S(X)$.

If the elements $x_\alpha : \alpha < \beta$ are already constructed, choose a dense subset Y_β of the sphere $S[x_\alpha : \alpha < \beta]$ with $\text{card}\, Y_\beta = \overline{\beta}$. Let $B(x,y)$ be the symmetric bilinear form corresponding to the functional q. Take $x_\beta \in S(X)$ so that

$$B(x_\alpha, x_\beta) = 0 \quad \text{for all } \alpha < \beta \tag{3}$$

and

$$f_y(x_\beta) = 0 \quad \text{for all } y \in Y_\beta, \tag{4}$$

where f_y is a functional attaining its norm at y.

Since X has property (1), this process can be continued up to $\text{dens}\, X$. Condition (2) follows from (3). Condition (4) guarantees that (x_α) is a transfinite basic sequence. ∎

Note that results similar to Proposition 1 for usual sequences but for functionals of arbitrary degree were obtained in [11], [9]. Unfortunately, the methods of [11], [9] do not work for transfinite sequences. It is easy to modify the proof of Proposition 1 for usual sequences.

PROPOSITION 2. *The kernel* $\ker(q)$ *of a continuous homogeneous quadratic functional q defined on an infinite-dimensional complex Banach space X with property* (1) *contains a subspace* $X_0 \subset \ker(q)$ *with* $\text{dens}\, X_0 = \text{dens}\, X$.

Proof. Let (x_α) be the transfinite sequence from Proposition 1. We can find a subset $I \subset \{\alpha : \alpha < \text{dens}\, X\}$ of size $|I| = \text{dens}\, X$ such that either $q(x_\alpha) = 0$ for all $\alpha \in I$ or else $q(x_\alpha) \ne 0$ for all $\alpha \in I$.

In the first case put $X_0 = [x_\alpha : \alpha \in I]$. By (2), $X_0 \subset \ker(q)$. Since (x_α) is a transfinite basic sequence, $\operatorname{dens} X_0 = \operatorname{dens} X$.

In the second case put $z_\alpha = x_\alpha / q(x_\alpha)$ for $\alpha \in I$. Let X_0 be the closed linear span of the elements

$$z_{\omega_0} + i z_{\omega_0+1}, \quad z_{\omega_0+2} + i z_{\omega_0+3}, \ \ldots, \ z_{2\beta} + i z_{2\beta+1}, \ \ldots$$

(we suppose the limit ordinals to be even). Condition (2) guarantees that $X_0 \subset \ker(q)$. Since (z_β) is a transfinite basic sequence, $\operatorname{dens} X_0 = \operatorname{dens} X$. ∎

The proofs of Propositions 1 and 2 imply

PROPOSITION 2′. *If X^* is weak* non-separable, then the kernel of any complex homogeneous quadratic continuous functional on X contains a non-separable subspace.*

REMARK 1. We cannot improve the condition $\operatorname{dens} X_0 = \operatorname{dens} X$ in Proposition 2 to, for example, separability of X/X_0. As a counterexample, take $X = l_2(\omega_1)$ and $q(x) = \sum a_\alpha^2$, where $x = (a_\alpha : \alpha < \omega_1)$. Every separable-codimensional subspace $X_0 \subset X$ contains uncountably many unit norm elements, so cannot be contained in $\ker(q)$.

Moreover, we shall show that the normed space $l_1^{\mathrm{f}}(\omega_1)$ of complex functions on $(0, \omega_1)$ with finite support and endowed with the l_1-norm has the following surprising property.

PROPOSITION 3. *Under the Continuum Hypothesis there is a continuous quadratic functional q on $l_1^{\mathrm{f}}(\omega_1)$ whose kernel contains separable linear subspaces only.*

LEMMA 1. *Suppose X is a complex normed space such that the kernel of each continuous quadratic functional q on X contains a non-separable linear subspace. Then for each bounded linear operator $T : X \to l_2$ there is a non-separable subspace $Y \subset X$ such that the closure H of $T(Y)$ in l_2 has infinite codimension in l_2.*

