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WEAKLY PROPER TORIC QUOTIENTS

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Abstract. We consider subtorus actions on complex toric varieties. A natural candidate for a categorical quotient of such an action is the so-called *toric quotient*, a universal object constructed in the toric category. We prove that if the toric quotient is *weakly proper* and if in addition the quotient variety is of expected dimension then the toric quotient is a categorical quotient in the category of algebraic varieties. For example, weak properness always holds for the toric quotient of a subtorus action on a toric variety whose fan has a convex support.

Introduction. In [Mu] D. Mumford introduced the notion of a categorical quotient for the action of an algebraic group G on an algebraic variety X. By definition this is a G-invariant morphism $p\colon X \to Y$ such that every G-invariant morphism from X to some algebraic variety factors uniquely through p. In general, such a categorical quotient need not exist. In this article we will consider subtorus actions on complex toric varieties. For a survey of results in this context see [AC;Ha4].

In this setting, a natural candidate for a categorical quotient has been constructed in the category of toric varieties, namely the so-called *toric quotient* (see [AC;Ha1]). The toric quotient is universal for toric morphisms from the given toric variety that are constant on the orbits of the subtorus action. Clearly, a necessary condition for the toric quotient to be categorical is surjectivity. But there are examples of toric quotients that are not surjective and hence not categorical. In fact, in [AC;Ha3, Section 5], an example of a subtorus action on a toric variety is given that does not admit a categorical quotient, not even in the category of algebraic or analytic spaces.

On the other hand, if the codimension of H in the big torus T of the toric variety is at most 2, then the toric quotient is always categorical (Corollary 4.3 in [AC;Ha3]). An important tool for the proof was to observe that the toric quotient in that case is *weakly proper*, i.e. it satisfies a certain weak lifting property for holomorphic germs of curves (the precise definition is given in Section 1).

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A morphism of complex algebraic varieties is weakly proper if and only if it is universally submersive. Another more geometric characterization is given by the following notion: We say that a morphism $p\colon X \to Y$ of algebraic varieties has the *curve covering property* if for every curve $Y' \subset Y$ there is a (not necessarily irreducible) curve $X' \subset X$ with $p(X') = Y'$ such that the morphism from X' to Y' defined by p has finite fibers. If in addition the curve covering property is preserved under any base change, then the morphism p has the *universal curve covering property*. If p is a surjective morphism over the field of complex numbers then this property is equivalent to weak properness (see Section 1).

For a toric morphism, weak properness has a very simple characterization in terms of fans, namely the property holds if and only if the associated lattice homomorphism induces a surjective map on the supports of the corresponding fans (see Section 2).

In this article we will show that weak properness is a sufficient condition for a toric quotient of expected dimension to be categorical. More precisely, our main result is the following (see Corollary 6.4):

THEOREM. *For a toric variety X and a subtorus H of the big torus T of X, let $p\colon X \to Y$ denote the toric quotient for the action of H on X. If p is weakly proper and* $\dim Y = \dim T/H$, *then the toric quotient is categorical.*

For example, weak properness is automatically satisfied if the fan associated to X has a convex support, or equivalently, if there is a proper toric morphism from X onto an affine toric variety. In fact, we can even show that in this case the dimension condition can be omitted (see Corollary 6.5):

THEOREM. *If the toric variety X corresponds to a fan with convex support, then for any subtorus action on X the toric quotient is a quotient in the category of algebraic varieties.*

In order to prove our results we proceed by induction on the number of steps in the construction of the toric quotient. The intermediate steps of this construction can be viewed as successive approximations of the quotient by non-separated prevarieties (see Section 6). Therefore it simplifies the arguments to work in the more general context of toric prevarieties. We prove a result stated in this context (see Theorem 6.2) and obtain the above theorems as corollaries.

Toric prevarieties are the non-separated analogues of toric varieties, i.e. complex algebraic prevarieties with an effective action of a torus having a dense orbit. In analogy to the separated case, there is a convex-geometrical description of toric prevarieties in terms of so-called *systems of fans* (see [AC;Ha2]). In Section 2 we briefly recall the basic facts on toric prevarieties and systems of fans.

In Section 3 we consider morphisms from toric prevarieties to algebraic varieties that are not necessarily toric. Every such morphism defines an equivalence relation on the cones occurring in the system of fans corresponding to the toric prevariety, and the supports of the equivalence classes form a partition of the support of the system of fans. The properties of this partition are analyzed in Sections 4 and 5. These sections contain the convex-geometrical lemmata that are needed for the proof of the main theorem that is carried out in Section 6.

As an application of the result, in Section 7 we give an example of a categorical quotient $p\colon X \to Y$ of a 4-dimensional toric variety by some $\mathbb{C}^*$-action where $\dim Y = 3$ that is not uniform in the sense of [Mu], i.e. such that for some open subset $U \subset Y$, the restriction $p\colon p^{-1}(U) \to U$ is not the categorical quotient for the induced $\mathbb{C}^*$-action.

1. Weak properness and the universal curve covering property. In this section we recall the definition of weak properness given in [AC;Ha3], and we give another interpretation of this notion in terms of a certain curve covering property. We start with the curve covering property, and we first consider algebraic prevarieties defined over an arbitrary algebraically closed field $\mathbb{K}$. Following the terminology used e.g. in [Bo], we do not require a prevariety to be irreducible. When we speak of a curve in a prevariety we mean a closed algebraic subset of pure dimension 1. So a curve in this sense is also not necessarily irreducible.

1.1. Definition. Let $p\colon X \to Y$ be a morphism of prevarieties. We say that p has the *curve covering property* (CCP) if for every irreducible curve $Y' \subset Y$ and every $y \in Y'$ there is an irreducible curve $X' \subset X$ such that $y \in p(X') \subset Y'$ and $p(X')$ is dense in Y'. If the curve covering property remains true even after any base change then we say that p has the *universal curve covering property*.

1.2. Example. Every surjective proper morphism of prevarieties has the universal curve covering property.

Proof. Let $p\colon X \to Y$ be a proper surjective morphism. Consider an irreducible curve Y' in Y and a point $y \in Y'$. Choose an irreducible curve X' in X such that $p(X')$ is dense in Y'. Since p is proper, the curve X' must intersect the fiber of y. ∎

In fact, the curve covering property is nothing but a geometric characterization of submersiveness:

1.3. Lemma. *A surjective morphism $p\colon X \to Y$ of algebraic prevarieties has the curve covering property if and only if it is submersive, i.e. if Y carries the quotient topology with respect to p.*

Proof. First suppose that the (CCP) holds, and consider a subset $U \subset Y$ whose preimage $p^{-1}(U)$ is open in X. Assume that the complement $A := Y \setminus U$ is not closed and choose a point $y \in \overline{A} \cap U$. Since $A = p(X \setminus p^{-1}(U))$ is constructible, there is an irreducible curve $C_Y \subset Y$ through y such that $C_Y \cap A$ is open and dense in C_Y. Using the (CCP) we can find an irreducible curve $C_X \subset X$ meeting the fiber $p^{-1}(y)$ in a point x such that $p(C_X)$ is dense in C_Y. That implies $x \in \overline{p^{-1}(A)} \setminus p^{-1}(A)$, contradicting the assumption that $p^{-1}(A)$ is closed.

Conversely, suppose that p is submersive and consider an irreducible curve C_Y through a point y in Y. Since p is surjective, the fiber of y is not empty. Moreover, since $C_Y \setminus \{y\}$ is not closed, the assumption implies that its preimage $p^{-1}(C_Y) \setminus p^{-1}(y)$ is also not closed. So there is a point x in the fiber of y which is contained in the closure of $p^{-1}(C_Y \setminus \{y\})$. This implies that there is a curve C_X through x in $p^{-1}(C_Y)$ intersecting the fiber of y only in a finite number of points. So the (CCP) is fulfilled. ∎

An example of a surjective morphism that does not have the curve covering property is the following:

1.4. Example. Let X denote the blow-up of $\mathbb{K}^2$ at the origin, and let $x = \infty$ be the point corresponding to the e_2-axis in the exceptional line. Then the morphism $p\colon X \setminus \{x\} \to \mathbb{K}^2$ defined by contracting the exceptional line is surjective. But there is no curve in $X \setminus \{x\}$ covering the e_2-axis near the origin.

Here is an example of a surjective morphism with curve covering property but such that the (CCP) does not hold universally.

1.5. Example. Let X denote the simple nodal curve in $\mathbb{K}^2$ defined by the equation $y^2 = x^2(x+1)$. Its normalization is given by $\nu\colon \mathbb{K}^1 \to X$, $t \mapsto (t^2-1, t(t^2-1))$. The map $p\colon \mathbb{K}^1 \setminus \{-1\} \to X$ defined by ν is surjective and the (CCP) holds. But base change of p via ν leads to a map that does not have the (CCP):

The fiber product of $\mathbb{K}^1$ and $\mathbb{K}^1 \setminus \{-1\}$ over X is the reducible subvariety $Y := \{(t,t);\, t \in \mathbb{K},\, t \neq -1\} \cup (-1,1)$ of $\mathbb{K}^2$, and for the projection $p_1\colon Y \to \mathbb{K}^1$ onto the first factor the (CCP) does not hold.

From now on, all prevarieties are assumed to be defined over $\mathbb{C}$. In this case the universal curve covering property has a local interpretation in terms of holomorphic germs of curves, and for this purpose we recall some definitions from [AC;Ha3]. A *local curve* in $x \in X$ is defined to be a holomorphic mapping germ $\gamma\colon \mathbb{C}_0 \to X'_x$, where X' is an algebraic curve in X through x. Let $p\colon X \to Y$ be a regular map of prevarieties. We say that a local curve $\widetilde{\gamma}\colon \mathbb{C}_0 \to X'_x$ at $x \in X$ is a *weak p-lifting* of a local curve $\gamma\colon \mathbb{C}_0 \to Y'_y$

at $y \in Y$ (where $Y' \subset Y$ is a curve through y) if there is a non-constant holomorphic mapping germ $\alpha\colon \mathbb{C}_0 \to \mathbb{C}_0$ and a commutative diagram

$$\begin{array}{ccc} \mathbb{C}_0 & \xrightarrow{\widetilde{\gamma}} & X'_x \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{C}_0 & \xrightarrow{\gamma} & Y'_y \end{array}$$

The map p is called *weakly proper* if any local curve in Y admits a weak p-lifting. A similar notion in the context of algebraic spaces was introduced by Kollár, the so-called weak lifting property for discrete valuation rings (see [Ko, Section 3]).

1.6. Proposition. *For a surjective morphism $p\colon X \to Y$ of complex algebraic prevarieties the following conditions are equivalent*:

(i) *p is weakly proper.*
(ii) *p has the universal curve covering property.*
(iii) *p is submersive, and this property is preserved by every base change.*

Proof. First note that surjectivity is preserved under base change. So the equivalence of the last two statements follows from Lemma 1.3. Moreover, weak properness is preserved under base change and clearly implies the (CCP). That shows that (i) implies (ii).

