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REGULAR STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES

BY

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Abstract. The concepts of statistical convergence of single and double sequences of complex numbers were introduced in [1] and [7], respectively. In this paper, we introduce the concept indicated in the title. A double sequence $\{x_{jk} : (j,k) \in \mathbb{N}^2\}$ is said to be regularly statistically convergent if (i) the double sequence $\{x_{jk}\}$ is statistically convergent to some $\xi \in \mathbb{C}$, (ii) the single sequence $\{x_{jk} : k \in \mathbb{N}\}$ is statistically convergent to some $\xi_j \in \mathbb{C}$ for each fixed $j \in \mathbb{N} \setminus \mathcal{S}_1$, (iii) the single sequence $\{x_{jk} : j \in \mathbb{N}\}$ is statistically convergent to some $\eta_k \in \mathbb{C}$ for each fixed $k \in \mathbb{N} \setminus \mathcal{S}_2$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are subsets of $\mathbb{N}$ whose natural density is zero. We prove that under conditions (i)–(iii), both $\{\xi_j\}$ and $\{\eta_k\}$ are statistically convergent to ξ. As an application, we prove that if $f \in L\log^+ L(\mathbb{T}^2)$, then the rectangular partial sums of its double Fourier series are regularly statistically convergent to $f(u,v)$ at almost every point $(u,v) \in \mathbb{T}^2$. Furthermore, if $f \in C(\mathbb{T}^2)$, then the regular statistical convergence of the rectangular partial sums of its double Fourier series holds uniformly on $\mathbb{T}^2$.

1. Regular convergence of double sequences. Let $\{x_{jk} : (j,k) \in \mathbb{N}^2\}$, $\mathbb{N} := \{0,1,2,\ldots\}$, be a double sequence of complex numbers. We recall that $\{x_{jk}\}$ is said to be *convergent in Pringsheim's sense* to some complex number ξ if for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$|x_{jk} - \xi| < \varepsilon \quad \text{if } \min\{j,k\} > n_0.$$

If this is the case, we write $\lim_{j,k\to\infty} x_{jk} = \xi$.

We present three examples which illustrate the weaknesses of the concept of convergence in Pringsheim's sense.

EXAMPLE 1. Let

$$x_{jk} := \begin{cases} \max\{j,k\} & \text{if } \min\{j,k\} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

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Clearly,

$$\lim_{j,k\to\infty} x_{jk} = 0, \qquad \sup_{(j,k)\in\mathbb{N}^2} |x_{jk}| = \infty.$$

This example shows that the convergence of a double sequence in Pringsheim's sense does not imply the boundedness of its terms.

Example 2. Let

$$x_{jk} := \frac{1}{2+\min\{j,k\}+(-1)^{j+k}}, \qquad (j,k)\in\mathbb{N}^2.$$

Clearly, $\lim_{j,k\to\infty} x_{jk} = 0$, but neither the single sequence $\{x_{jk} : k \in \mathbb{N}\}$ nor $\{x_{jk} : j \in \mathbb{N}\}$ is convergent for any fixed $j \in \mathbb{N}$ or $k \in \mathbb{N}$, respectively.

Example 3. The converse situation may also occur. Let

$$x_{jk} := \begin{cases} 1 & \text{if } j = k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\{x_{jk}\}$ is not convergent in Pringsheim's sense, but the single sequences $\{x_{jk} : k \in \mathbb{N}\}$ and $\{x_{jk} : j \in \mathbb{N}\}$ are convergent to 0 for each fixed $j \in \mathbb{N}$ and $k \in \mathbb{N}$, respectively.

Following Hardy [3], a double sequence $\{x_{jk}\}$ is said to be *regularly convergent* if $\{x_{jk}\}$ is convergent in Pringsheim's sense and, in addition, the single sequences $\{x_{jk} : k \in \mathbb{N}\}$ and $\{x_{jk} : j \in \mathbb{N}\}$ are convergent for each fixed $j \in \mathbb{N}$ and $k \in \mathbb{N}$, respectively. Clearly, the regular convergence of a double sequence implies the boundedness of its terms. Furthermore, Hardy proved that the limit of a regularly convergent double sequence $\{x_{jk}\}$ can be computed as the iterated limit of the sequence of the limits of the corresponding single sequences:

$$\lim_{j,k\to\infty} x_{jk} = \lim_{j\to\infty}\big(\lim_{k\to\infty} x_{jk}\big) = \lim_{k\to\infty}\big(\lim_{j\to\infty} x_{jk}\big).$$

We note that the definition of regular convergence was introduced in [3] only for double sequences, while for multiple sequences it can be found, for example, in [4, p. 34]. Furthermore, the notion of regular convergence was rediscovered by the present author in [5] and [6], where it was defined in another but equivalent form (and called "convergence in a restricted sense").