Proof. If T has finite-dimensional range, then it has non-separable kernel Y. Consequently, Y is a non-separable subspace of X such that H has infinite codimension in l_2. So we can assume that $T(X)$ is infinite-dimensional. In this case we can assume that $T(X)$ is dense in l_2. Consider the standard quadratic functional $q(x) = \sum a_n^2$ on l_2, where $x = (a_1, a_2, \ldots)$. It follows from our hypothesis that X contains a non-separable subspace $Y \subset X$ lying in the kernel of the functional $q \circ T$. Then $T(Y)$ lies in the kernel of q. We have to show that the closure H of $T(Y)$ has infinite codimension in l_2. For this consider the real subspace $\Re l_2 = \{x \in l_2 : x = \bar{x}\}$ of l_2 and observe that $H \cap \Re l_2 = \{0\}$. This implies that H has infinite codimension in l_2 as a real subspace, and consequently, H is infinite-codimensional in l_2. ∎

Proof of Proposition 3. Assume the Continuum Hypothesis. The family of closed subspaces of infinite codimension in the separable space l_2 has the size of the continuum and thus can be enumerated as $\{F_\alpha : \alpha < \omega_1\}$. By transfinite induction we can choose a bounded transfinite sequence $\{x_\alpha : \alpha < \omega_1\}$ in l_2 such that $x_\alpha \notin \bigcup_{\beta\le\alpha} \mathrm{lin}(F_\beta \cup \{x_\gamma : \gamma < \alpha\})$ for each ordinal $\alpha < \omega_1$ (the existence of x_α follows from the Baire theorem since x_α should avoid the countable union of linear spaces of infinite codimension in l_2). Evidently, we can choose this sequence so that $[x_\alpha : \alpha < \omega_1] = l_2$. Now define a bounded operator $T : l_1^{\mathrm{f}}(\omega_1) \to l_2$ by letting $T(f) = \sum_{\alpha<\omega_1} f(\alpha)x_\alpha$ for $f \in l_1^{\mathrm{f}}(\omega_1)$. Given a countable ordinal α consider the characteristic function $e_\alpha : \omega_1 \to \{0,1\}$ of $\{\alpha\}$ defined by $e_\alpha^{-1}(1) = \{\alpha\}$. This function e_α is an element of $l_1^{\mathrm{f}}(\omega_1)$. It follows from the choice of the sequence (x_α) that $T^{-1}(F_\alpha) \subset \mathrm{lin}\{e_\beta : \beta \le \alpha\}$ is separable in $l_1^{\mathrm{f}}(\omega_1)$.

Assuming that the closure of $T(Y)$ has infinite codimension in l_2 for some non-separable subspace $Y \subset l_1^{\mathrm{f}}(\omega_1)$, find an ordinal $\alpha < \omega_1$ with $T(Y) \subset F_\alpha$ and observe that $Y \subset T^{-1}(F_\alpha)$ is separable, which is a contradiction. ∎

We do not know if Proposition 3 is true without the Continuum Hypothesis. Also we do not know if the normed space $l_1^{\mathrm{f}}(\omega_1)$ in this proposition can be replaced by the Banach space $l_1(\omega_1)$.

However the following fact is true.

PROPOSITION 4. *Suppose that X is a Banach space all of whose subspaces of infinite codimension are separable. Then there is a continuous quadratic polynomial q on X whose kernel $q^{-1}(0)$ contains no non-separable linear subspace.*

Proof. Assuming the converse and applying Lemma 1 we conclude that for each bounded operator $T : X \to l_2$ there is a non-separable subspace $Y \subset X$ whose image $T(Y)$ has infinite-codimensional closure in l_2.

Observe that the space X admits a countable family of linear functionals separating points of X. Indeed, take any countable linearly independent subset F in the unit sphere S^* of the dual space X^*. Then the subspace $F^\perp = \{x \in X : \forall f \in F\ \ f(x) = 0\}$ of X has infinite codimension and thus is separable. Take any countable subset $E \subset S^*$ separating points of $F^\perp$. Then the countable set $F \cup E$ separates points of X. Using this countable set of functionals it is easy to construct an injective continuous operator $T : X \to l_2$ (for example, put $T(x) = (2^{-n} f_n(x))_{n\in\omega}$, where $\{f_n : n \in \omega\}$ is any enumeration of $F \cup E$).

It follows from the above discussion that X contains a non-separable subspace Y such that the closure of $T(Y)$ has infinite codimension in l_2. Then Y has infinite codimension in X and hence must be separable. This is a contradiction. ∎

In light of the previous proposition it should be mentioned that *the existence of a non-separable Banach space without non-separable infinite-codimensional subspaces* is a well-known open problem.

