COLLOQUIUM MATHEMATICUM

VOL. 102 2005 NO. 2

MULTIPARAMETER ADMISSIBLE SUPERADDITIVE PROCESSES

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Abstract. In this article some properties of Markovian mean ergodic operators are studied. As an application of the tools developed, and using the admissibility feature, a "reduction of order" technique for multiparameter admissible superadditive processes is obtained. This technique is later utilized to obtain a.e. convergence of averages $n^{-2}\sum_{i,j=0}^{n-1} f_{(i,j)}$ as well as their weighted version.

1. Introduction. If T and S are positive Markovian Dunford–Schwartz operators, then the "averages" $n^{-2}F_{(n,n)}$ converge almost everywhere for any bounded strongly (T,S)-superadditive process $F=\{F_{(m,n)}\}$ [Ç, Sm]. The same convergence result is obtained in [AK] assuming only superadditivity of the process for operators induced by measure preserving transformations. Furthermore, the definition of superadditivity in [AK] is more general than in [Ç] and [Sm]. Since the pointwise ergodic theorem need not hold for superadditive processes relative to general Markovian L_1-contractions, the results mentioned above are not valid if T and S are only L_1-contractions (or only L_∞-contractions). In [ÇL$_1$] ergodic theorems for one-parameter superadditive processes with respect to *mean ergodic* Markovian operators (which need not be L_∞-contractions) were obtained. Recently, some of these results have been extended to the setting of multiparameter strongly superadditive processes relative to mean ergodic operators [ÇL$_2$].

In proving all the theorems on the a.e. convergence of multiparameter superadditive processes mentioned above, the standard technique has been to dominate the given process by an *additive* process and then obtain the desired convergence result. In this article we will show (in Section 3) that one can dominate a two-parameter *admissible* superadditive process by a *one-parameter admissible* process with the same time constant. Consequently, one can utilize tools developed for one-parameter superadditive processes in their study. Since admissible processes define superadditive processes in the sense of [AK], the results obtained in Section 4 (Theorems 4.2 and 4.4)

2000 *Mathematics Subject Classification*: Primary 28D05; Secondary 37A99.

Key words and phrases: admissible processes, reduction of order, Markovian operators, mean ergodic operators, weighted ergodic theorem, bounded Besicovitch sequences.

generalize some of the almost everywhere convergence results in [AK] and in [DeK] to the setting of mean ergodic Markovian operators. It should be noted here that many of the results in [DeK] deal with norm convergence of multiparameter superadditive processes in the sense of [AK], which include admissible processes. Almost everywhere convergence results in [DeK] assume that either the operators are Dunford–Schwartz or there exists a positive invariant function, neither of which is assumed in Theorems 4.2 and 4.4 below. Also, some of the tools developed in Section 3 will be utilized to obtain a.e. convergence of weighted averages of admissible processes relative to commuting Markovian Dunford–Schwartz operators, which generalizes Theorem 1.4 in [JO].

2. Preliminaries. Throughout this paper, we will view a process $F = \{F_{(m,n)}\}$ as a family of functions $\{f_{(i,j)}\}$ with partial sums $F_{(m,n)} = \sum_{i,j=0}^{m-1,n-1} f_{(i,j)}$. Let (X, Σ, μ) be a σ-finite measure space and T be a positive linear operator on $L_1(X)$. A process $F = \{f_i\} \subset L_1(X)$ is called *T-admissible* if $Tf_i \leq f_{i+1}$ for all $i \geq 0$. Clearly, a T-admissible process defines a T-superadditive process, that is, $F_{m+n} \geq F_m + T^m F_n$ for all $m, n \geq 0$. Hence, any additive process $\{\sum_{i=0}^{n-1} T^i f\}$ is T-admissible. Similarly, for commuting positive operators T and S on L_1, a process $F = \{f_{(i,j)}\}_{\{i\geq 0, j\geq 0\}} \subset L_1(X)$ is called *(T,S)-admissible* if $Tf_{(i,j)} \leq f_{(i+1,j)}$ and $Sf_{(i,j)} \leq f_{(i,j+1)}$ for all $i, j \geq 0$. Such a (T,S)-admissible process also defines a (T,S)-superadditive process in the sense of [AK] (i.e. if we define $\overline{F}_I = T^{n_1-m_1} S^{n_2-m_2} F_{(n_1-m_1, n_2-m_2)}$ for every rectangle $I = [m_1, n_1] \times [m_2, n_2]$, where $m_1 < n_1$ and $m_2 < n_2$ are positive integers, then $\overline{F}_{I_1 \cup I_2} \geq \overline{F}_{I_1} + \overline{F}_{I_2}$ for disjoint rectangles I_1 and I_2.) Naturally, a (T,S)-admissible process is a (T,S)-superadditive process in the sense of [Ç, Sm]: for all $k, l, m, n \geq 0$,

$$F_{(m+k,n)} \geq F_{(m,n)} + T^m F_{(k,n)} \quad \text{and} \quad F_{(m,n+l)} \geq F_{(m,n)} + S^n F_{(m,l)}.$$

A superadditive process F is called *positive* if $F_{(m,n)} \geq 0$ for all $n \geq 1$, and *bounded* (with *time constant* γ_F) if

$$\gamma_F := \sup_{(m,n)>(0,0)} \frac{1}{mn} \|F_{(m,n)}\| < \infty.$$

When F is admissible, it is called *strongly bounded* if $\sup_{(i,j)\geq(0,0)} \|f_{(i,j)}\| < \infty$, and *dominated* if there exists $g \in L_1$ (called a *dominant*) such that, for all $m, n > 0$,

$$\frac{1}{mn} F_{(m,n)} \leq \frac{1}{mn} \sum_{i,j=0}^{m-1,n-1} T^i S^j g.$$

A dominated process is necessarily bounded, with $\int |g| \, d\mu \geq \gamma_F$ for every dominant g. A dominant g is called an *exact dominant* if $\int |g| \, d\mu = \gamma_F$.

