COLLOQUIUM MATHEMATICUM

VOL. 136 2014 NO. 2

HAAR WAVELETS ON THE LEBESGUE SPACES OF LOCAL FIELDS OF POSITIVE CHARACTERISTIC

BY

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Abstract. We construct the Haar wavelets on a local field K of positive characteristic and show that the Haar wavelet system forms an unconditional basis for $L^p(K)$, $1 < p < \infty$. We also prove that this system, normalized in $L^p(K)$, is a democratic basis of $L^p(K)$. This also proves that the Haar system is a greedy basis of $L^p(K)$ for $1 < p < \infty$.

1. Introduction. The concept of unconditional convergence is one of the classical topics of analysis. Usually, wavelets with some smoothness properties form unconditional bases for the Lebesgue spaces. Meyer [16] proved that wavelets with polynomial decay form unconditional bases for $L^p(\mathbb{R})$. Similar results with weaker hypotheses were proved in [9, 11, 25]. For the proofs of the unconditionalilty of the Haar basis for $L^p(\mathbb{T})$ and $L^p(\mathbb{R})$ we refer to [23] and [24], respectively.

The concept of wavelets can be generalized to many different setups. Dahlke [8] introduced this concept on locally compact abelian groups. This was generalized to abstract Hilbert spaces by Han, Larson, Papadakis and Stavropoulos [10]. Lemarié [15] extended the original concept to stratified Lie groups. J. Benedetto and R. Benedetto [6] developed a wavelet theory for local fields and related groups.

A field K equipped with a topology is called a local field if both the additive and multiplicative groups of K are locally compact abelian groups. Local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). Local fields of characteristic zero include the p-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin p-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and MRA (multiresolution analysis) theories are quite different.

In [14] Lang proved that if a wavelet ψ for the Cantor dyadic group G satisfies the property $|\psi(x) - \psi(y)| \leq C|x - y|$ for some constant C and for

2010 *Mathematics Subject Classification*: Primary 42C40; Secondary 42C15, 43A70, 11S85.
Key words and phrases: wavelet, multiresolution analysis, local field, unconditional basis, democratic basis, greedy basis.

DOI: 10.4064/cm136-2-1

all $x, y \in G$, then the corresponding wavelet basis is unconditional in $L^p(G)$ for $1 < p < \infty$. Chuong and Duong [7] proved that the Haar system is an unconditional basis for $L^r(\mathbb{Q}_p)$, $1 < r < \infty$.

In this article, we construct the analogue of Haar wavelets on a local field of positive characteristic. For some aspects of wavelet theory on such fields, we refer to [1, 2, 3, 4, 5, 12].

The article is organized as follows. In Section 2, we recall some basic facts about local fields and define a multiresolution analysis on a local field K of positive characteristic. As an example, we construct the Haar wavelets. In Section 3, we discuss the convergence properties of wavelet expansions with respect to the Haar wavelets. In particular, we show that the projections $P_j f$ of $f \in L^p(K)$ onto the resolution spaces V_j of the Haar MRA converge to f in the L^p-norm. We also discuss pointwise convergence properties of the projection operators. In Section 4, we prove that the Haar wavelets form an unconditional basis for the Lebesgue spaces $L^p(K)$, $1 < p < \infty$. In the last section, we show that the normalized Haar wavelet system is a greedy basis of $L^p(K)$ for $1 < p < \infty$.

2. Preliminaries on local fields. Let K be a field and a topological space. Then K is called a *locally compact field* or a *local field* if both K^+ and K^* are locally compact abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K respectively.

If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. So, by a local field we mean a field K which is locally compact, nondiscrete and totally disconnected.

We use the notation of the book [20] by Taibleson. Proofs of all the results stated in this section can be found in [20] and [18].

Let K be a local field. Since K^+ is a locally compact abelian group, we choose a Haar measure dx for K^+. If $\alpha \neq 0$, $\alpha \in K$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the *absolute value* or *valuation* of α. We also let $|0| = 0$.

The map $x \mapsto |x|$ has the following properties:

(a) $|x| = 0$ if and only if $x = 0$;
(b) $|xy| = |x|\,|y|$ for all $x, y \in K$;
(c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the *ultrametric inequality*. It follows that

$$|x + y| = \max\{|x|, |y|\} \quad \text{if } |x| \neq |y|. \tag{2.1}$$

The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. It is the unique maximal compact subring of K. Define $\mathfrak{P} = \{x \in K : |x| < 1\}$. The set $\mathfrak{P}$ is called the *prime ideal* in K. Since K is totally disconnected, the set of values $|x|$ as x varies over K is a discrete set of the form $\{s^k : k \in \mathbb{Z}\} \cup \{0\}$ for some $s > 0$. Hence, there is an element of $\mathfrak{P}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{P}$. Such an element is called a *prime element* of K. Note that as an ideal in $\mathfrak{D}, \mathfrak{P} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$.