Now let us assume (ii) and conclude (i). Without loss of generality we can also assume that Y is separated. Let γ be a local curve through a point $y \in Y$, and let C_Y denote the Zariski closure of the image of γ in Y. The local curve γ factors through the normalization $\widetilde{C}_Y$, and after a base change we can achieve that Y is normal and 1-dimensional.

By assumption there is a point x in the fiber of y and an irreducible curve C_X through x such that $p(C_X)$ is dense in Y. Consider the normalization $\widetilde{C}$ of C_X, and choose a point $\widetilde{x} \in \widetilde{C}$ above x. We have an induced dominant morphism from $\widetilde{C}$ to Y mapping $\widetilde{x}$ to y.

Now we can argue locally in the analytic category. The germs $\widetilde{C}_{\widetilde{x}}$ and Y_y are smooth and hence isomorphic to $\mathbb{C}_0$. The holomorphic germ of the morphism p looks like the germ $\mathbb{C}_0 \to \mathbb{C}_0$ defined by $z \mapsto z^n$ for some $n \in \mathbb{N}$, and similarly the germ of γ is of the form $\mathbb{C}_0 \to \mathbb{C}_0$, $z \mapsto z^m$ for some m. Since both germs commute, one can choose $\alpha = p$ and $\widetilde{\gamma} = \gamma$ to obtain the desired commutative diagram. So in fact (i) holds. ∎

As mentioned in the introduction, we want to further investigate weakly proper toric quotients. The main task will be to check the defining property of a categorical quotient. In this context the following factorization result is particularly useful (see Proposition 1.1 in [AC;Ha3]):

1.7. PROPOSITION. *Let $p\colon X \to Y$ be a weakly proper morphism of prevarieties, assume that Y is normal and let $f\colon X \to Z$ be a morphism into a variety Z. If f is constant on the fibers of p, then there is a unique morphism $\widetilde{f}\colon Y \to Z$ such that $f = \widetilde{f} \circ p$.* ∎

Note that for open surjections the above result is well known (see e.g. [Bo, II.6.2]). However, the statement is not true in general for arbitrary morphisms p.

2. Toric morphisms and weak properness. We now come to the toric setting. Since for the proof of our main result we need the non-separated analogues of toric varieties, we first briefly summarize the basic facts about toric prevarieties. Then we state a characterization of weak properness in terms of fans.

By definition a *toric prevariety* is a normal prevariety together with an effective action of an algebraic torus having a dense orbit. We also fix an embedding of the torus T in the toric prevariety X and denote the point in X corresponding to the identity element by x_0. A morphism $f\colon X \to X'$ of toric prevarieties with tori T and T' respectively is called a *toric morphism* if f maps T into T' and is equivariant with respect to the actions of T and T' respectively. In particular, the restriction map $f|_T\colon T \to T'$ is a group homomorphism, and we will refer to its kernel as the *kernel of* f and denote it by $\ker(f)$.

As in the separated case, one can associate to each toric prevariety a convex-geometrical object. More precisely, the category of toric prevarieties is equivalent to the category of affine systems of fans (see [AC;Ha2]). Let us recall the basic definitions. A *system of fans* in a lattice N is a finite collection $\mathcal{S} = (\Delta_{ij})_{i,j\in I}$ of fans Δ_{ij} in the lattice N with

$$\Delta_{ij} = \Delta_{ji} \quad \text{and} \quad \Delta_{ij} \cap \Delta_{jk} \subset \Delta_{ik}$$

for all i, j, k. Such a system of fans is called *affine* if for every $i \in I$ the fan Δ_{ii} consists of the faces of a single cone σ_{ii} in N.

Given an affine system of fans $\mathcal{S}$ in a lattice N, one can construct a toric prevariety $X_{\mathcal{S}}$ with torus $T = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ by taking the affine toric varieties X_i associated to the cones σ_{ii} in the lattice N, and glueing X_i and X_j along the open toric subvariety corresponding to the common subfan Δ_{ij} of Δ_{ii} and Δ_{jj} for every $i, j \in I$. The subfans Δ_{ij} induce a glueing relation on the set

$$\mathcal{F}(\mathcal{S}) := \{(\tau, i);\ i \in I,\ \tau \prec \sigma_{ii}\}$$

of labelled faces of the maximal cones occurring in $\mathcal{S}$, namely $(\sigma, i) \sim (\tau, j)$ if and only if $\sigma = \tau \in \Delta_{ij}$. There is a 1-1 correspondence between the set of equivalence classes $\Omega(\mathcal{S}) := \mathcal{F}(\mathcal{S})/\!\sim$ and the set of T-orbits in $X_{\mathcal{S}}$. More

precisely, every labelled face $(\tau, i) \in \mathcal{F}(\mathcal{S})$ defines a distinguished point x_τ in the toric variety X_i, and this point is identified with x_τ in X_j if and only if $\tau \in \Delta_{ij}$. Therefore every equivalence class $[\tau, i] \in \Omega(\mathcal{S})$ defines a distinguished point $x_{[\tau,i]}$ in $X_\mathcal{S}$, and the orbits $T \cdot x_{[\tau,i]}$, $[\tau, i] \in \Omega(\mathcal{S})$, form a partition of $X_\mathcal{S}$. The orbit structure is reflected by the partial ordering on $\Omega(\mathcal{S})$ given by $[\tau, j] \prec [\sigma, i]$ if and only if $\tau \prec \sigma$ and $[\tau, j] = [\tau, i]$. We have

$$T \cdot x_{[\sigma,i]} \subset \overline{T \cdot x_{[\tau,j]}} \;\Leftrightarrow\; [\tau, j] \prec [\sigma, i].$$

We will also need a description of toric morphisms in terms of systems of fans. For our purposes, however, it will suffice to consider toric morphisms from prevarieties to varieties. So let $X_\mathcal{S}$ be the toric prevariety arising from an affine system of fans $\mathcal{S}$ in a lattice N and let X_Δ denote the toric variety associated to a fan Δ in a lattice N'. Set

$$\mathcal{C}(\mathcal{S}) := \bigcup_{i \in I} \Delta_{ii}.$$

Then any toric morphism $f\colon X_\mathcal{S} \to X_\Delta$ corresponds to a lattice homomorphism $F\colon N \to N'$ with the property that for every $\tau \in \mathcal{C}(\mathcal{S})$ there is a cone $\sigma \in \Delta$ with $F_\mathbb{R}(\tau) \subset \sigma$. (Here $F_\mathbb{R}$ denotes the scalar extension of F to the real vector spaces generated by N and N'.) For later use we also introduce the following notation. For any given natural number k, we denote the subset of k-dimensional cones in $\mathcal{C}(\mathcal{S})$ by $\mathcal{C}(\mathcal{S})^k$.

The *support* of the system of fans $\mathcal{S}$ is defined to be $|\mathcal{S}| = \bigcup_{i \in I} |\Delta_{ii}|$. For toric morphisms weak properness can be characterized as follows (see Proposition 1.2 in [AC;Ha3]):

2.1. Proposition. *A toric morphism $f\colon X_\mathcal{S} \to X_\Delta$ from a toric prevariety to a toric variety is weakly proper if and only if the associated lattice homomorphism F induces a surjection on the supports of the corresponding systems of fans*,

$$F_\mathbb{R}(|\mathcal{S}|) = |\Delta|. \;\blacksquare$$

3. Partition of the support defined by a morphism. Let us consider a morphism $f\colon X \to Z$ from a toric prevariety $X = X_\mathcal{S}$ arising from an affine system of fans $\mathcal{S}$ in a lattice N to an algebraic variety Z that is not necessarily toric. As we will see, such a morphism defines an equivalence relation on the set $\mathcal{C}(\mathcal{S})$ of cones occurring in $\mathcal{S}$, and the supports of the equivalence classes form a finite partition of the support of $\mathcal{S}$. Whether or not f factors through a given toric morphism can be expressed in terms of this partition.

We will consider two elements of $\Omega(\mathcal{S})$ as equivalent with respect to f if the corresponding parametrized orbits are mapped by f to the same parametrized set in Z, or more precisely if the compositions of f with the

orbit maps of the corresponding distinguished points yield the same map on T.

3.1. Definition. The morphism f induces an equivalence relation on the set $\Omega(\mathcal{S})$, namely:

$$[\sigma, i] \sim_f [\sigma', j] \;\Leftrightarrow\; f(t \cdot x_{[\sigma,i]}) = f(t \cdot x_{[\sigma',j]}) \text{ for all } t \in T.$$

Note that if the same cone appears with two different labels in $\Omega(\mathcal{S})$, $[\sigma, i]$ and $[\sigma, j]$ say, then $[\sigma, i] \sim_f [\sigma, j]$. That is an immediate consequence of the following remark. For a cone σ, let σ° denote its relative interior. We observe:

3.2. Remark. *If for $[\sigma, i], [\sigma', j] \in \Omega(\mathcal{S})$ we have $\sigma^\circ \cap (\sigma')^\circ \neq \emptyset$, then $[\sigma, i] \sim_f [\sigma', j]$.*

Proof. Choose $v \in N \cap \sigma^\circ \cap (\sigma')^\circ$. Let λ_v denote the corresponding one-parameter subgroup of T and fix $t \in T$. Then $\lim_{s\to 0} t \cdot \lambda_v(s) \cdot x_0 = t \cdot x_{[\sigma,i]}$ in the affine chart X_i, whereas $\lim_{s\to 0} t \cdot \lambda_v(s) \cdot x_0 = t \cdot x_{[\sigma',j]}$ in X_j. Since f is continuous and we assumed Z to be separated, this implies the claim. ∎

So in fact, f induces an equivalence relation on the set $\mathcal{C}(\mathcal{S})$, and we will also denote this relation by $\sim_f$. We define the *support of the equivalence class* of $\sigma \in \mathcal{C}(\mathcal{S})$ by

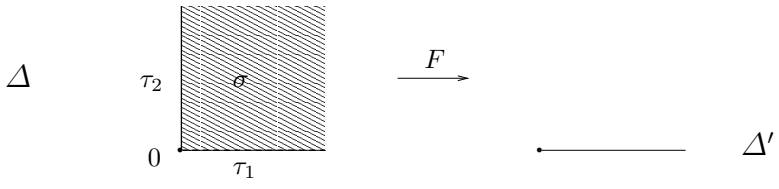
$$|\sigma|_f := \bigcup_{\sigma' \sim_f \sigma} (\sigma')^\circ.$$

As an immediate consequence of Remark 3.2 we obtain the following

3.3. Remark. *The subsets $|\sigma|_f$, $\sigma \in \mathcal{C}(\mathcal{S})$, form a partition of the support of $\mathcal{S}$.* ∎

3.4. Example. Consider a fan Δ in a lattice N', and let X_Δ denote the corresponding toric variety. Let $f\colon X_\mathcal{S} \to X_\Delta$ be a toric morphism, and let $F\colon N \to N'$ denote the associated lattice homomorphism. Then the equivalence classes of $\mathcal{C}(\mathcal{S})$ correspond to the elements of Δ that meet $F(|\mathcal{S}|)$, more precisely $\sigma \sim_f \sigma'$ if and only if there is a cone $\tau \in \Delta$ with $F(\sigma^\circ) \subset \tau^\circ \supset F((\sigma')^\circ)$. The supports of the equivalence classes are the sets $F^{-1}(\tau^\circ) \cap |\mathcal{S}|$, $\tau \in \Delta$.