REMARKS. 1. A superadditive process is positive ($F_{(m,n)} \geq 0 \; \forall m, n > 0$) whenever $F_{(1,1)} \geq 0$.

2. If F is a superadditive process, then it is the sum of an additive process and the positive superadditive process defined by

$$F'_{(m,n)} = F_{(m,n)} - \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} T^i S^j F_{(1,1)} \quad \text{for all } m, n > 0.$$

Furthermore, $F' = \{F'_{(m,n)}\}$ is bounded (dominated) if F is bounded (dominated).

A positive $L_1(X)$-contraction T is called a *Markovian operator* if $\int Tf \, d\mu = \int f \, d\mu$ for all $f \in L_1$. A linear operator T on $L_1(X)$ is called *mean ergodic* if $\lim n^{-1} \sum_{i=0}^{n-1} T^i f$ exists in L_1-norm for all $f \in L_1$.

All the results in this article can be stated in an arbitrary n-parameter setting. However, for simplicity, they are stated and proved in the two-parameter setting only. The set $\mathbb{Z}_+^2$ will be considered with partial order $(i,j) \leq (u,v)$ if $i \leq u$ and $j \leq v$; $(i,j) < (u,v)$ if $(i,j) \leq (u,v)$ and $(i,j) \neq (u,v)$.

In obtaining a.e. convergence of superadditive processes boundedness is an essential property. The following statement shows that, if T and S are Markovian, bounded (T,S)-admissible processes $F = \{f_{(i,j)}\} \subset L_1$ can be characterized to be those which are strongly bounded.

PROPOSITION 2.1. *Let T and S be commuting Markovian operators on L_1. A positive (T,S)-admissible process F is bounded if and only if it is strongly bounded.*

Proof. Obviously, if $\sup_{(i,j)>(0,0)} \|f_{(i,j)}\| < \infty$, then F is bounded. Conversely, if F is bounded, then

$$\begin{aligned}
\|f_{(i,j)}\| &= \frac{1}{mn} \sum_{k,l=0}^{m-1,n-1} \int T^k S^l f_{(i,j)} \quad \text{(by Markovian property)} \\
&\leq \frac{1}{mn} \sum_{i=k,j=l}^{m+k-1,n+l-1} \int f_{(i,j)} \quad \text{(by admissibility)} \\
&\leq \frac{1}{mn} \int [F_{(m+k,n+l)} - F_{(k,n+l)} - T^k F_{(m,l)}] \quad \text{(by superadditivity)} \\
&= \frac{1}{mn} \int [F_{(m+k,n+l)} - F_{(k,n+l)} - F_{(m,l)}] \quad \text{(by Markovian property)} \\
&\leq \frac{1}{mn} \int F_{(m+k,n+l)} \\
&= \frac{(m+k)(n+l)}{mn} \left[\frac{1}{(m+k)(n+l)} \int F_{(m+k,n+l)} \right].
\end{aligned}$$

Letting $m \to \infty$, $n \to \infty$, and applying the fact that $\gamma_F = \lim \frac{1}{mn} \int F_{(m,n)}$ when F is bounded [Ç, Sm] yields the assertion. ■

In the one-parameter case, any bounded superadditive process (relative to a Markovian operator) admits an exact dominant [ASu]. On the other hand, this is not the case for multiparameter processes. Indeed, it is known that an arbitrary multiparameter superadditive process need not have an exact dominant, even if it is relative to a pair of measure preserving transformations [AK, Sm]. However, when admissible processes are considered, one can provide a general method of constructing examples of multiparameter superadditive processes relative to Markovian operators with exact dominants. For, let $\{v_{ij}\} \subset L_1^+$ and $v \in L_1^+$ be such that $v_{ij} \uparrow v$ a.e. as $i, j \to \infty$. If T and S are Markovian operators on L_1, then define $f_{(i,j)} = T^i S^j v_{ij}$. The resulting process $F = \{f_{(i,j)}\}$ is (T, S)-admissible with $\gamma_F = \sup_{(i,j)>(0,0)} \|f_{(i,j)}\| = \|v\| < \infty$. Furthermore, for any $m, n \geq 1$, we have

$$\frac{1}{mn} \sum_{i,j=0}^{m-1,n-1} f_{(i,j)} \leq \frac{1}{mn} \sum_{i,j=0}^{m-1,n-1} T^i S^j v.$$

Hence, v is an exact dominant for F.

In ergodic theory, one often encounters examples of Markovian operators induced by (nonsingular) point transformations. However, one can also construct examples of interesting Markovian operators in terms of integral operators. If $K : X \times X \to \mathbb{R}$ is a *stochastic kernel*, i.e. a measurable function satisfying $K(x, y) \geq 0$ and $\int_X K(x, y)\, \mu(dx) = 1$, then the operator $T_K : L_1(X) \to L_1(X)$ defined by $T_K f(x) = \int_X K(x, y) f(y)\, \mu(dy)$, $f \in L_1(X)$, is Markovian. For example, let $X = [0, 1)$ with Lebesgue measure, and define stochastic kernels $K(x, y)$ and $L(x, y)$ on $X \times X$ by letting $K(x, y) = L(x, y) = 3/4$ on $A \times C$ and $B \times C$; $K(x, y) = L(x, y) = 3/2$ on $C \times C$; and

$$K(x,y) = \begin{cases} 0 & \text{on } A \times B,\ B \times A,\ C \times A,\ C \times B, \\ 3 & \text{on } A \times A,\ B \times B, \end{cases}$$