It can be proved that $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{P}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{P}$ is isomorphic to a finite field $\mathrm{GF}(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. For a proof of this fact we refer to [20].

For a measurable subset E of K, let $|E| = \int_K \chi_E(x)\,dx$, where χ_E is the characteristic function of E and dx is the Haar measure of K normalized so that $|\mathfrak{D}| = 1$. Then it is easy to see that $|\mathfrak{P}| = q^{-1}$ and $|\mathfrak{p}| = q^{-1}$ (see [20]). It follows that if $x \neq 0$, and $x \in K$, then $|x| = q^k$ for some $k \in \mathbb{Z}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{P} = \{x \in K : |x| = 1\}$. If $x \neq 0$, we can write $x = \mathfrak{p}^k x'$, with $x' \in \mathfrak{D}^*$. Let $\mathfrak{P}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in K : |x| \leq q^{-k}\}$, $k \in \mathbb{Z}$. These are called *fractional ideals*. Each $\mathfrak{P}^k$ is compact and open and is a subgroup of K^+ (see [18]).

If K is a local field, then there is a nontrivial, unitary, continuous character χ on K^+. It can be proved that K^+ is self-dual (see [20]). Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is nontrivial on $\mathfrak{P}^{-1}$. We can find such a character by starting with any nontrivial character and rescaling. We will define such a character for a local field of positive characteristic. For $y \in K$, we define $\chi_y(x) = \chi(yx)$, $x \in K$.

DEFINITION 2.1. If $f \in L^1(K)$, then the *Fourier transform* of f is the function $\hat{f}$ defined by

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}\,dx.$$

Similarly to the standard Fourier analysis on the real line, one can prove the following results:

(a) The map $f \mapsto \hat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and $\|\hat{f}\|_\infty \leq \|f\|_1$.
(b) If $f \in L^1(K)$, then $\hat{f}$ is uniformly continuous.
(c) If $f \in L^1(K) \cap L^2(K)$, then $\|\hat{f}\|_2 = \|f\|_2$.

To define the Fourier transform of a function in $L^2(K)$, we introduce the functions Φ_k. For $k \in \mathbb{Z}$, let Φ_k be the characteristic function of $\mathfrak{P}^k$. For $f \in L^2(K)$, let $f_k = f\Phi_{-k}$ and define

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\leq q^k} f(x)\,\overline{\chi_\xi(x)}\,d\xi,$$

where the limit is taken in $L^2(K)$. It turns out that the Fourier transform is unitary on $L^2(K)$ (see Theorem 2.3 in [20]).

A set of the form $h + \mathfrak{P}^k$ will be called a *sphere* with centre h and radius q^{-k}. It follows from the ultrametric inequality that if S and T are two spheres in K, then either S and T are disjoint or one sphere contains the other.

Let χ_u be any character on K^+. Since $\mathfrak{D}$ is a subgroup of K^+, the restriction $\chi_u|_{\mathfrak{D}}$ is a character on $\mathfrak{D}$. Also, as characters on $\mathfrak{D}$, $\chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. That is, $\chi_u = \chi_v$ if $u + \mathfrak{D} = v + \mathfrak{D}$ and $\chi_u \neq \chi_v$ if $(u+\mathfrak{D}) \cap (v+\mathfrak{D}) = \emptyset$. Hence, if $\{u(n)\}_{n=0}^{\infty}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K^+, then $\{\chi_{u(n)}\}_{n=0}^{\infty}$ is a list of distinct characters on $\mathfrak{D}$. It is proved in [20] that this list is complete. That is, we have the following proposition.

Proposition 2.2. *Let $\{u(n)\}_{n=0}^{\infty}$ be a complete list of (distinct) coset representatives of $\mathfrak{D}$ in K^+. Then $\{\chi_{u(n)}\}_{n=0}^{\infty}$ is a complete list of (distinct) characters on $\mathfrak{D}$. Moreover, it is a complete orthonormal system on $\mathfrak{D}$.*

Given such a list of characters $\{\chi_{u(n)}\}_{n=0}^{\infty}$, we define the *Fourier coefficients* of $f \in L^1(\mathfrak{D})$ as

$$\hat{f}(u(n)) = \int_{\mathfrak{D}} f(x)\, \overline{\chi_{u(n)}(x)}\, dx.$$

The series $\sum_{n=0}^{\infty} \hat{f}(u(n))\chi_{u(n)}(x)$ is called the *Fourier series* of f. From the standard L^2-theory for compact abelian groups we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\int_{\mathfrak{D}} |f(x)|^2\, dx = \sum_{n=0}^{\infty} |\hat{f}(u(n))|^2.$$

Also, if $f \in L^1(\mathfrak{D})$ and $\hat{f}(u(n)) = 0$ for all $n = 0, 1, 2, \ldots$, then $f = 0$ a.e.