2. Statistical convergence of single and double sequences. From now on, we allow that certain terms of a single sequence $\{x_j : j \in \mathbb{N}\}$ in question are not defined; or, in other words, one may set $x_j := *$ for certain $j \in \mathbb{N}$, where "$*$" indicates an unspecified symbol. Except these unspecified terms, all the other terms of the sequence $\{x_j\}$ are assumed to be complex numbers. The significance of this agreement will be clear in Section 3.

We recall that such a sequence $\{x_j : j \in \mathbb{N}\}$ is said to be *statistically convergent* to some $\xi \in \mathbb{C}$, in symbols: st-$\lim_{j\to\infty} x_j = \xi$, if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m+1} |\{j \le m : |x_j - \xi| \not\le \varepsilon\}| = 0,$$

where by $j \le m$ we mean that $j = 0, 1, \ldots, m$; by $|x_j - \xi| \not\le \varepsilon$ we mean that either $x_j = *$ or x_j is defined but $|x_j - \xi| > \varepsilon$; and by $|\mathcal{S}|$ we mean the cardinality of the set $\mathcal{S} \subseteq \mathbb{N}$. It is plain that if a sequence $\{x_j\}$ of complex numbers is convergent in the ordinary sense, then it is statistically convergent to the same limit. The converse implication is trivially not true.

We recall that the *natural* (or *asymptotic*) *density* of a set $\mathcal{S} \subseteq \mathbb{N}$ is defined by

$$d_1(\mathcal{S}) := \lim_{m\to\infty} \frac{1}{m+1} |\{j \le m : j \in \mathcal{S}\}|,$$

provided that the limit on the right-hand side exists. Accordingly, the concept of statistical convergence can be reformulated in terms of the natural density as follows. A sequence $\{x_j : j \in \mathbb{N}\}$ is statistically convergent to some $\xi \in \mathbb{C}$ if and only if for every $\varepsilon > 0$,

$$d_1(\{j \in \mathbb{N} : |x_j - \xi| \not\le \varepsilon\}) = 0. \tag{2.1}$$

In particular, we have

$$d_1(\{j \in \mathbb{N} : x_j \text{ is not defined}\}) = 0.$$

Clearly, (2.1) is equivalent to the following:

$$d_1(\{j \in \mathbb{N} : x_j \text{ is defined and } |x_j - \xi| \le \varepsilon\}) = 1.$$

Now, it is reasonable to define the *natural density* of a set $\mathcal{S} \subseteq \mathbb{N}^2$ as follows:

$$d_2(\mathcal{S}) := \lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} |\{j \le m \text{ and } k \le n : x_{jk} \in \mathcal{S}\}|,$$

provided that this limit exists in Pringsheim's sense. We recall that a double sequence $\{x_{jk} : (j,k) \in \mathbb{N}^2\}$, certain terms of which may be undefined, while all the others are complex numbers, is said to be *statistically convergent* to some $\xi \in \mathbb{C}$, if for every $\varepsilon > 0$,

$$d_2(\{(j,k) \in \mathbb{N}^2 : |x_{jk} - \xi| \not\le \varepsilon\}) = 0. \tag{2.2}$$

If this is the case, we write st-$\lim_{j,k\to\infty} x_{jk} = \xi$.

A few remarks and examples are appropriate here.

It is well known that the convergence in Pringsheim's sense of a double sequence does not imply the boundedness of its terms. On the other hand, the statistical convergence of a double sequence $\{x_{jk}\}$ implies the *statistical boundedness* of its terms in the sense that there exists a positive number B

such that

$$d_2(\{(j,k) \in \mathbb{N}^2 : |x_{jk}| \not\leq B\}) = 0. \tag{2.3}$$

Indeed, by (2.2) the number $B := |\xi| + \varepsilon$ is suitable in (2.3) for any $\varepsilon > 0$.