$$L(x,y) = \begin{cases} 0 & \text{on } A \times A,\ B \times B,\ C \times A,\ C \times B, \\ 3 & \text{on } A \times B,\ B \times A, \end{cases}$$

where $A = [0, 1/3)$, $B = [1/3, 2/3)$, and $C = [2/3, 1)$. Then, for $f \in L_1[0, 1)$,

$$\begin{aligned} T_K f(x) = 3\Big[&\chi_A(x) \int_A f\, dy + \chi_B(x) \int_B f\, dy \\ &+ \left(\frac{1}{4}\chi_A(x) + \frac{1}{4}\chi_B(x) + \frac{1}{2}\chi_C(x)\right) \int_C f\, dy\Big], \end{aligned}$$

$$T_L f(x) = 3\Big[\chi_B(x)\int_A f\,dy + \chi_A(x)\int_B f\,dy + \Big(\frac{1}{4}\chi_A(x) + \frac{1}{4}\chi_B(x) + \frac{1}{2}\chi_C(x)\Big)\int_C f\,dy\Big],$$

and hence, T_K and T_L are commuting Markovian operators. Furthermore, both T_K and T_L are mean ergodic operators with $T_K 1 \neq 1$, and $T_L 1 \neq 1$. As observed in the preceding paragraph, if $\{v_{ij}\} \subset L_1^+$ is a sequence with $v_{ij} \uparrow v \in L_1^+$ a.e. as $i, j \to \infty$, then the family $\{f_{(i,j)}\}_{i,j}$, where $f_{(i,j)} = T_K^i T_L^j v_{ij}$, is a (T_K, T_L)-admissible process. In general, if $K(x,y)$ and $L(x,y)$ are two stochastic kernels satisfying

$$\int_X K(x,z)L(z,y)\,\mu(dz) = \int_X L(x,z)K(z,y)\,\mu(dz),$$

then T_K and T_L are commuting Markovian operators.

3. Reduction of dimension for admissible processes. In this section we will utilize the construction of the Brunel operator [B] to obtain a one-parameter admissible process dominating a given two-parameter one. Since the Brunel operator is quite well known in the literature, we will just outline its construction and refer the reader to [K] for details.

Let $\phi(x) = 1 - \sqrt{1-x}$ and let $\sum_{i=0}^{\infty} \alpha_i^{(1)} x^i$ be its series expansion. If $\sum_{i=0}^{\infty} \alpha_i^{(k)} x^i$ is the series expansion of $[\phi(x)]^k$, then the coefficients $\alpha_i^{(k)}$ are given by

$$\alpha_i^{(k)} = \begin{cases} 0 & \text{if } i < k, \\ \dfrac{k}{2i}\, 2^{k+1-2i} \dbinom{2i-k-1}{i-1} & \text{if } i \geq k. \end{cases}$$

Let $\Phi(x) = (1/x)\phi(x)$, and let T and S be two linear operators on L_1. Then the *Brunel operator* U associated with T and S is defined to be the operator

$$U = \Phi(T)\Phi(S) = \sum_{i,j=0}^{\infty} \alpha_{i+1}^{(1)} \alpha_{j+1}^{(1)} T^i S^j.$$

It is a linear operator on L_1. Note that $\sum_{i,j=0}^{\infty} \alpha_{i+1}^{(1)} \alpha_{j+1}^{(1)} = 1$. Also,

$$U^k = [\Phi(T)]^k [\Phi(S)]^k = \sum_{i,j=0}^{\infty} \alpha_{i+k}^{(k)} \alpha_{j+k}^{(k)} T^i S^j \quad \text{for any } k \geq 1.$$

PROPOSITION 3.1. *Let T and S be bounded, commuting linear operators on L_1, and let U be the associated Brunel operator. Then*:

(a) *If T and S are Markovian operators, then U is also Markovian.*
(b) *If T and S are mean ergodic L_1-contractions, then U is also mean ergodic.*

Proof. (a) Since T and S are Markovian, for any $m, n \geq 0$, and for any $f \in L_1$, we have

$$\int \sum_{i,j=0}^{m,n} \alpha_{i+1}^{(1)} \alpha_{j+1}^{(1)} T^i S^j f = \sum_{i,j=0}^{m,n} \alpha_{i+1}^{(1)} \alpha_{j+1}^{(1)} \int T^i S^j f = \sum_{i,j=0}^{m,n} \alpha_{i+1}^{(1)} \alpha_{j+1}^{(1)} \int f.$$

The term on the right hand side is dominated by $\int f$. Therefore, by the monotone convergence theorem, the term on the left hand side converges to $\int Uf$ while the term on the right converges to $\int f$. Hence, U is Markovian.

(b) First observe that U is a convex combination of the (discrete) semigroup of operators $\{T^i S^j\}_{i,j\geq 0}$. By hypothesis, the operators T, S are positive L_1-contractions, and T^*, S^* are positive L_∞-contractions. Since T and S commute, by the Brunel–Falkowitz Lemma [K],

$$\begin{aligned} F(T) \cap F(S) &= \{f \in L_1 : f = Tf = Sf\} = \{f \in L_1 : f = Uf\} = F(U), \\ F_*(T) \cap F_*(S) &= \{\phi \in L_\infty : \phi = T^*\phi = S^*\phi\} = \{\phi \in L_\infty : \phi = U^*\phi\} \\ &= F_*(U). \end{aligned}$$

Mean ergodicity of T and S implies, by Sine's criterion of mean ergodicity [K], that $F(T) \cap F(S)$ separates $F_*(T) \cap F_*(S)$. Consequently, $F(U)$ separates $F_*(U)$ as well. Again, Sine's criterion implies that U is mean ergodic. ■

REMARKS. 1. If, furthermore, the operators in Proposition 2.1 are L_∞-contractions, then U is an L_∞-contraction as well [Ç, DuSc].