For example, let us look at the toric morphism $f\colon \mathbb{C}^2 \to \mathbb{C}$ given by the projection on the first factor. The system of fans corresponding to $\mathbb{C}^2$ is the fan $\Delta := \{\sigma, \tau_1, \tau_2, 0\}$ in $\mathbb{Z}^2$, where $\sigma := \text{cone}(e_1, e_2)$, $\tau_i := \text{cone}(e_i)$



for $i = 1, 2$, and $\mathbb{C}$ arises from the fan $\Delta' := \{\mathrm{cone}(e_1), 0\}$ in $\mathbb{Z}$. The lattice homomorphism corresponding to f is the projection $F\colon \mathbb{Z}^2 \to \mathbb{Z}$. So in this case we have $0 \sim_f \tau_2$ and $\tau_1 \sim_f \sigma$. The supports of the equivalence classes in $|\mathcal{S}| = \sigma$ are $|\tau_2|_f = \tau_2$ and $|\sigma|_f = \sigma^\circ \cup \tau_1^\circ$.

For a given cone $\sigma \in \mathcal{C}(\mathcal{S})$, let $T(|\sigma|_f)$ denote the subtorus of T corresponding to the sublattice obtained by intersecting N with the linear subspace $\mathrm{lin}\,|\sigma|_f$ generated in $N_\mathbb{R}$ by the set $|\sigma|_f$. Then $T(|\sigma|_f)$ is generated by all isotropy subgroups $T_{x_{[\sigma',i]}}$, where $[\sigma', i] \in \Omega(\mathcal{S})$ and $\sigma' \sim_f \sigma$.

3.5. Remark. *We have $f(t \cdot t' \cdot x_{[\sigma,i]}) = f(t \cdot x_{[\sigma,i]})$ for every $t' \in T(|\sigma|_f)$. In particular, f is invariant with respect to the action of $T(|0|_f)$.*

Proof. To see this, choose cones $\sigma_1, \ldots, \sigma_s$ in the f-equivalence class of σ that generate $\mathrm{lin}\,|\sigma|_f$ as a vector space. If $[\sigma_j, i_j] \in \Omega(\mathcal{S})$, then $f(t \cdot x_{[\sigma_j,i_j]}) = f(t \cdot x_{[\sigma,i]})$ for every $t \in T$, since $\sigma_j \sim_f \sigma$. So we can conclude that $f(t' \cdot t \cdot x_{[\sigma,i]}) = f(t \cdot x_{[\sigma,i]})$ for every t' in the stabilizer T_j of the point $x_{[\sigma_j,i_j]}$. Since the subtori T_j generate $T(|\sigma|_f)$, that implies the claim. ∎

Let us now consider a dominating toric morphism $p\colon X_\mathcal{S} \to X_\Delta$ to some toric variety X_Δ, and assume that the associated lattice homomorphism $P\colon N \to N'$ is surjective. That means that the kernel of the homomorphism of tori $T \to T'$ induced by P is connected. We will denote this kernel by $\ker(p) \subset T$. Assume that p is weakly proper. With the above notations we can describe the fibers of p as follows:

3.6. Lemma. *Two points x, y lie in the same fiber of p if and only if there are elements $[\sigma_1, i], [\sigma_2, j] \in \Omega(\mathcal{S})$ with $\sigma_1 \sim_p \sigma_2$, $t \in T$ and $t_1 \in T(|\sigma_1|_p) \cdot \ker(p)$ with $x = t \cdot x_{[\sigma_1,i]}$ and $y = t \cdot t_1 \cdot x_{[\sigma_2,j]}$.*

Proof. Let $p|_T\colon T \to T'$ denote the restriction homomorphism of p to the big tori of $X_\mathcal{S}$ and X_Δ respectively. Any point $z \in X_\Delta$ is of the form $z = t' \cdot x_{\sigma'}$ for some $t' \in T'$ and $\sigma' \in \Delta$. Then the p-fiber of the point z is

$$p^{-1}(z) = p^{-1}(t' \cdot x_{\sigma'}) = \bigcup_{P_\mathbb{R}(\sigma)^\circ \subset (\sigma')^\circ} (p|_T)^{-1}(t' \cdot T'_{x_{\sigma'}}) \cdot x_{[\sigma,i]}$$

(see [AC;Ha3, Proposition 3.5]). As described in Example 3.4, since p is surjective, the given cone $\sigma' \in \Delta$ defines a p-equivalence class, represented by $\sigma_1 \in \mathcal{C}(\mathcal{S})$ say, and $|\sigma_1|_p = |\mathcal{S}| \cap P_\mathbb{R}^{-1}((\sigma')^\circ)$. We obtain

$$p^{-1}(t' \cdot x_{\sigma'}) = \bigcup_{\sigma \sim_p \sigma_1} (p|_T)^{-1}(t' \cdot T'_{x_{\sigma'}}) \cdot x_{[\sigma,i]}.$$

To prove the lemma it suffices to show that

$$(p|_T)^{-1}(T'_{x_{\sigma'}}) = T(|\sigma_1|_p) \cdot \ker(p).$$

First note that the subtorus $T'_{x_{\sigma'}}$ corresponds to the sublattice of N' defined by $\lin \sigma'$. By Proposition 2.1, since p is weakly proper we have $P_{\mathbb{R}}(|\mathcal{S}|) = |\Delta|$. That implies $P_{\mathbb{R}}(|\sigma_1|_p) = (\sigma')^\circ$ and hence $P_{\mathbb{R}}(\lin |\sigma_1|_p) = \lin \sigma'$. Therefore $P_{\mathbb{R}}^{-1}(\lin \sigma') = \lin |\sigma_1|_p + \ker(P_{\mathbb{R}})$. Since we assumed P to be surjective, that proves the claim. ∎

From this description of the fibers of a weakly proper toric morphism we obtain the following factorization criterion.

3.7. Lemma. *Let $f\colon X_{\mathcal{S}} \to Z$ be a morphism to a variety Z, and let $p\colon X_{\mathcal{S}} \to Y$ be a toric morphism with the universal curve covering property to a toric variety Y such that* $\ker(p)$ *is connected. Then the following statements are equivalent*:

(i) *The morphism f factors through p.*
(ii) *$f(t_1 \cdot x_{[\sigma_1,i]}) = f(t_2 \cdot x_{[\sigma_2,j]})$ for all $[\sigma_1, i], [\sigma_2, j] \in \Omega(\mathcal{S})$ with $\sigma_1 \sim_p \sigma_2$ and $t_1, t_2 \in T$ with $t_2^{-1} t_1 \in T(|\sigma_1|_p) \cdot \ker(p)$.*
(iii) *f is* $\ker(p)$*-invariant, and for $\sigma, \sigma' \in \mathcal{C}(\mathcal{S})$, whenever $\sigma \sim_p \sigma'$ then $\sigma \sim_f \sigma'$.*

Proof. By Proposition 1.7, f factors through p if and only if f is constant on the fibers of p. From the above description of the fibers of p it follows immediately that (i) and (ii) are equivalent. Now assume that (i) holds. Then in particular f is $\ker(p)$-invariant. Moreover, it can be read off directly from (ii) that $\sigma \sim_f \sigma'$ whenever $\sigma \sim_p \sigma'$.

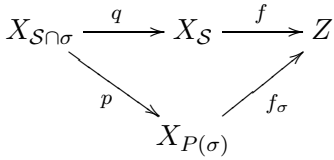
Conversely, assume that (iii) holds, and let $\sigma \in \mathcal{C}(\mathcal{S})$. Then the f-equivalence class of σ contains the p-equivalence class of σ, and hence $T(|\sigma|_p) \subset T(|\sigma|_f)$. Now (ii) follows from Remark 3.5. ∎

We want to apply this lemma to the following situation. Let $\mathcal{S}$ be an affine system of fans in a lattice N and let σ be a convex cone in N, not necessarily contained in $\mathcal{C}(\mathcal{S})$ but with $\sigma \subset |\mathcal{S}|$. Let σ_0 denote the minimal face of σ. Then σ_0 is a linear subspace of $N_{\mathbb{R}}$, and the sublattice $\sigma_0 \cap N$ defines a subtorus H of the big torus of $X_{\mathcal{S}}$.

Define a system of fans $\mathcal{S} \cap \sigma = (\Delta'_{ij})_{i,j \in I}$ in N by setting $\sigma'_{ii} := \sigma_{ii} \cap \sigma$ and $\Delta'_{ij} := \{\tau \cap \sigma;\ \tau \in \Delta_{ij}\}$. Then the identity homomorphism id_N defines a toric morphism $q\colon X_{\mathcal{S} \cap \sigma} \to X_{\mathcal{S}}$, and the projection $P\colon N \to N/(\sigma_0 \cap N)$ defines a toric morphism $p\colon X_{\mathcal{S} \cap \sigma} \to X_{P(\sigma)}$ to the affine toric variety $X_{P(\sigma)}$ associated to the cone $P(\sigma)$ in $N/(\sigma_0 \cap N)$.

3.8. Corollary. *Let $f\colon X_{\mathcal{S}} \to Z$ be a morphism to some variety Z, and assume that for every face τ of σ, the set τ° is contained in the support of an f-equivalence class of $\mathcal{C}(\mathcal{S})$. Then the morphism f is H-invariant, and there is a unique morphism $f_\sigma\colon X_{P(\sigma)} \to Z$ such that the following diagram*

is commutative:

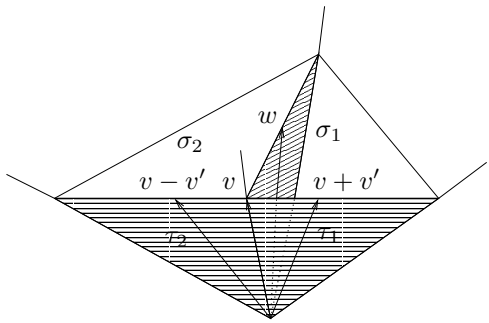


Proof. The minimal face σ_0 is a linear subspace and therefore it coincides with its relative interior. By assumption, σ_0 is contained in the support of an f-equivalence class, and that means that $\sigma_0 \subset |0|_f$. Therefore $H \subset T(|0|_f)$, and Remark 3.5 implies that f is H-invariant.