It is plain that if a double sequence $\{x_{jk}\}$ of complex numbers is convergent in Pringsheim's sense, then it is statistically convergent to the same limit. The converse implication is trivially not true.

EXAMPLE 4. Let $\mathbb{N}_+ := \{1, 2, \ldots\}$ and

$$x_{jk} := \begin{cases} 1 & \text{if } k = \ell 2^j,\ \ell \in \mathbb{N}_+,\ j \in \mathbb{N}, \\ & \text{or } j = \ell 2^k,\ \ell \in \mathbb{N}_+,\ k \in \mathbb{N}; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\text{st-lim}_{j,k\to\infty} x_{jk} = 0$, while $\text{st-lim}_{k\to\infty} x_{jk}$ does not exist for any fixed $j \in \mathbb{N}_+$ and $\text{st-lim}_{j\to\infty} x_{jk}$ does not exist for any fixed $k \in \mathbb{N}_+ := \{1, 2, \ldots\}$.

EXAMPLE 5. The converse situation may also occur. Let

$$x_{jk} := \begin{cases} 1 & \text{if } k = j+1, j+2, \ldots, 2j \text{ and } j \in \mathbb{N}_+, \\ & \text{or } j = k+1, k+2, \ldots, 2k \text{ and } k \in \mathbb{N}_+; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\text{st-lim}_{j,k\to\infty} x_{jk}$ does not exist, while

$$\underset{k\to\infty}{\text{st-lim}}\, x_{jk} = \underset{j\to\infty}{\text{st-lim}}\, x_{jk} = 0 \quad \text{for every fixed } j \in \mathbb{N} \text{ and } k \in \mathbb{N},$$

respectively.

We note that the term "statistical convergence" first appeared in the paper [1] by Fast, where this concept was attributed to Steinhaus [9], who used the term "asymptotic convergence". However, it was Antoni Zygmund who proved theorems on the statistical convergence of Fourier series in the first edition of his book "Trigonometric Series" in 1935, where the term "almost convergence" was used in place of "statistical convergence". (See [10, Vol. 2, pp. 181 and 188].)

Some basic properties of statistical convergence were proved by Schoenberg [8] in 1959 and by Fridy [2] in 1985 in the case of single sequences, and by the present author in 2003 in the case of multiple sequences.

3. Regular statistical convergence of double sequences. Again we consider a double sequence $\{x_{jk} : (j,k) \in \mathbb{N}^2\}$, certain terms of which may be undefined, while all the others are complex numbers. We say that $\{x_{jk}\}$ is *regularly statistically convergent* to some $\xi \in \mathbb{C}$ (or alternatively, we may say that the statistical convergence of $\{x_{jk}\}$ is regular) if the following three conditions are satisfied:

(i) the double sequence $\{x_{jk}\}$ is statistically convergent to ξ,

(ii) the single sequence $\{x_{jk} : k \in \mathbb{N}\}$ is statistically convergent to some $\xi_j \in \mathbb{C}$ for each fixed $j \in \mathbb{N} \setminus \mathcal{S}_1$, where $d_1(\mathcal{S}_1) = 0$,

(iii) the single sequence $\{x_{jk} : j \in \mathbb{N}\}$ is statistically convergent to some $\eta_k \in \mathbb{C}$ for each fixed $k \in \mathbb{N} \setminus \mathcal{S}_2$, where $d_1(\mathcal{S}_2) = 0$.

Our main result reads as follows.

THEOREM 1. *If a double sequence $\{x_{jk}\}$ is regularly statistically convergent to some $\xi \in \mathbb{C}$, then*

$$\text{st-}\lim_{j\to\infty} \xi_j = \xi \quad \textit{and} \quad \text{st-}\lim_{k\to\infty} \eta_k = \xi. \tag{3.1}$$

Proof. By condition (i) of regular statistical convergence, for all $\varepsilon > 0$ and $\delta > 0$ there exists $m_0 = m_0(\varepsilon, \delta) \in \mathbb{N}$ such that

$$\frac{1}{(m+1)(n+1)} \left|\{j \le m \text{ and } k \le n : |x_{jk} - \xi| \not\le \varepsilon/2\}\right| < \delta,$$

or equivalently,

$$\frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \left|\{k \le n : |x_{jk} - \xi| \not\le \varepsilon/2\}\right| < \delta \quad \text{ if } m, n > m_0. \tag{3.2}$$