2. Another proof of mean ergodicity of U, shorter but requiring more technical tools, is given in [ÇL$_2$].

In the same fashion as the construction of U above, one can construct a one-parameter family of U-admissible processes dominating a given two-parameter (T, S)-admissible process. For a given positive (T, S)-admissible process, $F = \{f_{(i,j)}\} \subset L_1$, define a one-parameter family of functions $G = \{g_k\}$ as $g_0 = 0$, and for $k \geq 1$,

$$g_k = \sum_{i,j=0}^{\infty} \alpha_{i+k}^{(k)} \alpha_{j+k}^{(k)} f_{(i,j)}.$$

By the monotone convergence theorem, $g_k \in L_1$ for all $k \geq 0$.

PROPOSITION 3.2. *Let T and S be linear, positive, bounded, commuting operators on L_1, and let $F = \{f_{(i,j)}\}$ be a strongly bounded (T, S)-admissible process. Then the family $G = \{g_k\}_{k\geq 0}$ is a strongly bounded U-admissible* *process, where U is the associated Brunel operator. Furthermore, if T and S are Markovian, then G has the same time constant as F.*

Proof. By strong boundedness of F,

$$\|g_k\| = \int \Big| \sum_{i,j=0}^{\infty} \alpha_{i+k}^{(k)} \alpha_{j+k}^{(k)} f_{(i,j)} \Big| \, d\mu \le \gamma_F \sum_{i,j=0}^{\infty} \alpha_{i+k}^{(k)} \alpha_{j+k}^{(k)} \le \gamma_F .$$

Hence, G is also strongly bounded. This also shows, by Proposition 2.1, that if T and S are Markovian, then $\gamma_G = \gamma_F$. Now, for any $k > 0$,

$$\begin{aligned} Ug_k &= \sum_{i,j=0}^{\infty} \alpha_{i+1}^{(1)} \alpha_{j+1}^{(1)} T^i S^j g_k = \sum_{i,j=0}^{\infty} \alpha_{i+1}^{(1)} \alpha_{j+1}^{(1)} \Big[\sum_{u,v=0}^{\infty} \alpha_{u+k}^{(k)} \alpha_{v+k}^{(k)} T^i S^j f_{(u,v)} \Big] \\ &\le \sum_{i,j=0}^{\infty} \alpha_{i+1}^{(1)} \alpha_{j+1}^{(1)} \Big[\sum_{u,v=0}^{\infty} \alpha_{u+k}^{(k)} \alpha_{v+k}^{(k)} f_{(u+i,v+j)} \Big] \quad \text{(by admissibility of } F) \\ &= \sum_{i,j=0}^{\infty} \alpha_{i+k+1}^{(k+1)} \alpha_{j+k+1}^{(k+1)} f_{(i,j)} = g_{k+1}, \end{aligned}$$

which shows that $\{g_k\}$ is U-admissible. ∎

In obtaining the main result of this article, we will use an adaptation of the reduction of dimension procedure (see [K] or [B]), and then use the resulting one-parameter superadditive process. This procedure, although its main ingredients are well known in the literature, is new in the setting of admissible processes.

The following well known lemma is the key ingredient of the reduction of dimension method (see [K, Lemma 3.3, p. 213] for details):

LEMMA 3.3. *There exists a constant $C > 0$ such that*

$$\frac{1}{[\sqrt{n}+1]} \sum_{k=0}^{[\sqrt{n}+1]-1} \alpha_{k+u}^{(k)} \alpha_{k+v}^{(k)} > \frac{C}{n^2}$$

for $0 \le u, v < n$, where $[\sqrt{n}+1]$ is the integer part of $\sqrt{n}+1$.

THEOREM 3.4. *Let T and S be commuting positive L_1-contractions and $F = \{f_{(i,j)}\} \subset L_1$ be a positive strongly bounded (T,S)-admissible process. Then there exists a constant C, independent of the operators and the process, a positive L_1-contraction U and a strongly bounded U-admissible process $G = \{g_k\}$, with the same time constant as F, such that*

$$\frac{1}{n^2} F_{(n,n)} \le \frac{C}{k} \sum_{i=0}^{k-1} g_i \qquad (k = k(n,F)).$$

Proof. By Lemma 3.3, with $k = k(n,F) = [\sqrt{n}+1]$,

$$\frac{1}{[\sqrt{n}+1]} \sum_{k=0}^{[\sqrt{n}+1]-1} \sum_{i,j=0}^{\infty} \alpha_{k+i}^{(k)} \alpha_{k+j}^{(k)} f_{(i,j)} \geq \frac{1}{[\sqrt{n}+1]} \sum_{i,j=0}^{n-1} \sum_{k=0}^{[\sqrt{n}+1]-1} \alpha_{k+i}^{(k)} \alpha_{k+j}^{(k)} f_{(i,j)} \geq \frac{C}{n^2} F_{(n,n)},$$

where the constant C is independent of the operators T and S and F. The other assertions follow from Proposition 2.2. ∎

Remark. It follows from Propositions 3.1 and 3.2 that, if T and S are commuting mean ergodic Markovian operators on L_1 and F is a bounded (T,S)-admissible process, then G is also a bounded U-admissible process, where U is a mean ergodic Markovian operator on L_1. Furthermore, if T and S are L_∞-contractions, so is U.

4. Admissible processes relative to Markovian mean ergodic operators. In this section, the results of the previous section, in particular Theorem 3.4, will be utilized to obtain a.e. convergence of the averages of the (ordinary and weighted) multiparameter admissible processes.

We start with the maximal ergodic theorem for multiparameter admissible processes.