By construction, P is surjective and weakly proper. So we can apply Lemma 3.7 to conclude that the morphism $f \circ q$ factors uniquely through p. That means that there is a morphism $f_\sigma \colon X_{P(\sigma)} \to Z$ with $f \circ q = f_\sigma \circ p$. ∎

4. Closures of cones in the support. The following two sections contain preparations for the proof of the main result given in Section 6. First we analyze morphisms from a toric variety corresponding to a fan with just two maximal cones in a special position.

4.1. Lemma. *Let Δ be a fan with only two maximal cones σ_1 and σ_2. Assume that there are faces τ_i of σ_i and vectors $v \in (\tau_1 \cap \tau_2)^\circ \cap N$, $v + v' \in \tau_1^\circ \cap N$, $v - v' \in \tau_2^\circ \cap N$, $w \in (\sigma_1 \cap \sigma_2)^\circ \cap N$ with $v + v' + w \in \sigma_1^\circ$. Let $f \colon X_\Delta \to Z$ be a morphism such that $\sigma_1 \sim_f (\sigma_1 \cap \sigma_2)$. Then $\tau_1 \sim_f (\tau_1 \cap \tau_2)$.*



Proof. Set $\varrho := \tau_1 \cap \tau_2$. We have to show that for every $t \in T$,

$$f(t \cdot x_\varrho) = f(t \cdot x_{\tau_1}) \, .$$

For a given $t \in T$, consider the morphism $f_t \colon X_\Delta \to Z$ defined by $f_t(x) := f(t \cdot x)$. Then clearly the morphism f_t satisfies the same assumption as f, i.e. $\sigma_1 \sim_{f_t} (\sigma_1 \cap \sigma_2)$. Therefore it suffices to show the claim for $t = 1$.

Let V denote the closure of the orbit $T \cdot x_\varrho$ in X_Δ. Then V is a toric variety with respect to the torus T/T_{x_ϱ}, and it corresponds to the fan obtained by projecting the star of ϱ in Δ to $N/(\operatorname{lin} \varrho \cap N)$. The lattice homomorphism $F \colon \mathbb{Z}^2 \to N/(\operatorname{lin} \varrho \cap N)$ defined by $F(e_1) = \overline{v'}$ and $F(e_2) = \overline{w}$ yields a toric

morphism $\varphi\colon \mathbb{P}_1 \times \mathbb{C} \to V$ with the following properties:

$$\varphi([r_0, r_1], s) = \begin{cases} \lambda_{v'}(r_1/r_0) \cdot \lambda_w(s) \cdot x_\varrho & \text{if } r_0, r_1, s \neq 0, \\ \lambda_w(s) \cdot x_{\tau_1} & \text{if } r_1 = 0,\ s \neq 0, \\ \lambda_{v'}(r_1/r_0) \cdot x_{\sigma_1 \cap \sigma_2} & \text{if } r_0, r_1 \neq 0,\ s = 0. \end{cases}$$

Now consider the composition $\psi = f \circ \varphi\colon \mathbb{P}_1 \times \mathbb{C} \to Z$. By definition, $v' \in \operatorname{lin} \sigma_1$ and hence $\lambda_{v'}(\mathbb{C}^*) \subset T(|\sigma_1|)$. Since we assumed $\sigma_1 \sim_f (\sigma_1 \cap \sigma_2)$, this implies $f(\lambda_{v'}(r) \cdot x_{\sigma_1 \cap \sigma_2}) = f(x_{\sigma_1})$ for all $r \in \mathbb{C}^*$. So $\psi(\mathbb{P}_1 \times \{0\}) = f(x_{\sigma_1})$, i.e. ψ contracts the curve $\mathbb{P}_1 \times \{0\}$ to a point. By Lemma 4.2 below, we conclude that ψ in fact does not depend on the first coordinate, and for $s = 1$ we obtain $f(x_\varrho) = f(x_{\tau_1})$. ∎

In the above proof we used a general fact about morphisms from $\mathbb{P}_1 \times \mathbb{C}$.

4.2. Lemma. *Let $\psi\colon \mathbb{P}_1 \times \mathbb{C} \to Z$ be a morphism to a variety Z with $\psi(\mathbb{P}_1 \times \{0\}) = z$ for some $z \in Z$. Then ψ is constant on $\mathbb{P}_1 \times \{s\}$ for every $s \in \mathbb{C}$, i.e. ψ does not depend on the first coordinate.*

Proof. Choose an open affine neighbourhood W of z in Z and set $Y := (\mathbb{P}_1 \times \mathbb{C}) \setminus \psi^{-1}(W)$. Consider the projection $\operatorname{pr}\colon \mathbb{P}_1 \times \mathbb{C} \to \mathbb{C}$. Since $\mathbb{P}_1$ is complete, $\operatorname{pr}(Y)$ is closed in $\mathbb{C}$. Moreover, $0 \notin \operatorname{pr}(Y)$. So $W_0 := \mathbb{C} \setminus \operatorname{pr}(Y)$ is an open neighbourhood of 0 in $\mathbb{C}$, and by definition

$$\mathbb{P}_1 \times W_0 = \operatorname{pr}^{-1}(W_0) \subset \psi^{-1}(W).$$

So by restriction we obtain a morphism $\psi\colon \mathbb{P}_1 \times W_0 \to W$. Since we chose W to be affine, ψ maps $\mathbb{P}_1 \times \{s\}$ to a point for every $s \in W_0$. So for continuity reasons, ψ does not depend on the first coordinate. ∎

Now we consider a system of fans $\mathcal{S}$ with convex support, and a morphism $f\colon X_\mathcal{S} \to Z$ to some variety Z. We apply Lemma 4.1 to prove the following:

4.3. Proposition. *Let $\sigma \subset |\mathcal{S}|$ be a rational (not necessarily strictly) convex cone. Suppose that σ° is contained in the support of an f-equivalence class of $\mathcal{C}(\mathcal{S})$. Then for every face τ of σ, the relative interior τ° is also contained in the support of an f-equivalence class.*

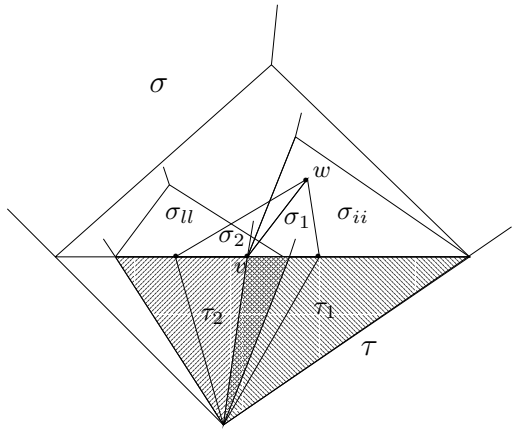
Proof. By induction on $n := \dim \sigma$ we will show that the assertion is true for all one-codimensional faces of σ. If σ is one-dimensional there is nothing to show. So assume that $n \geq 2$, and let τ be a face of σ of dimension $n-1$. Without loss of generality we can assume that $\dim \sigma_{ii} = \dim \sigma$ for all i since the cones of maximal dimension cover σ. We reduce the induction step to proving the following

Claim. *For every cone $\tau_1 \in \mathcal{C}(\mathcal{S} \cap \tau)^{n-1}$ we have $\tau_1 \cap \tau^\circ \subset |\tau_1|_f$.*

From this claim it follows that $|\tau_1|_f \cap \tau^\circ$ is relatively closed in τ°. The $(n-1)$-dimensional cones in $\mathcal{C}(\mathcal{S} \cap \tau)$ cover τ, and we obtain a partition of τ° into relatively closed subsets of the form $|\tau_1|_f \cap \tau^\circ$, where $\tau_1 \in \mathcal{C}(\mathcal{S} \cap \tau)^{n-1}$.

Since τ° is connected this implies that one of the subsets actually equals τ°, and that means that τ° is contained in the support of an f-equivalence class.

So to prove the proposition it suffices to show the claim. Assume the claim were not true. Then since all the cones in question are rational, we can find a rational vector v in the boundary of τ_1 with $v \in \tau^\circ \setminus |\tau_1|_f$. Note that v may be zero if the cone τ is not strictly convex.

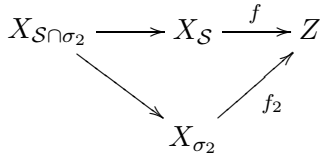


Now choose a vector $v' \in \operatorname{lin}\tau$ such that $v+v' \in \tau_1^\circ$. For sufficiently small ε the ball $B_\varepsilon(v)$ of radius ε in $\operatorname{lin}\tau$ around v is already covered by those cones in $\mathcal{C}(\mathcal{S}\cap\tau)^{n-1}$ that contain v. Therefore if we choose v' of length $\le \varepsilon$, then $v - v' \in \tau_2$ for some $\tau_2 \in \mathcal{C}(\mathcal{S}\cap\tau)^{n-1}$ containing v.

Moreover, for $i = 1, 2$ we can construct $(n-1)$-dimensional rational cones $\tau_i' \subset \tau_i$ such that $v + v' \in (\tau_1')^\circ$, $v - v' \in (\tau_2')^\circ$ and $\tau_1' \cap \tau_2' = \operatorname{cone}(v)$ is a common face of τ_1' and τ_2'. To do that, we first choose a hyperplane through v separating $v+v'$ and $v-v'$, and then choose appropriate simplicial rational cones around $v + v'$ and $v - v'$ respectively that lie entirely on one side of the hyperplane, and then form the convex hull with v.

By assumption there are n-dimensional cones σ_{ii} and σ_{ll} having τ_1 and τ_2 respectively as a face. Now choose $w \in \sigma_{ii}^\circ \cap N$, and set $\sigma_1 := \operatorname{cone}(\tau_1', w)$ and $\sigma_2 := \operatorname{cone}(\tau_2', w)$. Let X_{σ_i} denote the toric variety associated to σ_i in N $(i = 1, 2)$. Since the cone σ_1 is contained in the cone σ_{ii}, the identity on N defines a toric morphism from X_{σ_1} to the affine chart $X_i := X_{\sigma_{ii}}$ of $X_{\mathcal{S}}$. The composition with f yields a morphism f_1 from X_{σ_1} to Z.