Now we consider the zeros of functionals generated by sequences of linear functionals.

PROPOSITION 5. *Let X be a (real or complex) Banach space with property* (1) *and $\varphi(t_1, t_2, \ldots)$ be an arbitrary function of countably many variables such that $\varphi(0, 0, \ldots) = 0$. Then for any sequence $f_1, f_2, \ldots$ from X^* the kernel of the functional $\varphi(f_1(x), f_2(x), \ldots)$ contains a separable-codimensional subspace.*

Proof. Let X_0 be the subspace of common zeros of all f_n. It is clear that $X_0 \subset \ker \varphi(f_1(x), f_2(x), \ldots)$. Since X has property (1), X/X_0 is separable. ∎

Given a (real or complex) Banach space X denote by $\mathcal{P}_A(X)$ the space of approximable functionals equal to the completion of finite sums of finite products of linear functionals in the uniform topology [8, p. 85].

COROLLARY. *If X is a (real or complex) Banach space X with property* (1), *then the kernel of each functional from $\mathcal{P}_A(X)$ contains a separable-codimensional subspace.*

3. Real case. In this section we consider real Banach spaces.

Following [13] we say that a Banach space X has the *controlled separable projection property* (CSPP) if for any countable subsets $E \subset X$ and $F \subset X^*$ there exists a separable-valued projection P in X with $\|P\| = 1$, $PX \supset E$ and $P^*X^* \supset F$.

However, the condition $\|P\| = 1$ is not essential (see [10]). Every WCG space has CSPP. This property is stronger than separable complementation property; as an example one can take $l_1(\omega_1)$. This space, as any space with unconditional basis, has the separable complementation property, but because $l_1(\omega_1)^*$ is weak* separable, $l_1(\omega_1)$ does not have CSPP. We do not know about the connection between CSPP and property (1).

Let us make two simple observations.

LEMMA 2. *Let $X = Y \oplus Z$, where Y and Z admit positive quadratic continuous functionals. Then X admits a positive quadratic continuous functional as well. In particular, if Y is separable and Z admits a positive quadratic continuous functional, then so does X.*

LEMMA 3. *If a real-valued continuous function f on a two-dimensional normed space takes values of distinct signs, then f vanishes at some nonzero element.*

Proposition 6. *Let X be a real Banach space with CSPP. If X admits no positive quadratic continuous functional, then every quadratic continuous functional q on X vanishes on some non-separable subspace.*

Proof. Let $B(x,y)$ be the symmetric bilinear form corresponding to q. Let us construct, by induction, a transfinite basic sequence of elements $x_\alpha :$ $1 \le \alpha < \omega_1$ in X such that for all $\alpha \ge \beta$,

$$B(x_\alpha, x_\beta) = 0. \tag{5}$$

By Lemma 3, there exists $x_1 \neq 0$ for which $B(x_1, x_1) = 0$.

Suppose the elements $x_\beta : 1 \le \beta < \alpha,\ \alpha < \omega_1$, are already constructed. Putting $E = \{x_\beta : \beta < \alpha\}$ and $F = \{f_\beta : \beta < \alpha\}$, where $f_\beta(x) = B(x_\beta, x)$, we find a separable-valued projection P in X with $\|P\| = 1,\ PX \supset E$ and $P^*X^* \supset F$. Since X admits no positive quadratic continuous functional, by Lemmas 2 and 3, there is an element $x_\alpha \in \ker P,\ x_\alpha \neq 0$, for which $B(x_\beta, x_\alpha) = 0$ for $\beta < \alpha$. Obviously, condition (5) for x_β is satisfied. By construction, the x_α form a transfinite basic sequence, hence $X_0 = [x_\alpha : \alpha < \omega_1]$ is non-separable. Condition (5) guarantees that $X_0 \subset \ker(q)$. ■

Remark 3. One cannot improve ω_1 to a larger cardinal in Proposition 6. As a counterexample we can take $l_3(\omega_1) \oplus l_2(\omega_2)$. Proposition 6 is connected with the following three-space problem: *Assume that for a subspace Y of a Banach space X there exist continuous linear injective operators from Y and X/Y into a Hilbert space. Does there exist a continuous linear injective operator from X into a Hilbert space?* In particular, suppose X/Y has weak* separable dual and there is a continuous linear injective operator from Y into a Hilbert space. Does there exist a continuous linear injective operator from X into a Hilbert space?

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