Theorem 4.1. *Let T and S be commuting mean ergodic Markovian operators on L_1 and $F = \{f_{(i,j)}\} \subset L_1$ be a positive bounded (T,S)-admissible process. Then, for any $\lambda > 0$,*

$$\mu\left\{x \in X : \limsup_n \frac{1}{n^2} F_{(n,n)} > \lambda\right\} \leq \frac{C}{\lambda} \gamma_F,$$

for some constant C that does not depend on F and the operators.

Proof. By Proposition 2.1, Proposition 3.1 and Theorem 3.4 there exists a constant C, independent of the operators, a mean ergodic Markovian operator U on L_1, and a bounded U-admissible process $G = \{g_k\}$, with the same time constant as F, such that

$$\frac{1}{n^2} F_{(n,n)} \leq \frac{C}{k} \sum_{i=0}^{k-1} g_i.$$

Since G has an exact dominant δ (with $\int \delta = \gamma_F$ [ASu]), it follows that

$$\frac{1}{n^2} \sum_{i,j=0}^{n-1} f_{(i,j)} \leq \frac{C}{k} \sum_{i=0}^{k-1} U^k \delta.$$

It is known that $\lim_n k^{-1} \sum_{i=0}^{k-1} U^k f$ exists a.e. for all $f \in L_1$ [ÇL$_1$, Theorem 1.5]. Hence, we have a maximal ergodic inequality for one-parameter additive processes relative to mean ergodic Markovian operators. Now, it follows easily from this maximal inequality that, for any $\lambda > 0$,

$$\mu\left\{x \in X : \limsup_n \frac{1}{n^2} F_{(n,n)} > \lambda\right\} \leq \frac{C}{\lambda} \gamma_F. \ \blacksquare$$

Almost everywhere convergence of the averages of the (ordinary and weighted) multiparameter admissible processes is a consequence of this maximal ergodic theorem. First, we will show the convergence of ordinary averages.

THEOREM 4.2. *Let T and S be commuting mean ergodic Markovian operators on L_1 and $F = \{f_{(i,j)}\} \subset L_1$ be a bounded (T,S)-admissible process. Then*

$$\lim_{n\to\infty} \frac{1}{n^2} F_{(n,n)} \quad \textit{exists a.e. and in } L_1.$$

Proof. Since the assertion of the theorem is true for additive processes ([ÇL$_1$, Theorems 2.8 and 2.2(i)]), we can assume that F is positive. By Theorem 2.2 in [Ç], for a given $\varepsilon > 0$, we can find a positive integer n_0 such that

$$\frac{1}{n_0^2} \int F_{(n_0,n_0)}\, d\mu > \gamma_F - \varepsilon.$$

Clearly the operators T^{n_0} and S^{n_0} are mean ergodic and Markovian (see also [ÇL$_1$]). Hence, the process $H = \{H_{(m,n)}\}$, where

$$H_{(m,n)} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} T^{n_0 i} S^{n_0 j} F_{(n_0,n_0)},$$

is a (T^{n_0}, S^{n_0})-additive process, so $n^{-2} H_{(n,n)}$ converges a.e. and in norm. For m and n large enough, let $m = kn_0 + r$ and $n = ln_0 + s$, where k, l, r and s are positive integers with $r, s < n_0$. Then, since T and S are Markovian, we obtain

$$\begin{aligned}\frac{1}{mn} \int H_{(m,n)}\, d\mu &\geq \frac{1}{mn} \int \sum_{i,j=0}^{k-1,l-1} T^{n_0 i} S^{n_0 j} F_{(n_0,n_0)}\, d\mu \\ &= \frac{kl}{mn} \int F_{(n_0,n_0)}\, d\mu > \gamma_F - \varepsilon.\end{aligned}$$

Therefore, the superadditive process $F' = F - H$ has time constant $\gamma_{F'} = \gamma_F - \gamma_H < \varepsilon$. First, we will obtain a.e. convergence. Indeed, if $f_* = \liminf_n n^{-2} F_{(n,n)}$ and $f^* = \limsup_n n^{-2} F_{(n,n)}$, then we see that, since $n^{-2} H_{(n,n)}$ converges a.e. as $n \to \infty$,

$$f^* - f_* \leq 2 \limsup_n \frac{1}{n^2} (F_{(n,n)} - H_{(n,n)}) \leq 2 \limsup_n \frac{1}{n^2} F'_{(n,n)}.$$

Therefore, by Theorem 4.1, for any $\lambda > 0$, if $E = \{x : f^* - f_* > \lambda\}$, then

$$\mu(E) \le \mu\Big\{x : \limsup_n \frac{1}{n^2} F'_{(n,n)}(x) > \lambda\Big\} \le \frac{C}{\lambda}\gamma_{F'} < \frac{C}{\lambda}\varepsilon.$$

Since ε is arbitrary, we have $\mu(E) = 0$, proving a.e. convergence.

For the norm convergence we will adopt the method of proof used in [ÇL$_2$]. By Theorem 3.4, $n^{-2}F_{(n,n)} \le Ck^{-1}\sum_{i=0}^{k-1} g_k$ for some strongly bounded U-admissible process $G = \{g_k\}$, where U is a mean ergodic Markovian operator. Any strongly bounded admissible process has an exact dominant; hence, by Theorem 2.2(i) in [ÇL$_1$] (one-parameter version), $k^{-1}\sum_{i=0}^{k-1} g_k$ converges in L_1-norm to a U-invariant function $g^* \in L_1$. By Propositions 2.1 and 2.3, $g^* \in F(T) \cap F(S)$ as well. If $C = \{g^* > 0\}$ and $D = X - C$, then $\|1_D n^{-2}F_{(n,n)}\|_1 \to 0$. Therefore, it is enough to prove the norm convergence on C. Since $g^* \in F(T) \cap F(S)$, $L_1(C)$ is both T- and S-invariant. Let $\widehat{T} = T|_{L_1(C)}$ and $\widehat{S} = S|_{L_1(C)}$, and define $\widehat{F}_{(m,n)} = 1_C F_{(m,n)}$. Then $\widehat{F} = \{\widehat{F}_{(m,n)}\}$ is bounded and $(\widehat{T}, \widehat{S})$-superadditive. Then $n^{-2}\widehat{F}_{(n,n)}$ converges in norm by Theorem 5.2 in [DeK], since the equivalent finite measure $g^*\,d\mu$ is invariant for $\widehat{T}$ and $\widehat{S}$. ■