The cone σ_2 has the following properties: $\sigma_2 \cap \tau = \tau_2' \subset \tau_j$ and $\sigma_2 \setminus \tau \subset \sigma^\circ$. Therefore the relative interior of every face of σ_2 is contained in an f-equivalence class. Note that σ_2 is strictly convex, since by construction $v' \notin \operatorname{lin} v$. Hence by Corollary 3.8, f defines a morphism f_2 from X_{σ_2} to Z such that the following diagram is commutative:



Now consider the toric variety X_Δ corresponding to the fan with the two maximal cones σ_1 and σ_2 in N. It follows from the above commutative diagram that the two morphisms f_1, f_2 coincide on the intersection of the two maximal affine charts, and so they glue together to a morphism $f'\colon X_\Delta \to Z$. Since $\sigma_1^\circ, \sigma_2^\circ \subset \sigma^\circ$, by assumption we have $\sigma_1 \sim_{f'} (\sigma_1 \cap \sigma_2)$.

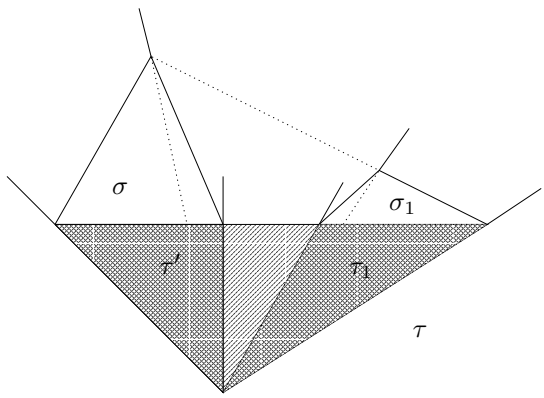
The cones σ_i with the faces τ_i' satisfy all the conditions of Lemma 4.1, and we obtain $\tau_1' \sim_{f'} \varrho$, where $\varrho := \tau_1' \cap \tau_2' = \mathrm{cone}(v)$ and hence $v \in |\tau_1'|_{f'}$. Since $(\tau_1')^\circ \cap \tau_1^\circ \neq \emptyset$, it follows that $v \in |\tau_1|_f$, which is a contradiction. ∎

5. The convex hull of two cones. As in the previous section we consider a system of fans with convex support $|\mathcal{S}|$ in a lattice N. Let $\sigma, \tau \in \mathcal{C}(\mathcal{S})$ be two cones with $\sigma \cap \tau^\circ \neq \emptyset$. Then since we assumed $|\mathcal{S}|$ to be convex, we have $\sigma + \tau \subset |\mathcal{S}|$. Moreover, it follows that $\sigma^\circ \subset (\sigma+\tau)^\circ = \sigma^\circ + \tau^\circ$.

A first observation is the following:

5.1. Lemma. *Let $f\colon X_{\mathcal{S}} \to Z$ be a morphism from the toric prevariety $X_{\mathcal{S}}$ associated to $\mathcal{S}$ to some variety Z. Suppose that σ_1 is a cone in $\mathcal{C}(\mathcal{S})$ with $\sigma_1 \cap \tau^\circ \neq \emptyset$ and $(\sigma_1)^\circ \subset \sigma^\circ + \tau^\circ$. Then $\sigma_1 \sim_f \sigma$.*

Proof. The assumptions on σ_1 imply that there is a face τ_1 of σ_1 with $\tau_1^\circ \cap \tau^\circ \neq \emptyset$. Similarly, there is a face τ' of σ with $(\tau')^\circ \cap \tau^\circ \neq \emptyset$. Therefore $\tau_1 \sim_f \tau \sim_f \tau'$. Now choose $v_1 \in \sigma_1^\circ \cap (\sigma^\circ + \tau^\circ)$, and write v_1 in the form $v_1 = w + v'$, where $w \in \sigma^\circ$, $v' \in \tau^\circ$.



Suppose that $[\sigma, i], [\tau, j], [\sigma_1, k] \in \Omega(\mathcal{S})$ and fix $t \in T$. Since $\tau_1 \sim_f \tau \sim_f \tau'$, for all $s \in \mathbb{C}^*$ the following holds:

$$f(t\cdot\lambda_{v_1}(s)\cdot x_{[\tau_1,k]}) = f(t\cdot\lambda_{v_1}(s)\cdot x_{[\tau,j]}) = f(t\cdot\lambda_w(s)\cdot x_{[\tau,j]})$$
$$= f(t\cdot\lambda_w(s)\cdot x_{[\tau',i]}).$$

In the affine chart $X_{[\sigma_1,k]}$ we have $\lim_{s\to\infty} t\cdot\lambda_{v_1}(s)\cdot x_{[\tau_1,k]} = t\cdot x_{[\sigma_1,k]}$. On the other hand, in the affine chart $X_{[\sigma,i]}$ we have $\lim_{s\to\infty} t\cdot\lambda_w(s)\cdot x_{[\tau',i]} = t\cdot x_{[\sigma,i]}$. That implies $f(t\cdot x_{[\sigma_1,k]}) = f(t\cdot x_{[\sigma,i]})$ for all t, and hence $\sigma\sim_f\sigma_1$. ∎

5.2. Proposition. *Let $\mathcal{S} = (\Delta_{ij})_{i,j\in I}$ be an affine system of fans with convex support, and let σ,τ be cones in $\mathcal{C}(\mathcal{S})$ with $\sigma\cap\tau^\circ\neq\emptyset$. Then either*

(i) *there is a nonzero linear subspace $L\subset\sigma+\tau$ and a cone $\sigma_1\in\mathcal{C}(\mathcal{S})$ such that $L\subset|\sigma_1|_f$ for every morphism $f\colon X_{\mathcal{S}}\to Z$ to a variety Z, or*

(ii) *$(\sigma+\tau)^\circ\subset|\sigma|_f$ for every morphism $f\colon X_{\mathcal{S}}\to Z$ to a variety Z.*

Proof. We argue by induction on the dimension of the support $n := \dim\operatorname{lin}|\mathcal{S}|$ and the number $|\mathcal{C}(\mathcal{S})^n|$ of n-dimensional cones in $\mathcal{C}(\mathcal{S})$. Suppose that the first assertion does not hold. Note that this implies that the analogous assertion also does not hold for any system of the form $\mathcal{S}\cap\delta$ obtained by intersecting $\mathcal{S}$ with a cone $\delta\subset|\mathcal{S}|$.

So without loss of generality we may assume that $|\mathcal{S}| = \sigma+\tau$. Moreover, it suffices to show assertion (ii) under the additional assumption

$$\sigma+\tau = \varrho+\tau \quad \text{for some ray } \varrho \text{ of } \sigma. \tag{1}$$

To see this, suppose that (ii) is true whenever the extra condition (1) holds. Choose a ray $\varrho\in\sigma\setminus\tau$ and consider the cone $\sigma' := \varrho+(\sigma\cap\tau)$. Our assumption implies that $(\sigma'+\tau)^\circ\subset|\sigma|_f$ for every morphism $f\colon X_{\mathcal{S}}\to Z$ to a variety Z. And if $v\in\sigma\cap\tau^\circ$, then any point $w\in\operatorname{cone}(\varrho,v)^\circ\subset\sigma'$ is contained in the relative interior of $\tau+\varrho$. Therefore we can replace τ in $\mathcal{C}(S)$ by $\tau+\varrho$, glued to any other cone along the origin only. By recursion over the rays we obtain the claim.

Now we further reduce the situation to the special case that

$$\dim(\sigma\cap\tau) = \dim\tau. \tag{2}$$

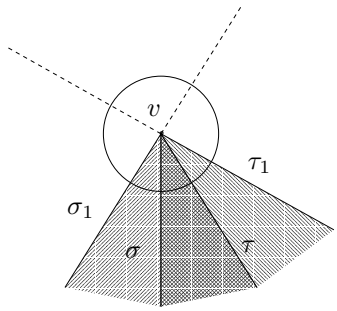
Suppose that assertion (ii) always holds if (2) is true. Consider a pair of cones σ,τ as in the proposition and satisfying condition (1). Then $\dim\tau = n$ or $n-1$, where $n = \dim|\mathcal{S}|$. Choose a point $v_1\in\sigma^\circ$. Since $|\mathcal{S}| = \sigma+\tau$, we can find an n-dimensional cone $\sigma_1\in\mathcal{C}(\mathcal{S})$ containing v_1 such that condition (2) holds for σ_1 and σ. So using (ii) we may replace σ in $\mathcal{S}$ by $\sigma_1+\sigma$. In other words, we can assume that $\dim\sigma = n$.

If also τ is n-dimensional we obtain $\tau^\circ\cap\sigma^\circ\neq\emptyset$, and that implies condition (2) for σ and τ. Otherwise τ must be a facet of $|\mathcal{S}|$. In that case consider a point $v\in\sigma\cap\tau^\circ$. As above we can find an n-dimensional cone $\tau_1\in\mathcal{C}(\mathcal{S})$ containing v such that condition (2) holds for τ_1 and τ. So using (ii) we may assume that τ_1 contains τ as a facet. From the fact that $\tau_1\subset\varrho+\tau$ we

can conclude that $\mathrm{cone}(v,\varrho)$ meets τ_1 in its relative interior. That implies $\sigma \cap \tau_1^\circ \neq \emptyset$, and we can replace τ by τ_1. Since $\dim \tau_1 = n$, condition (2) follows as before.

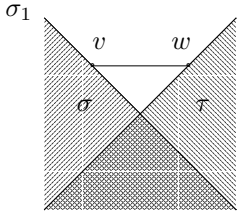
From now on conditions (1) and (2) are assumed to be satisfied. So we are left with two possibilities: either $\dim \tau = \dim \sigma$ or $\dim \tau = \dim \sigma - 1$.

Let us first consider the case $\dim \tau = \dim \sigma$. If $\sigma \cup \tau$ is convex then there is nothing to show, since then we have $(\sigma \cup \tau)^\circ = \sigma^\circ \cup \tau^\circ \subset |\sigma|_f$.



(For consider $v \in \partial\sigma \cap \partial\tau$. Then there are facets $\sigma_1 \prec \sigma$ and $\tau_1 \prec \tau$ with $v \in \sigma_1 \cap \tau_1$. Choose defining hyperplanes $u \in \sigma^\vee$ and $w \in \tau^\vee$ with $u^\perp \cap \sigma = \sigma_1$ and $w^\perp \cap \tau = \tau_1$. Here $\sigma^\vee$ and $\tau^\vee$ denote the dual cones of σ and τ respectively. In a ball of sufficiently small radius around v in $\mathrm{lin}\,\sigma$ we find a point v' with $u(v') < 0$ and $w(v') < 0$ and hence $v' \notin \tau \cup \sigma$. This shows that v cannot lie in the relative interior of $\sigma \cup \tau$.)

So assume that $\sigma \cup \tau$ is not convex. Then there is a facet $\sigma_1 \prec \sigma$ defined by a hyperplane $u \in \sigma^\vee$ such that $u^\perp \cap \tau^\circ \neq \emptyset$ and $\sigma_1 \not\subset \tau$.