Since $d_1(\mathcal{S}_1) = 0$, for every $\delta > 0$ there exists $m_1 = m_1(\delta) \in \mathbb{N}$ such that

$$\frac{1}{m+1} \left|\{j \le m : j \in \mathcal{S}_1\}\right| < \delta \quad \text{ if } m > m_1. \tag{3.3}$$

Also by (ii), for all $j \in \mathbb{N}\setminus\mathcal{S}_1$, $\varepsilon > 0$, and $\delta > 0$ there exists $n_j = n_j(\varepsilon, \delta) \in \mathbb{N}$ such that

$$\frac{1}{n+1} \left|\{k \le n : |x_{jk} - \xi_j| \not\le \varepsilon/2\}\right| < \delta,$$

or equivalently,

$$\ell_j := \left|\{k \le n : |x_{jk} - \xi_j| \le \varepsilon/2\}\right| > (1-\delta)(n+1) \quad \text{ if } n > n_j. \tag{3.4}$$

Without loss of generality, we may assume that $0 < \delta \le 1/2$. For the time being, fix $m \in \mathbb{N}$ so that $m > \widetilde{m}_0 := \max\{m_0, m_1\}$, and let

$$n > \widetilde{n}_0 := \max\{m_0, n_j : j \in [0, m] \setminus \mathcal{S}_1\}. \tag{3.5}$$

Now, assume that $|\xi_j - \xi| > \varepsilon$ for some $j \in [0, m] \setminus \mathcal{S}_1$. If, in addition, $|x_{jk} - \xi_j| \le \varepsilon/2$ for some $k \in \mathbb{N}$, then

$$|x_{jk} - \xi| \ge |\xi_j - \xi| - |x_{jk} - \xi_j| > \varepsilon/2.$$

Taking into account (3.4), we find that

$$\left|\{k \le n : |x_{jk} - \xi| > \varepsilon/2\}\right| \ge \ell_j \quad \text{ if } |\xi_j - \xi| > \varepsilon \text{ and } n > n_j.$$

By this inequality and (3.5), we have

$$\text{(3.6)}\qquad |\{j\in[0,m]\setminus\mathcal{S}_1 : |\xi_j-\xi|>\varepsilon\}| \le \sum_{j\in[0,m]\setminus\mathcal{S}_1}\frac{1}{\ell_j}\,|\{k\le n : |x_{jk}-\xi|>\varepsilon/2\}| \quad \text{if } n>\widetilde{n}_0.$$

Combining (3.2)–(3.4), (3.6) and the fact that $\delta\le 1/2$ gives

$$\begin{aligned}\frac{1}{m+1}\,|\{j\le m : |\xi_j-\xi|\not\le\varepsilon\}| &\le \frac{1}{m+1}\,|\{j\le m : j\in\mathcal{S}_1\}|\\ &\quad+\frac{1}{m+1}\sum_{j\in[0,m]\setminus\mathcal{S}_1}\frac{1}{\ell_j}\,|\{k\le n : |x_{jk}-\xi|>\varepsilon/2\}|\\ &<\delta+\frac{1}{(1-\delta)(m+1)(n+1)}\sum_{j=0}^{m}|\{k\le n : |x_{jk}-\xi|\not\le\varepsilon/2\}|\\ &<\delta+\frac{\delta}{1-\delta}\le 3\delta \quad \text{if } m>\widetilde{m}_0 \text{ and } n>\widetilde{n}_0.\end{aligned}$$

Now, letting $m\to\infty$ yields

$$\limsup_{m\to\infty}\frac{1}{m+1}\,|\{j\le m : |\xi_j-\xi|\not\le\varepsilon\}|\le 3\delta.$$

Since $\delta>0$ can be chosen as small as we want, it follows that

$$\lim_{m\to\infty}\frac{1}{m+1}\,|\{j\le m : |\xi_j-\xi|\not\le\varepsilon\}|=0,$$

which is valid for every $\varepsilon>0$. This proves the first equality in (3.1).

The second equality in (3.1) can be proved in an analogous way. ∎

In Section 4, we shall use a version of Theorem 1 when the $x_{jk}:\mathcal{D}\to\mathbb{C}$ are functions for $(j,k)\in\mathbb{N}^2$ with a common domain $\mathcal{D}$ and exhibiting uniform statistical convergence in each of (i)–(iii) of the definition of regular statistical convergence. For our purpose, it is enough to consider the particular case when the exceptional sets $\mathcal{S}_1$ and $\mathcal{S}_2$, indicated in the definition of regular statistical convergence, are empty.