Remark. Both Theorem 4.1 and Theorem 4.2 are proved in [ÇL$_2$] for bounded strongly superadditive processes (in the sense of [Sm]). However, the novelty here is that: (i) the maximal inequality needed for the proof is obtained via the dominating one-parameter *superadditive* process (which is constructed in Section 2) with the same time constant, (ii) the notion of superadditivity used here is less restrictive than the one in [ÇL$_2$].

Another feature of (T, S)-admissible processes is that, when T and S are *invertible* and power bounded (i.e., $\sup_{n\in\mathbb{Z}} \|T^n\| < \infty$ and $\sup_{n\in\mathbb{Z}} \|S^n\| < \infty$), then the process is always dominated.

Lemma 4.3. *Let T and S be positive commuting invertible power bounded operators on L_1, and let $F \subset L_1$ be a bounded (T, S)-admissible process. Then F has an exact dominant.*

Proof. We will use the idea of [Sa]. For any $A \in \Sigma$ with $\mu(A) < \infty$, we have $\{\int T^k\chi_A\,d\mu\}_{k\ge0} \in l_\infty$. Hence the Banach limit $\mathrm{Lim}(\int T^k\chi_A\,d\mu)$ exists. Define

$$m(A) = \mathrm{Lim}\Big(\int T^k\chi_A\,d\mu\Big).$$

For $A \in \Sigma$ with $\mu(A) = \infty$, define $m(A) = \lim_k m(A_k)$, where $\{A_k\} \subset \Sigma$ are such that $A_k \subset A_{k+1}$ for all $k \ge 1$ with $A_k \uparrow A$ and $\mu(A_k) < \infty$ for all $k \ge 1$. Then m defines a σ-finite measure on Σ satisfying $m \sim \mu$. The shift invariance of the Banach limit implies that T is a Markovian operator on $L_1(m)$ (see [Sa] for details). In the same manner, this time using the sequence

$\{\int S^k\chi_A dm\}_{k\geq 0} \in l_\infty$ and the Banach limit, we construct a σ-finite measure ν on Σ satisfying $\nu \sim m$ such that S is a Markovian operator on $L_1(\nu)$. Now, since T is Markovian on $L_1(m)$, for any $A \in \Sigma$ with $\mu(A) < \infty$,

$$\begin{aligned}\int T\chi_A\, d\nu = \nu(T\chi_A) &= \mathrm{Lim}\left[\int S^n(T\chi_A)\, dm\right] = \mathrm{Lim}\left[\int T(S^n\chi_A)\, dm\right]\\ &= \mathrm{Lim}\left[\int S^n\chi_A\, dm\right] = \nu(A) = \int \chi_A\, d\nu.\end{aligned}$$

Hence, by an approximation argument, it follows that T is also Markovian on $L_1(\nu)$. Since $\mu \sim \nu$, there exists a strictly positive measurable function p such that $\nu(A) = \int \chi_A p\, d\mu$. Also, $f' \in L_1(\nu)$ if and only if $pf' \in L_1(\mu)$, and the operators $\widehat{T}$ and $\widehat{S}$ in $L_1(\nu)$ corresponding to T and S in $L_1(\mu)$ are given by $\widehat{T}f' = p^{-1}T(pf')$ and $\widehat{S}f' = p^{-1}S(pf')$. Furthermore, $\int \widehat{T}f'\, d\nu = \int f'\, d\nu$ and $\int \widehat{S}f'\, d\nu = \int f'\, d\nu$. If $F' = \{f'_{i,j}\}$, where $f'_{i,j} = p^{-1}f_{i,j}$, then F' is a $(\widehat{T}, \widehat{S})$-admissible process. Let the sequence $\{v_{(i,j)}\}_{i,j\geq 0} \subset L_1$ be defined by $v_{(i,j)} = \widehat{T}^{-i}\widehat{S}^{-j}f'_{(i,j)}$. Then it follows that $v_{(i+1,j)} \geq v_{(i,j)}$ and $v_{(i,j+1)} \geq v_{(i,j)}$, and hence $v_{(i,j)} \uparrow v$ for some function $v \in L_1$. Furthermore, since $v_{(i,j)} \leq v$ for all $i, j \geq 0$, we have $f'_{(i,j)} \leq \widehat{T}^i\widehat{S}^j v$ for all $i, j \geq 0$. This also implies that $\|f'_{(i,j)}\|_{L_1(\nu)} \leq \|\widehat{T}^i\widehat{S}^j v\|_{L_1(\nu)} \uparrow \|v\|_{L_1(\nu)}$. Therefore, v is an exact dominant for F'. Now, $\|f_{(i,j)}\|_{L_1(\mu)} = \|f'_{(i,j)}\|_{L_1(\nu)}$ implies that

$$\sup_{i,j} \|f_{(i,j)}\|_{L_1(\mu)} = \sup_{i,j} \|f'_{(i,j)}\|_{L_1(\nu)} = \|v\|_{L_1(\nu)} = \|pv\|_{L_1(\mu)} < \infty.$$

Also,

$$f'_{(i,j)} \leq \widehat{T}^i\widehat{S}^j v = \frac{1}{p}\,[T^i(\widehat{S}^j v)p] = \frac{1}{p}\left[T^i\left(\frac{1}{p}(S^j(pv)\right)p\right] = \frac{1}{p}\,T^iS^j(pv).$$