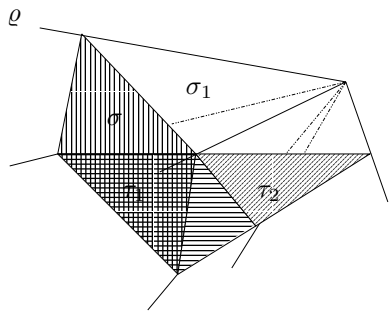


(To see this, choose points $v \in \partial\sigma$ and $w \in \partial\tau$ such that the segment $[v,w]$ joining v and w intersects $\sigma \cup \tau$ only in $\{v,w\}$. The point v lies on a facet $\sigma_1 \prec \sigma$, and since $w \notin \sigma$, we can choose a defining hyperplane $u \in \sigma^\vee$ of σ_1 such that $u(w) < 0$. On the other hand, $u(\tau^\circ \cap \sigma^\circ) > 0$, and so $u^\perp \cap \tau^\circ \neq \emptyset$.)

Now we decompose $\sigma + \tau$ along the defining hyperplane $u^\perp$ of σ_1. Since $\sigma_1 \not\subset \tau$ and $\sigma = \varrho + (\sigma \cap \tau)$, we have $\varrho \subset \sigma_1$ and therefore $\sigma + \tau = \sigma_1 + \tau$. Set

$$\tau_0 := \tau \cap u^\perp, \qquad \tau_1 := \{v \in \tau;\, u(v) \geq 0\}, \qquad \tau_2 := \{v \in \tau;\, u(v) \leq 0\}.$$

Then τ_1 and τ_2 are cones that intersect in the common face τ_0. Moreover, since $\sigma + \tau = \sigma_1 + \tau$, we have $\sigma + \tau = (\sigma_1 + \tau_1) \cup (\sigma_1 + \tau_2) = (\sigma + \tau_1) \cup (\sigma_1 + \tau_2)$.



Note that σ and τ_1 are again both n-dimensional cones, whose intersection is also n-dimensional. So in particular, $\sigma \cap \tau_1^\circ \neq \emptyset$. Moreover, $|\mathcal{C}(\mathcal{S})^n| \geq |\mathcal{C}(\mathcal{S} \cap (\sigma + \tau_1))^n|$. In fact, we can reduce the problem to considering the pair of cones σ and τ_1 in $\mathcal{S} \cap (\sigma + \tau_1)$.

Because if $(\sigma + \tau_1)^\circ$ is contained in the support of some f-equivalence class, then it follows from Proposition 4.3 that the same is true for its face $\tau' := \sigma_1 + \tau_0 \subset u^\perp$. This implies $(\sigma_1 + \tau_0)^\circ \subset |\sigma_1|_f$, and so by our general assumption, τ' must be strictly convex. It follows from our choice of u that $\emptyset \neq u^\perp \cap \tau^\circ \subset \tau_0^\circ \subset (\tau')^\circ$, and therefore $\sigma_1 \sim_f \tau$. Consider the system of fans $\mathcal{S}'$ obtained from $\mathcal{S} \cap (\sigma_1 + \tau_2)$ by adding the fan of faces of τ', and glueing τ' to the other cones along the zero cone. Since σ lies on the other side of the hyperplane $u^\perp$, we have $|\mathcal{C}(\mathcal{S}')^n| < |\mathcal{C}(\mathcal{S})^n|$. We also know that $\tau_2 \cap (\tau')^\circ \neq \emptyset$. It follows by induction that $(\tau_2 + \sigma_1)^\circ = (\tau_2 + \tau')^\circ \subset |\tau|_f$. Therefore it suffices to show the claim for the pair of cones σ and τ_1 in $\mathcal{S} \cap (\sigma + \tau_1)$.

If $\sigma \cup \tau_1$ is not convex, then we can again decompose $\sigma + \tau_1$ along a defining hyperplane meeting τ_1° and reduce the problem to considering the cone σ together with a smaller cone τ_1' as above. After repeating this procedure a finite number of times, we arrive at a pair of cones such that their union is convex.

Now let us consider the second case, namely that $\dim \tau = n - 1$ and $\dim(\tau \cap \sigma) = n - 1$, and suppose that $\tau \not\subset \sigma$. If there is a cone $\sigma' \in \mathcal{C}(\mathcal{S})^n$ with $(\sigma')^\circ \cap \sigma^\circ \neq \emptyset$ and $\sigma' \not\subset \sigma$, then we conclude, by the assertion of the first case applied to the pair of cones σ, σ' in $\mathcal{S} \cap (\sigma + \sigma')$, that $(\sigma + \sigma')^\circ \subset |\sigma|_f$. It follows again from the general assumption that $\sigma'' := \sigma + \sigma'$ is strictly convex. By replacing the two fans of faces of σ and σ' respectively in $\mathcal{S}$ by the single fan of faces of σ'', where σ'' is glued to all the other cones only along the zero cone, we obtain a system of fans $\mathcal{S}'$ that contains strictly fewer n-dimensional cones than $\mathcal{S}$. So by induction the claim follows.

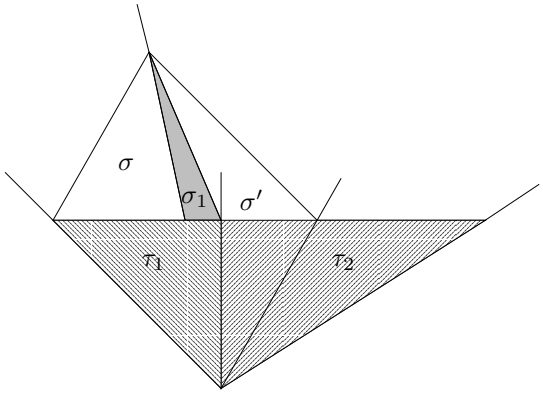
From now on, assume that for every cone $\sigma' \in \mathcal{C}(\mathcal{S})^n$ that is not contained in σ, we have $\sigma' \cap \sigma^\circ = \emptyset$. By assumption, the intersection $\sigma \cap \tau$ contains a point of τ°. Therefore we can find a facet σ_1 of σ with $\sigma \cap \tau^\circ \neq \emptyset$ and

$\sigma_1 \not\subset \tau$. (To see this, note that $\sigma \cap \tau \subsetneq \tau$. So there is a facet $\varrho_1 \prec \sigma \cap \tau$ with $\varrho_1^\circ \subset \tau^\circ$. Since $\sigma = \varrho + (\sigma \cap \tau)$, where $\varrho \not\subset \mathrm{lin}(\sigma \cap \tau)$, $\sigma_1 := \varrho + \varrho_1$ is a facet of σ with the desired properties.)

As in the previous case, we decompose $\sigma + \tau$ along a defining hyperplane of σ_1. We set

$$\tau_0 := \tau \cap u^\perp, \quad \tau_1 := \{v \in \tau;\, u(v) \geq 0\}, \quad \tau_2 := \{v \in \tau;\, u(v) \leq 0\},$$

and observe that τ_1 and τ_2 are cones intersecting in the common face τ_0. And, since $\sigma + \tau = \sigma_1 + \tau$, we again have $\sigma + \tau = (\sigma_1 + \tau_1) \cup (\sigma_1 + \tau_2) = (\sigma + \tau_1) \cup (\sigma_1 + \tau_2)$.



Moreover, since the n-dimensional cones in $\mathcal{C}(\mathcal{S})$ cover σ_1, we can choose a cone $\sigma' \in \mathcal{C}(\mathcal{S})^n$ through a point $v \in \sigma_1 \cap \tau^\circ$ such that $\dim(\sigma_1 \cap \sigma') = n-1$. Because of our assumptions on the position of the n-dimensional cones relative to σ, that implies in particular that $\sigma' \subset \sigma_1 + \tau_2$.

One consequence is that $|\mathcal{C}(\mathcal{S} \cap (\sigma + \tau_1))^n| < |\mathcal{C}(\mathcal{S})^n|$. So by induction we have $(\sigma + \tau_1)^\circ \subset |\sigma|_f$. Furthermore, the face $\sigma_1 + \tau_0$ of $\sigma + \tau_1$ must be contained in an f-equivalence class. Since $\sigma_1 \cap \tau_0^\circ \neq \emptyset$, that means $(\sigma_1 + \tau_0)^\circ \subset |\sigma_1|_f$. By Lemma 5.1, $\sigma_1 \sim_f \sigma$, and therefore in fact $(\sigma_1 + \tau_0)^\circ \subset |\sigma|_f$.

Now consider the system of fans $\mathcal{S}'$ obtained from $\mathcal{S} \cap (\sigma_1 + \sigma')$ by adding the fan of faces of $\tau' := \sigma_1 + \tau_0$ and glueing τ' to the other cones along the zero cone. Since $\sigma' \cap \sigma_1^\circ \neq \emptyset$ and $\dim \sigma_1 = \dim \tau$, we have $\sigma' \cap (\tau')^\circ \neq \emptyset$. Because $|\mathcal{C}(\mathcal{S}')^n| < |\mathcal{C}(\mathcal{S})^n|$, we conclude by induction that $(\tau' + \sigma')^\circ \subset |\sigma'|_f = |\sigma|_f$. Here the equality is again a consequence of Lemma 5.1. Altogether we find that the relative interior of the convex cone

$$\sigma'' := (\sigma + \tau_1) \cup (\tau' + \sigma')$$

is contained in $|\sigma|_f$. Replacing the fans of faces of σ and σ' in $\mathcal{S}$ as above by the fan of faces of σ'', glued to the other cones along zero, we end up with a system of fans $\mathcal{S}''$ that has strictly fewer n-dimensional cones than $\mathcal{S}$. So by induction, applied to the pair of cones σ'' and τ, the claim follows. This ends the proof. ∎

6. Weakly proper quotients. In this section we prove the main result of this paper, namely that if the toric quotient of a toric variety with respect to a subtorus action is weakly proper and the quotient variety is of expected dimension, then the toric quotient is even categorical.

In order to simplify our arguments, we work in the more general context of toric prevarieties and obtain the announced result as a corollary of a statement in this context.

First we note that the construction of toric quotients given in [AC;Ha1] also proves the existence of a separated quotient in the toric category for a subtorus action on a toric prevariety. More precisely, we have the following:

6.1. PROPOSITION. *Let X be a toric prevariety with big torus T, and let H be a subtorus of T. Then there is an H-invariant toric morphism $q\colon X \to Y$ to some separated toric variety Y such that every H-invariant toric morphism from X to a toric variety factors uniquely through q.*

If X is separated then q is precisely the toric quotient of X by H. If X is not separated, we call the morphism q the *separated toric quotient* of X by H.