THEOREM 1*. *Let $\{x_{jk}:\mathcal{D}\to\mathbb{C}\}$ be a double sequence of functions such that*

(i) *the double sequence $\{x_{jk}\}$ is statistically convergent to some function $\xi:\mathcal{D}\to\mathbb{C}$ uniformly on $\mathcal{D}$;*
(ii) *the single sequence $\{x_{jk}:k\in\mathbb{N}\}$ is statistically convergent to some function $\xi_j:\mathcal{D}\to\mathbb{C}$ uniformly on $\mathcal{D}$ for each fixed $j\in\mathbb{N}$;*
(iii) *the single sequence $\{x_{jk}:j\in\mathbb{N}\}$ is statistically convergent to some function $\eta_k:\mathcal{D}\to\mathbb{C}$ uniformly on $\mathcal{D}$ for each fixed $k\in\mathbb{N}$.*

Then we have (3.1) *uniformly on $\mathcal{D}$.*

Proof. It goes along the same lines as the proof of Theorem 1. In particular, the proof of the first equality in (3.1) hinges on the fact that inequalities (3.2) and (3.4) hold uniformly on $\mathcal{D}$. ∎

4. An application to double Fourier series. Let $f = f(u, v)$ be a periodic function in each variable and integrable in Lebesgue's sense on the two-dimensional torus $\mathbb{T}^2 := [-\pi, \pi) \times [-\pi, \pi)$. We recall that the double Fourier series of f is defined by

$$f(u, v) \sim \sum\sum_{(j,k)\in\mathbb{Z}^2} \widehat{f}(j, k) e^{i(ju+kv)}, \tag{4.1}$$

where $\mathbb{Z} := \{\dots, -1, 0, 1, 2, \dots\}$ and the Fourier coefficients $\widehat{f}(j, k)$ are defined by

$$\widehat{f}(j, k) := \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} f(u, v) e^{-i(ju+kv)}\, du\, dv, \qquad (j, k) \in \mathbb{Z}^2. \tag{4.2}$$

We consider the (symmetric) *rectangular partial sums*

$$s_{mn}(f, u, v) := \sum_{|j|\le m} \sum_{|k|\le n} \widehat{f}(j, k) e^{i(ju+kv)}, \qquad (m, n) \in \mathbb{N}^2,$$

of the series in (4.1). The reader is referred to [10, Vol. 2, Ch. 17] for more details.

We proved in [7] that if

$$\iint_{\mathbb{T}^2} |f(u, v)|(\log^+ |f(u, v)|)\, du\, dv < \infty,$$

in symbols,

$$f \in L \log^+ L(\mathbb{T}^2), \quad \text{where} \quad \log^+ |f| := \max\{0, \log |f|\},$$

then

$$\operatorname*{st-lim}_{m,n\to\infty} s_{mn}(f, u, v) = f(u, v) \quad \text{if } (u, v) \in \mathbb{T}^2 \setminus W_1, \tag{4.3}$$

where $\lambda_2(W_1) = 0$ and $\lambda_2(\cdot)$ denotes the Lebesgue measure on the real plane $\mathbb{R}^2$. Furthermore, we also proved in [7] that if $f \in C(\mathbb{T}^2)$, then (4.3) holds uniformly on $\mathbb{T}^2$.

Now, we shall prove that stronger conclusions are valid under the same conditions.

THEOREM 2. (i) *If $f \in L \log^+ L(\mathbb{T}^2)$, then the statistical convergence in* (4.3) *is regular at almost every point $(u, v) \in \mathbb{T}^2$.*

(ii) *If $f \in C(\mathbb{T}^2)$, then the statistical convergence in* (4.3) *is regular uniformly on $\mathbb{T}^2$.*

Proof. According to what we have said just before stating Theorem 2, it remains to check conditions (ii) and (iii) in the definition of regular statistical convergence. To this end, we fix $m \in \mathbb{N}$ and consider the single series

$$s_{mn}(f,u,v) = \sum_{|j|\le m} e^{iju} \sum_{|k|\le n} \widehat{f}(j,k)e^{ikv}, \qquad n \in \mathbb{N}. \tag{4.4}$$

The statistical convergence of this sequence as $n \to \infty$ for each fixed $m \in \mathbb{N}$ follows from the statistical convergence of the single sequence

$$\sum_{|k|\le n} \widehat{f}(j,k)e^{ikv}, \qquad n \in \mathbb{N}, \tag{4.5}$$

for each fixed $j \in \mathbb{Z}$.