Hence, $f_{(i,j)} = pf'_{(i,j)} \leq T^iS^j(pv)$, which proves that pv is an exact dominant for F. ∎

Now we are ready to prove a.e. convergence for (T, S)-admissible processes F, where T and S are positive commuting invertible operators:

Theorem 4.4. *Let T and S be commuting positive invertible power bounded mean ergodic operators on L_1 and $F = \{f_{(i,j)}\} \subset L_1$ be a strongly bounded (T, S)-admissible process. Then*

$$\lim_{n\to\infty} \frac{1}{n^2}\, F_{(n,n)} \quad \textit{exists a.e. and in } L_1.$$

Proof. Since T and S are commuting positive invertible power bounded mean ergodic operators on L_1, they can be viewed as commuting positive invertible mean ergodic contractions (in an equivalent norm). Then, by Theorem 2.8 in [ÇL$_1$], the averages $n^{-2}\sum_{i,j=0}^{n-1} T^iS^jf$ converge a.e. for all $f \in L_1$. Therefore the assertion of the theorem is true for additive processes, and hence, for the rest of the proof we can assume that F is positive. By Lemma 4.3 there exists $v \in L_1$ such that $f_{(i,j)} \le T^i S^j v$ for all $i, j \ge 0$. Hence $\sum_{i,j=0}^{m-1,n-1} f_{(i,j)} \le \sum_{i,j=0}^{m-1,n-1} T^i S^j v$. As observed in Lemma 4.3, we can view T and S as commuting mean ergodic Markovian operators (in an equivalent norm). Now, the process in Section 3 yields a positive mean ergodic operator U on L_1 and a U-additive process $G = \{U^k v\}$ which dominates F in the sense that

$$\frac{1}{n^2} F_{(n,n)} \le \frac{1}{n^2} \sum_{i,j=0}^{n-1,n-1} T^i S^j v \le \frac{C}{k} \sum_{k=0}^{[\sqrt{n}+1]-1} U^k v.$$

U is mean ergodic by Proposition 2.1. Hence Theorem 1.5 in [ÇL$_1$] implies that the averages $k^{-1} \sum_{i=0}^{k-1} U^k v$ converge a.e. Therefore we have a maximal inequality for $G = \{U^k v\}$. Now the assertion follows via the method of proof of Theorem 4.2. ∎

Remark. When T is a non-Markovian, not necessarily invertible mean ergodic operator, a bounded T-superadditive process need not have an exact dominant.

If T and S are commuting positive mean ergodic contractions (not necessarily invertible) then a (T,S)-superadditive process may have a dominant (see [AK]). In that case, following the same lines of proof as in Theorem 4.4 above and using Theorem 2.5 in [ÇL$_1$] a.e. convergence is obtained. We will state the theorem only.

Theorem 4.5. *Let T and S be commuting positive mean ergodic operators on L_1 and $F = \{f_{(i,j)}\} \subset L_1$ be a strongly bounded (T,S)-admissible process. If the dominating one-parameter process G has a dominant, then*

$$\lim_{n\to\infty} \frac{1}{n^2} F_{(n,n)} \quad \textit{exists a.e. and in } L_1.$$

Remark. A condition that implies the existence of an exact dominant for a given U-superadditive process G, where U is a positive L_1-contraction, is that

$$\lim_n \left\| \frac{1}{n} \sum_{k=1}^{n} (G_k - UG_{k-1}) \right\| < \infty \qquad \text{[BSu]}.$$

As observed by various authors (see [Sm]), however, this condition may not be easy to check for a given multiparameter superadditive process.

Next, we turn to weighted averages of multiparameter admissible processes and obtain the a.e. convergence of the averages

$$\frac{1}{n^2} \sum_{i,j=0}^{n-1} a_{ij} f_{(i,j)},$$

where $F = \{f_{(i,j)}\}$ is an admissible process relative to commuting Markovian operators T and S, which are also L_∞-contractions, and $\mathbf{a} = \{a_{ij}\}$ is a two-parameter bounded Besicovitch sequence. C. Ryll-Nardzewski initiated the study of almost everywhere convergence of weighted averages along (single parameter) bounded Besicovitch sequences [R-N]. The multiparameter case was studied by Jones–Olsen [JO]. Two-parameter bounded Besicovitch sequences are defined as $\{a_{ij}\} \in l_\infty$ for which there exists a sequence of (two-parameter) trigonometric polynomials ϕ_ε such that

$$\limsup_{m,n\to\infty} \frac{1}{mn} \sum_{i,j=0}^{m-1,n-1} |a_{ij} - \phi_\varepsilon(i,j)| < \varepsilon.$$

It is known that if T and S are commuting Dunford–Schwartz operators and $\mathbf{a}$ is a bounded Besicovitch sequence, then the weighted averages $n^{-2}\sum_{i,j=0}^{n-1} a_{ij}T^iS^jf$ converge a.e. for all $f \in L_1$ ([JO, Theorem 1.4]).