Proof. This statement follows directly from the existence of a quotient fan of a system of cones proved in [AC;Ha1]. Suppose that X arises from an affine system of fans $\mathcal{S}$ in a lattice N, and let L be the primitive sublattice of N corresponding to H. The set $\mathcal{C}(\mathcal{S})$ of cones of $\mathcal{S}$ satisfies the definition of a system of N-cones given in [AC;Ha1]. Therefore by Theorem 2.3 of [AC;Ha1] there is a well defined fan Δ in a lattice $\widetilde{N}$, the so-called *quotient fan* of $\mathcal{C}(\mathcal{S})$ by L, with the following universal property: Whenever we have a fan Δ' in a lattice N' and a lattice homomorphism $F\colon N \to N'$ with $L \subset \ker(F)$, such that every cone of $\mathcal{C}(\mathcal{S})$ is mapped into a cone of Δ', then there is a unique map of fans from Δ to Δ' such that the diagram commutes. If we translate this property back into the language of toric prevarieties, we get exactly the universal property stated in the proposition. ∎

As we have just proved, a separated toric quotient for a subtorus action on a toric prevariety always exists. Moreover, there is even an algorithm to construct this quotient from the combinatorial data associated to the toric prevariety and the subtorus.

For the description of the algorithm we need to specify the combinatorial data. So let $\mathcal{S}$ be the system of fans in a lattice N, associated to a given toric prevariety. Let $L \subset N$ be the primitive sublattice corresponding to a given subtorus H. Then the separated toric quotient of X by H is of the form $q\colon X_{\mathcal{S}} \to X_{\Delta}$, where Δ is the above-mentioned *quotient fan* of the set of cones of $\mathcal{S}$ by the sublattice L.

The construction of the quotient fan Δ consists of the following steps (see [AC;Ha1]):

Initialization: Let $P\colon N \to N/L$ denote the projection. Let S_1 denote the set of maximal elements of $\{P_{\mathbb{R}}(\sigma);\ \sigma \in \mathcal{C}(\mathcal{S})\}$ with respect to inclusion of sets.

Loop, Step $l > 1$: If possible choose $\tau, \tau' \in S_{l-1}$ such that $\tau \cap \tau'$ is not a face of τ' and let $\varrho' \prec \tau'$ denote the smallest face containing $\tau \cap \tau'$. Now let S_l be the set of maximal elements of $S_{l-1} \cup \{\tau + \varrho'\}$. Otherwise set $n := l-1$ and stop the procedure.

Output: Let Σ be the set of all faces of the cones of S_n. By construction, the set Σ is a *quasi-fan*, i.e. the cones in Σ may not be strictly convex, but otherwise all the axioms of a fan are satisfied. We call Σ the quotient quasi-fan of $\mathcal{C}(\mathcal{S})$ by L.

Final step: The minimal element $V(\Sigma)$ of Σ is a linear subspace of $(N/L)_{\mathbb{R}}$, and $V(\Sigma) \cap N/L = L'$ is a sublattice. Let $P'\colon N/L \to (N/L)/L'$ denote the projection. Then the desired quotient fan is the fan $\Delta := \{P'_{\mathbb{R}}(\tau);\ \tau \in \Sigma\}$ in the quotient lattice $(N/L)/L'$ of N/L.

Now we can state our main theorem in the context of toric prevarieties.

6.2. Theorem. *Let $X = X_{\mathcal{S}}$ be a toric prevariety arising from an affine system of fans $\mathcal{S}$ in a lattice N, and let H be the subtorus corresponding to a given primitive sublattice $L \subset N$. Let $P\colon N \to N/L$ denote the projection and let Σ be the quotient quasi-fan of the set of cones of $\mathcal{S}$ by L introduced above. If we assume $P(|\mathcal{S}|) = |\Sigma|$, then the separated toric quotient q of X with respect to H has the following universal property* (CA): *Every H-invariant morphism from X to a variety factors uniquely through q.*

In particular, if X is separated and hence arises from a fan Δ_1 in N, this theorem says:

6.3. Corollary. *If $P(|\Delta_1|) = |\Sigma|$, then the toric quotient of X by H is a quotient in the category of algebraic varieties.*

Our main applications of Theorem 6.2 are the following two corollaries.

6.4. Corollary. *For a toric variety X and a subtorus H of the big torus T, let $p\colon X \to Y$ denote the toric quotient. If p is weakly proper and the quotient space Y is of expected dimension, i.e.* $\dim Y = \dim(T/H)$, *then the toric quotient is a quotient in the category of algebraic varieties.*

Proof. Suppose that X arises from a fan Δ_1 in a lattice N. Since $\dim Y = \dim(T/H) = \dim(N/L)_{\mathbb{R}}$, the quotient fan Δ of Δ_1 by the lattice L corresponding to H lives in the space N/L. That means that the quotient quasi-fan Σ in this case is already a fan and the final step in the above algorithm is not necessary. So here Σ and Δ coincide.

Moreover, the projection $P\colon N \to N/L$ is the lattice homomorphism associated to the toric quotient p. So if p is weakly proper, then by Proposition 2.1, we have $P(|\Delta_1|) = |\Delta|$, and Theorem 6.2 applies. ∎

6.5. Corollary. *Let X be a toric variety corresponding to a fan with convex support. That means that there is a proper toric morphism from X onto an affine toric variety. Then the toric quotient of X by any subtorus of the big torus is always a quotient in the category of algebraic varieties.*

Proof. Suppose that X arises from a fan Δ_1 with convex support. Then any projection from Δ_1 to a quotient quasi-fan is automatically surjective, since the cones of any quotient quasi-fan are obtained by successively forming convex hulls. ∎

Now we come to the proof of the main theorem.

Proof of Theorem 6.2. First we reduce the situation to a special case. Denote by $\widehat{H}$ the largest subtorus of T such that every H-invariant morphism from X to some variety is in fact $\widehat{H}$-invariant. Then in particular, the separated toric quotient q is $\widehat{H}$-invariant, and that means that $H \subset \widehat{H} \subset \ker(q)$. Here, as before, $\ker(q) \subset T$ denotes the kernel of the homomorphism of tori induced by q.

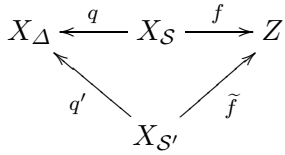
Note that the separated toric quotients with respect to H and $\widehat{H}$ coincide. Moreover, the assumption $P(|\mathcal{S}|) = |\Sigma|$ implies that $\widehat{P}(|\mathcal{S}|) = |\widehat{\Sigma}|$, where $\widehat{P}$ denotes the projection modulo the lattice $\widehat{L}$ corresponding to $\widehat{H}$, and $\widehat{\Sigma}$ denotes the quotient quasi-fan of $\mathcal{C}(S)$ by $\widehat{L}$. So for this proof we can assume that $H = \widehat{H}$.

We claim that moreover we can assume $H = 1$ or equivalently $L = 0$. To see this, let $f\colon X_{\mathcal{S}} \to Z$ be an H-invariant morphism to some variety Z. Consider the algebraic quotients of the affine charts $p_i\colon X_{\sigma_{ii}} \to X_{\sigma_{ii}}//H$. Since any good quotient is categorical, the restriction of f to $X_{\sigma_{ii}}$ factors uniquely through p_i. Therefore we obtain morphisms $\widetilde{f_i}\colon X_{\sigma_{ii}}//H \to Z$ with $f_i = \widetilde{f_i} \circ p_i$.

In particular, f is $\ker(p_i)$-invariant for every i, and therefore by assumption $\ker(p_i) = H$ for all i. That means that $X_{\sigma_{ii}}//H$ is the affine toric variety defined by the cone $\tau_i := P_{\mathbb{R}}(\sigma_{ii})$ in $N' := N/L$, and all the cones τ_i are strictly convex.

If we glue all the affine charts X_{τ_i} along the open orbit $T' = T/H$, we obtain a toric prevariety X' that in some sense is a first non-separated approximation of the quotient variety X_Δ. The prevariety X' corresponds to the system of fans $\mathcal{S}' = (\Delta'_{ij})_{i,j\in I}$ in N', where Δ'_{ii} is the fan of faces of τ_i and $\Delta'_{ij} = \{0\}$ for all $i \neq j$. We can view the quotient fan Δ of $\mathcal{C}(\mathcal{S})$ by L as the quotient fan of $\mathcal{C}(\mathcal{S}')$ by the zero lattice. Let $q'\colon X_{\mathcal{S}'} \to X_\Delta$ denote the separated toric quotient of $X' = X_{\mathcal{S}'}$ by the trivial group.

The morphisms $\widetilde{f}_i$ coincide on the open orbit T', and hence they glue together to a morphism $\widetilde{f}\colon X_{\mathcal{S}'} \to Z$. Since for every i, the morphism f agrees with $\widetilde{f}\circ p_i$ on the affine chart $X_{\sigma_{ii}}$, we conclude that f factors through q if and only if $\widetilde{f}$ factors through q':



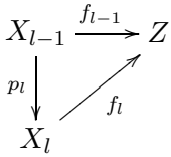
Therefore from now on we may assume that $H = \widehat{H} = 1$. In this case the condition in Theorem 6.2 says that $|\mathcal{S}| = |\Sigma|$.

Let $f\colon X_\mathcal{S} \to Z$ be a morphism to a variety. Let us have a closer look at the algorithmic construction of the quotient fan Δ of $\mathcal{C}(S)$ by the zero lattice. The first set S_1 simply consists of the maximal cones in $\mathcal{C}(S)$. Let X_1 denote the toric prevariety obtained from the maximal affine charts of $X_\mathcal{S}$, but now only glued along the open orbit. The identity on the affine charts induces a toric morphism $p_1\colon X_1 \to X_\mathcal{S}$. Set $f_1 := f \circ p_1\colon X_1 \to Z$.

By induction on the index l counting the steps in the algorithm we now prove the following for $l > 1$:

(i) All the cones in S_l are strictly convex. Hence we may define a toric prevariety X_l from the affine toric varieties corresponding to the cones in S_l by glueing them along the open orbit T.

(ii) There is a morphism $f_l\colon X_l \to Z$ such that we have a commutative diagram



where p_l denotes the toric morphism induced by the identity on N.

So let $l > 1$, and assume that the induction hypothesis holds for all previous steps. Suppose that S_l was obtained from S_{l-1} by replacing a cone $\tau \in S_{l-1}$ by $\widetilde{\tau} = \tau + \varrho'$, where $\varrho' \prec \tau' \in S_{l-1}$ is such that $(\tau \cap \tau')^\circ \subsetneq (\varrho')^\circ$. Then, in particular, $\tau \cap (\varrho')^\circ \neq \emptyset$.

Let $\mathcal{S}_l$ denote the system of fans associated to X_l. By the general assumption, $|\mathcal{S}_l| = |\mathcal{S}_{l-1}|$ and hence $\widetilde{\tau} \subset |\mathcal{S}_{l-1}|$. Since we assumed that $\widehat{H} = 1$, we can apply Proposition 5.2 to $\mathcal{S}_{l-1} \cap \widetilde{\tau}$ to conclude that $\widetilde{\tau}^\circ = \tau^\circ + (\varrho')^\circ \subset |\tau|_{f_{l-1}}$.