By (4.2) and Fubini's theorem, we observe that

$$\widehat{f}(j,k) = \frac{1}{2\pi}\int_{\mathbb{T}} \left(\frac{1}{2\pi}\int_{\mathbb{T}} f(u,v)e^{-iju}\,du\right)e^{-ikv}\,dv = \widehat{f_j}(k), \qquad j,k \in \mathbb{Z},$$

where

$$f_j(v) := \frac{1}{2\pi}\int_{\mathbb{T}} f(u,v)e^{-iju}\,du, \qquad j \in \mathbb{Z},\ v \in \mathbb{T}. \tag{4.6}$$

Again by Fubini's theorem, we have

$$\int_{\mathbb{T}} |f_j(v)|\,dv \le \frac{1}{2\pi}\int_{\mathbb{T}}\left(\int_{\mathbb{T}} |f(u,v)|\,du\right)dv = \iint_{\mathbb{T}^2} |f(u,v)|\,du\,dv < \infty.$$

That is, $f_j \in L(\mathbb{T})$ and its Fourier series is of the form

$$f_j(v) \sim \sum_{k\in\mathbb{Z}} \widehat{f}(j,k)e^{ikv}, \qquad j \in \mathbb{Z}. \tag{4.7}$$

Consequently, the sums in (4.5) are the (symmetric) partial sums of the Fourier series occurring in (4.7). Making use of [7, Theorem 3] gives

$$\operatorname*{st-lim}_{n\to\infty} \sum_{|k|\le n} \widehat{f}(j,k)e^{ikv} = f_j(v) \quad \text{if } v \in \mathbb{T}\setminus V_j, \tag{4.8}$$

where $\lambda_1(V_j) = 0$ for each $j \in \mathbb{Z}$ and $\lambda_1(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}$. Taking into account the relation between (4.4) and (4.5) explained above, we conclude that

$$\operatorname*{st-lim}_{n\to\infty} s_{mn}(f,u,v) = \sum_{|j|\le m} f_j(v)e^{iju} \quad \text{if } (u,v) \in \mathbb{T}^2\setminus W_2, \tag{4.9}$$

for each fixed $m \in \mathbb{N}$, where

$$W_2 := \mathbb{T} \times \bigcup_{j\in\mathbb{Z}} V_j.$$

Clearly, we have $\lambda_2(W_2) = 0$. Thus, condition (ii) in the definition of regular statistical convergence is satisfied.

The fulfillment of condition (iii) there can be checked in an analogous way. This time, let

$$g_k(u) := \frac{1}{2\pi} \int_{\mathbb{T}} f(u,v) e^{-ikv}\, dv, \qquad k \in \mathbb{Z},\ u \in \mathbb{T}. \tag{4.10}$$

Then $g_k \in L(\mathbb{T})$ for each $k \in \mathbb{Z}$ and the symmetric counterpart of (4.8) reads as follows:

$$\operatorname*{st-lim}_{m\to\infty} \sum_{|j|\le m} \widehat{f}(j,k) e^{iju} = g_k(u) \quad \text{ if } u \in \mathbb{T} \setminus U_k, \tag{4.11}$$

where $\lambda_1(U_k) = 0$ for each $k \in \mathbb{Z}$. It follows that

$$\operatorname*{st-lim}_{m\to\infty} s_{mn}(f,u,v) = \sum_{|k|\le n} g_k(u) e^{ikv} \quad \text{ if } (u,v) \in \mathbb{T}^2 \setminus W_3, \tag{4.12}$$

for each fixed $n \in \mathbb{N}$, where

$$W_3 := \Big(\bigcup_{k\in\mathbb{Z}} U_k \Big) \times \mathbb{T}.$$

Clearly, $\lambda_2(W_3) = 0$.

Taking into account (4.3), (4.9) and (4.12) proves part (i) of Theorem 2.