Theorem 4.6. *Let T and S be commuting Markovian Dunford–Schwartz operators on L_1 and $F = \{f_{(i,j)}\} \subset L_1$ be a bounded (T,S)-admissible process. If $\mathbf{a}$ is a bounded Besicovitch sequence, then*

$$\lim_n \frac{1}{n^2} \sum_{i,j=0}^{n-1} a_{ij} f_{(i,j)} \quad \textit{exists a.e.}$$

Proof. As in Theorem 4.2, since the result is valid for additive processes, we can assume that $f_{(i,j)} \geq 0$ for all $i,j \geq 0$. Now, for a fixed integer $m > 1$, define

$$g^m_{(i,j)}(x) = \begin{cases} T^{i-m}S^{j-m}f_{(m,m)}(x) & \text{for } i,j > m, \\ f_{(i,j)}(x) & \text{for } 0 \leq i \leq m \text{ or } 0 \leq j \leq m. \end{cases}$$

Let $h^m_{(i,j)}(x) = f_{(i,j)}(x) - g^m_{(i,j)}(x)$, which is nonnegative by admissibility. Then, for each m, the processes $G^m = \{g^m_{(i,j)}\}$ and $H^m = \{h^m_{(i,j)}\}$ are bounded (T,S)-admissible processes, where G^m is additive (ignoring first m-terms, if necessary). Clearly, $\gamma_{G^m} = \|f_{(m,m)}\| \leq \gamma_F$. Now, for $i,j > m$,

$$\begin{aligned} h^m_{(i,j)} &= f_{(i,j)} - T^{i-m}S^{j-m}f_{(m,m)} \\ &= \sum_{u=0}^{i-m-1} T^u(f_{(i-u,j)} - Tf_{(i-u-1,j)}) \\ &\quad + T^{i-m-1}\sum_{v=0}^{j-m-1} S^v(f_{(m,j-v)} - Sf_{(m,j-v-1)}). \end{aligned}$$

Therefore, $\|h^m_{(i,j)}\| = \|f_{(i,j)}\| - \|f_{(m,m)}\|$, which implies that $\gamma_{H^m} \le \gamma_F - \|f_{(m,m)}\|$. Observe that $\|f_{(m,m)}\| \uparrow \gamma_F$ as $m \to \infty$. Since $\mathbf{a} \in l_\infty$,

$$\left|\frac{1}{n^2}\sum_{i,j=0}^{n-1} a_{ij} h^m_{(i,j)}\right| \le \|\mathbf{a}\|_\infty \frac{1}{n^2}\sum_{i,j=0}^{n-1} h^m_{(i,j)}.$$

By Theorem 3.4, there exists a constant C independent of the operators, a Markovian operator U which is also an L_∞-contraction, and a U-admissible process $\{h_u\}$ with the same time constant as H^m, i.e. $\gamma_F - \|f_{(m,m)}\|$, such that $n^{-2}\sum_{i,j=0}^{n-1} h^m_{(i,j)} \le Ck^{-1}\sum_{u=0}^{k-1} h_u$. Hence, by Theorem 4.1, for any $\lambda > 0$,

$$\mu\left\{x : \limsup_n \frac{1}{n^2}\sum_{i,j=0}^{n-1} h^m_{(i,j)}(x) > \lambda\right\} \le \frac{C}{\lambda}\gamma_{H^m}.$$

Now, if

$$f_* = \liminf_n \frac{1}{n^2}\sum_{i,j=0}^{n-1} a_{ij} f_{(i,j)}, \qquad f^* = \limsup_n \frac{1}{n^2}\sum_{i,j=0}^{n-1} a_{ij} f_{(i,j)},$$

then, for any $m > 1$,

$$\begin{aligned}0 \le |f^* - f_*| &\le 2\limsup_n \left|\frac{1}{n^2}\sum_{i,j=0}^{n-1} a_{ij}(f_{(i,j)} - g^m_{(i,j)})\right| \\ &\le 2\|\mathbf{a}\|_\infty \limsup_n \frac{1}{n^2}\sum_{i,j=0}^{n-1} h^m_{(i,j)}.\end{aligned}$$

Therefore, for any $\lambda > 0$, if $E = \{x : |(f^* - f_*)(x)| > \lambda\}$, then

$$\mu(E) \le \mu\left\{x : \limsup_n \frac{1}{n^2}\sum_{i,j=0}^{n-1} h^m_{(i,j)}(x) > \frac{\lambda}{2\|\mathbf{a}\|_\infty}\right\} \le \frac{C\|\mathbf{a}\|_\infty}{\lambda}\gamma_{H^m}.$$

Since $\gamma_{H^m} \downarrow 0$ as $m \to \infty$, we have $\mu(E) = 0$, proving the assertion. ■

Remark. The assertion of Theorem 4.6 is also valid if $\mathbf{a} \in l_\infty$ is any sequence which is good a.e. for (multiparameter) additive processes.

Theorem 4.7. *Let T and S be commuting L_1-contractions whose moduli $|T|$, $|S|$ are mean ergodic and commute. If* $\mathbf{a}$ *is a bounded Besicovitch sequence*, *then* $\lim_n n^{-2}\sum_{i,j=0}^{n-1} a_{ij}T^iS^jf$ *exists a.e. for every* $f \in L_1$.

Proof. It was proved in [ÇL$_1$] that under the assumptions of the theorem the averages $\lim_n n^{-2}\sum_{i,j=0}^{n-1} T^iS^jf$ converge a.e. for every $f \in L_1$. We can assume, if necessary by a change of measure, that μ is finite. Since $|\lambda T| = |T|$ and $|\lambda S| = |S|$ for any $|\lambda| = 1$, by Theorem 2.8 in [ÇL$_1$] we see that, if $|\lambda_1| = 1$, and $|\lambda_2| = 1$, then $\lim_n n^{-2}\sum_{i,j=0}^{n-1} \lambda_1\lambda_2 T^iS^jf$ exists a.e.

for every $f \in L_\infty$. Hence, by linearity, the same averages converge a.e. with λ_i replaced by (two-parameter) trigonometric polynomials. It follows from the definition of bounded Besicovitch sequences that the assertion is valid if $f \in L_\infty$. Since the Brunel operator U is mean ergodic by Proposition 3.1, $\lim_n n^{-1} \sum_{k=0}^{n-1} U^k f$ converges a.e. for any $f \in L_1$. Hence we have a maximal inequality as in Theorem 4.6, and the result follows from the Banach Principle. ∎

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Received 25 February 2004;
revised 1 February 2005 (4432)