So by Proposition 4.3, the relative interior of any face of $\widetilde{\tau}$ is contained in some f_{l-1}-equivalence class. If $\widetilde{\tau}$ contained a linear subspace, then this would imply that f_{l-1} for any choice of f would be invariant with respect

to the corresponding subtorus, and that contradicts the assumption that $\widehat{H} = 1$.

Therefore in fact $\widetilde{\tau}$ must be strictly convex, and f_{l-1} by Corollary 3.8 defines a morphism on $X_{\widetilde{\tau}}$. Hence we can extend f_{l-1} to a morphism $f_l\colon X_{\mathcal{S}_l} \to Z$ as desired. This proves the induction claims (i) and (ii).

It remains to show that the morphism $f_n\colon X_{\mathcal{S}_n} \to Z$ corresponding to the last step of the loop factors through the quotient variety X_{Δ}.

As we have just seen, in our special case the cones in S_n are strictly convex. Therefore S_n is in fact the set of maximal cones of the quotient fan Δ. The toric prevariety $X_{\mathcal{S}_n}$ is obtained from the affine charts X_τ, where $\tau \in S_n$, by glueing them along the open orbit T. The morphism $f_n\colon X_{\mathcal{S}_n} \to Z$ induces morphisms $f_\tau\colon X_\tau \to Z$ that coincide on the open dense orbit T.

We can also view the X_τ as affine charts of X_Δ. But in this variety, for any pair $\tau, \tau' \in S_n$, the charts X_τ and $X_{\tau'}$ are glued along the common subset $X_{\tau\cap\tau'}$. Since Z is separated, and the morphisms f_τ and $f_{\tau'}$ agree on the open dense orbit, they even agree on $X_{\tau\cap\tau'}$. So they fit together to a morphism $\widetilde{f}\colon X_\Delta \to Z$ and $f_n = \widetilde{f} \circ p$, where $p\colon X_{\mathcal{S}_n} \to X_\Delta$ is the toric morphism induced by the identity on N. This ends the proof. ∎

As a corollary, we also obtain the following statement about actions of subtori of small codimension that was proved in [AC;Ha3] as Theorem 4.1.

6.6. Proposition. *Let $H \subset T$ be a subtorus of codimension ≤ 2. Then for any separated toric quotient of a toric prevariety with big torus T with respect to H the universal property* (CA) *holds.*

Proof. Let $p\colon X \to Y$ denote the separated toric quotient of a toric prevariety X by H. If $\dim T/H = 1$, then the conditions of Theorem 6.2 are automatically satisfied for p, and therefore (CA) holds.

Now assume that $\dim T/H = 2$. Then the projections of the maximal cones of Δ in N/L are at most 2-dimensional, and therefore the convex hull of two overlapping projections either equals their union or equals the whole plane $N_{\mathbb{R}}/L_{\mathbb{R}}$. So if the variety Y is 2-dimensional, then the algorithm for constructing the quotient fan shows that here p also automatically satisfies the conditions of the theorem.

If $\dim Y \leq 1$, then among the cones of the given toric prevariety $X = X_{\mathcal{S}}$ there must be a chain of cones $\sigma_1, \ldots, \sigma_r$ such that $P_{\mathbb{R}}(\sigma_i^\circ) \cap P_{\mathbb{R}}(\sigma_{i+1}^\circ) \neq \emptyset$, $\tau := \bigcup_{i=1}^{r-1} P_{\mathbb{R}}(\sigma_i)$ is strictly convex, and $\tau \cup P_{\mathbb{R}}(\sigma_r)$ contains a one-dimensional linear subspace L'. Assume that $[\sigma_i, i] \in \Omega(\mathcal{S})$ and let $X_i := X_{[\sigma_i,i]}$ denote the corresponding affine chart in X. Let $X' := \bigcup_{i=1}^{r-1} X_i \subset X$ denote the open toric subprevariety of X. The toric morphism $p'\colon X' \to X_\tau$ is weakly proper and hence (CA) holds for p'.

Now let $f\colon X \to Z$ be an H-invariant morphism. Then $f|_{X'}$ factors uniquely through p', i.e. there is a unique morphism $\widetilde{f_1}\colon X_\tau \to Z$ such that $f = \widetilde{f_1} \circ p'$ on X'. The toric quotient $p''\colon X_{\sigma_r} \to X_{\tau_r}$ (where $\tau_r := P_{\mathbb{R}}(\sigma_r)$) is even an affine quotient and hence categorical. So we similarly obtain a unique morphism $\widetilde{f_2}\colon X_{\tau_r} \to Z$ such that $f = \widetilde{f_2} \circ p''$ on X_{σ_r}. The morphisms $\widetilde{f_1}$ and $\widetilde{f_2}$ glue together to a morphism $\widetilde{f}\colon X_{\mathcal{S}'} \to Z$, where $\mathcal{S}'$ denotes the affine system of fans obtained from τ and τ_r as maximal cones by glueing them only along the zero cone. We have $f = \widetilde{f} \circ q$, where $q\colon X' \cup X_{\sigma_r} \to X_{\mathcal{S}'}$ denotes the natural toric morphism.

Since $\tau^\circ \cap \tau_r^\circ \neq \emptyset$, we conclude from Proposition 5.2 that either $(\tau+\tau_r)^\circ \subset |\tau|_{\widetilde{f}}$ or there exists a non-trivial linear subspace $L \subset \tau + \tau_r$ that is contained in the support of an $\widetilde{f}$-equivalence class. Since $L' \subset \tau + \tau_r$, in any case we have a one-dimensional linear subspace contained in the support of an $\widetilde{f}$-equivalence class, and $\widetilde{f}$ is invariant with respect to the associated subtorus H' of the big torus T' of $X_{\mathcal{S}}$. So f is invariant with respect to $q^{-1}(H')$. That shows that in fact p is the separated toric quotient of X with respect to a torus of codimension at most 1, and we are back in the first case. ∎

7. Example. In this section we give an example of a $\mathbb{C}^*$-action on a 4-dimensional toric variety with 3-dimensional toric quotient space, such that the toric quotient $p\colon X \to Y$ is categorical, but not uniform in the sense of Mumford, i.e. there is a saturated T-stable open subset U of X such that the restriction of p to U is not the categorical quotient of U. Note that [AC;Ha3] also contains an example of a categorical quotient that is not uniform. But in that example the dimension of the quotient space is not maximal but strictly less than $\dim(T/H)$.

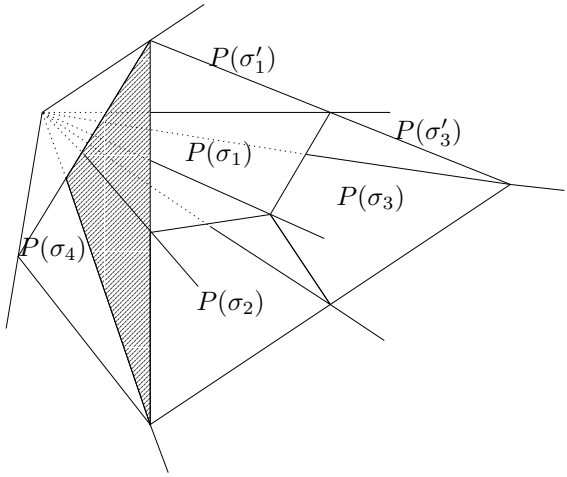
Let $X = X_\Delta$ be the 4-dimensional toric variety associated to the fan Δ in $\mathbb{Z}^4$ with the following four maximal cones:

$$\begin{aligned}
\sigma_1 &:= \operatorname{cone}(e_3, e_2+e_3, e_1+e_2+e_3, e_1+e_3),\\
\sigma_2 &:= \operatorname{cone}(e_1, e_1+e_3, e_1+e_2+e_3, e_1+e_2),\\
\sigma_3 &:= \operatorname{cone}(e_2, e_1+e_2, e_1+e_2+e_3, e_2+e_3),\\
\sigma_4 &:= \operatorname{cone}(e_1, e_1+e_3-e_2, e_3, e_4).
\end{aligned}$$

Let $P\colon \mathbb{Z}^4 \to \mathbb{Z}^3$ denote the linear map given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then the diagram below shows the images of the maximal cones in $\mathbb{Z}^3$. Note that $P(|\Delta|)$ in this case is a convex cone.



Let Y denote the toric variety associated to the fan in $\mathbb{Z}^3$ with the two maximal cones $\tau_1 := \bigcup_{i=1}^3 P_{\mathbb{R}}(\sigma_i)$ and $\tau_2 := P_{\mathbb{R}}(\sigma_4)$. The lattice homomorphism P defines a toric morphism $p\colon X \to Y$, and in fact p is the toric quotient of X by the action of the one-dimensional subtorus H corresponding to the kernel of P, which is generated by $(1,0,1,-1)^T$ in $(\mathbb{C}^*)^4$.

Since p is weakly proper and $\dim Y = \dim X - 1$, the toric quotient in this case is even categorical. However, the quotient does not have the base-change property. Consider the open subset $U := X_{\sigma_1'} \cup X_{\sigma_3'}$ of X consisting of the two affine charts corresponding to the cones $\sigma_1' = \operatorname{cone}(e_3, e_2 + e_3) \prec \sigma_1$ and $\sigma_3' := \operatorname{cone}(e_2 + e_3, e_2) \prec \sigma_3$. This subset is saturated, and the image under p is the affine toric subvariety of Y corresponding to the cone generated by e_2, e_3 in $\mathbb{Z}^3$.

On the other hand, the toric variety U admits a good quotient, and its quotient space is $U/\!/H = X_{P(\sigma_1')} \cup X_{P(\sigma_3')}$. So in particular, $U/\!/H$ is not affine. Therefore the restriction of p to U cannot be the categorical quotient of U by H.

REFERENCES

[AC;Ha1] A. A'Campo-Neuen and J. Hausen, *Quotients of toric varieties by the action of a subtorus*, Tôhoku Math. J. 51 (1999), 1–12.

[AC;Ha2] —, —, *Toric prevarieties and subtorus actions*, Geom. Dedicata 87 (2001), 35–64.

[AC;Ha3] —, —, *Examples and counterexamples for existence of categorical quotients*, J. Algebra 231 (2000), 67–85.

[AC;Ha4] —, —, *Toric invariant theory*, in: Algebraic Group Actions and Quotients (Wykno, 2000), J. Wiśniewski (ed.), Hindawi, 2004, 1–37.

[Bo] A. Borel, *Linear Algebraic Groups*, 2nd ed., Springer, New York, 1991.

[Ko] J. Kollár, *Quotient spaces modulo algebraic groups*, Ann. of Math. 145 (1997), 33–79.

[Mu] D. Mumford, *Geometric Invariant Theory*, Springer, Berlin, 1965.

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