In order to prove Theorem 2(ii), it is enough to notice that if $f \in C(\mathbb{T}^2)$, then f_j defined in (4.6) and g_k defined in (4.10) are in $C(\mathbb{T})$ for each $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$, respectively. By [7, Theorem 3], the statistical convergence in (4.8) and (4.11) is uniform on $\mathbb{T}$ for each $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$, respectively; and the rest of the proof is the same as in the case of (i). ∎

The following corollary of Theorems 1 and 2 is worth formulating.

Corollary. (i) *If $f \in L \log^+ L(\mathbb{T}^2)$, then*

$$\begin{aligned} \operatorname*{st-lim}_{m,n\to\infty} s_{mn}(f,u,v) &= \operatorname*{st-lim}_{m\to\infty}\big(\operatorname*{st-lim}_{n\to\infty} s_{mn}(f,u,v)\big) \\ &= \operatorname*{st-lim}_{n\to\infty}\big(\operatorname*{st-lim}_{m\to\infty} s_{mn}(f,u,v)\big) \\ &= f(u,v) \quad \textit{if } (u,v) \in \mathbb{T}^2 \setminus W, \end{aligned} \tag{4.13}$$

where $\lambda_2(W) = 0$.

(ii) *If $f \in C(\mathbb{T}^2)$, then each of the five statistical limits in* (4.13) *exists uniformly on $\mathbb{T}^2$.*

Proof. (i) In case $f \in L \log^+ L(\mathbb{T}^2)$, (4.13) follows from Theorem 1, by making use of (4.8), (4.9), (4.11) and (4.12), and setting $W := W_1 \cup W_2 \cup W_3$.

(ii) In case $f \in C(\mathbb{T}^2)$, the uniform existence of the first (double) statistical limit in (4.13) was proved in [7, Theorem 3]. The uniform existence of the second and third inner (single) statistical limits is obvious from (4.8), (4.9), (4.11) and (4.12). Finally, the uniform existence of the second and third outer (single) statistical limits is guaranteed by Theorem 1^*. ■

5. Regular statistical convergence of double series. We consider double series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} x_{jk} \tag{5.1}$$

of complex numbers with rectangular partial sums

$$s_{mn} := \sum_{j=0}^{m}\sum_{k=0}^{n} x_{jk}, \qquad (m,n) \in \mathbb{N}^2.$$

The definition of s_{mn} evidently excludes the case when certain terms x_{jk} of the double series are undefined. That is, this time the case $x_{jk} = *$ cannot occur for any $(j,k) \in \mathbb{N}^2$.

It is easy to see that the regular convergence of the double sequence $\{s_{mn}\}$ is equivalent to the following three conditions:

(i) the double sequence $\{s_{mn}\}$ is convergent in Pringsheim's sense,
(ii) the "row series" $\sum_{k=1}^{\infty} x_{jk}$ is convergent for each fixed $j \in \mathbb{N}$,
(iii) the "column series" $\sum_{j=1}^{\infty} x_{jk}$ is convergent for each fixed $k \in \mathbb{N}$.

Now, if these conditions (i)–(iii) are satisfied, we say that series (5.1) is *regularly convergent*. It is plain that every absolutely convergent double series is regularly convergent. The converse implication is not true, as the following example shows.

EXAMPLE 6. Let

$$x_{jk} := \frac{(-1)^{j+k}}{1+\max\{j,k\}}, \qquad (j,k) \in \mathbb{N}^2.$$

Then series (5.1) is regularly convergent, but not absolutely.

We proved in [5] that series (5.1) is regularly convergent if and only if for every $\varepsilon > 0$ there exists $m_0 = m_0(\varepsilon)$ such that

$$\Big|\sum_{j=a_1}^{b_1}\sum_{k=a_2}^{b_2} x_{jk}\Big| < \varepsilon \quad \text{if } \max\{a_1,a_2\} > m_0, \tag{5.2}$$

where $0 \le a_1 \le b_1$ and $0 \le a_2 \le b_2$ are arbitrary integers. Roughly speaking, condition (5.2) says that any "remaining sum of rectangular shape" is as small as we want whenever the lower left corner of the rectangle $[a_1,b_1] \times [a_2,b_2]$ is far enough from the origin in the coordinate system of the lattice points $(j,k) \in \mathbb{N}^2$.

We raise the following

Problem. Characterize the regular statistical convergence of double series of complex numbers in a way analogous to (5.2) which characterizes the regular convergence of double series.

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