These results hold irrespective of the ordering of the characters. We now proceed to impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$.

Note that $\mathfrak{D}/\mathfrak{P}$ is isomorphic to the finite field $\mathrm{GF}(q)$ and $\mathrm{GF}(q)$ is a c-dimensional vector space over the field $\mathrm{GF}(p)$. We choose a set $\{1 = \epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{c-1}\} \subset \mathfrak{D}^*$ such that $\mathrm{span}\{\epsilon_j\}_{j=0}^{c-1} \cong \mathrm{GF}(q)$.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}_0$ such that $0 \le n < q$, we have

$$n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, \quad 0 \le a_k < p,\ k = 0, 1, \ldots, c-1.$$

Define

$$u(n) = (a_0 + a_1\epsilon_1 + \cdots + a_{c-1}\epsilon_{c-1})\mathfrak{p}^{-1}. \tag{2.2}$$

Note that $\{u(n) : n = 0, 1, \ldots, q-1\}$ is a complete set of coset representatives of $\mathfrak{D}$ in $\mathfrak{P}^{-1}$. Now, for $n \in \mathbb{N}_0$, write

$$n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s, \quad 0 \le b_k < q,\ k = 0, 1, \ldots, s,$$

and set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.3}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But it follows that

$$u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s) \quad \text{if } r \geq 0,\ k \geq 0 \text{ and } 0 \leq s < q^k.$$

In the following proposition we list some properties of $\{u(n) : n \in \mathbb{N}_0\}$ which will be used later. For a proof, we refer to [2].

Proposition 2.3. *For $n \in \mathbb{N}_0$, let $u(n)$ be defined as in* (2.2) *and* (2.3). *Then:*

(a) *$u(n) = 0$ if and only if $n = 0$. If $k \geq 1$, then $|u(n)| = q^k$ if and only if $q^{k-1} \leq n < q^k$.*
(b) *$\{u(k) : k \in \mathbb{N}_0\} = \{-u(k) : k \in \mathbb{N}_0\}$.*
(c) *For a fixed $l \in \mathbb{N}_0$, we have $\{u(l) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$.*

For brevity, we will write $\chi_n = \chi_{u(n)}$ for $n \in \mathbb{N}_0$. As mentioned before, $\{\chi_n : n \in \mathbb{N}_0\}$ is a complete set of characters on $\mathfrak{D}$.

Let K be a local field of characteristic $p > 0$ and $\epsilon_0, \epsilon_1, \ldots, \epsilon_{c-1}$ be as above. We define a character χ on K as follows (see [27]):

$$\chi(\epsilon_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \neq 1. \end{cases}$$

Note that χ is trivial on $\mathfrak{D}$ but nontrivial on $\mathfrak{P}^{-1}$.

The following result, proved in [1], will be used later.

Proposition 2.4. *For $0 \leq r, s \leq q-1$,*

$$\frac{1}{q} \sum_{t=0}^{q-1} \chi\big((u(r) - u(s))\mathfrak{p}u(t)\big) = \delta_{r,s}.$$

In order to be able to define the concepts of multiresolution analysis and wavelet on local fields, we need analogous notions of translation and dilation. Since $\bigcup_{j\in\mathbb{Z}} \mathfrak{p}^{-j}\mathfrak{D} = K$, we can regard $\mathfrak{p}^{-1}$ as the dilation (note that $|\mathfrak{p}^{-1}| = q$), and since the set $\Lambda = \{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K, it can be treated as the translation set. Note that it follows from Proposition 2.3 that the translation set Λ is a subgroup of K^+ even though it is indexed by $\mathbb{N}_0$.

For $f \in L^2(K)$, $j \in \mathbb{Z}$, and $k \in \mathbb{N}_0$, we define

$$f_{j,k}(x) = q^{j/2} f(\mathfrak{p}^{-j}x - u(k)), \quad j \in \mathbb{Z},\ k \in \mathbb{N}_0.$$

It is easy to see that

$$(f_{j,k})^\wedge(\xi) = q^{-j/2}\, \overline{\chi_k(\mathfrak{p}^j \xi)}\, \hat{f}(\mathfrak{p}^j \xi).$$

A finite set $\{\psi^m : m = 1, 2, \ldots, M\} \subset L^2(K)$ is called a *set of basic wavelets* of $L^2(K)$ if the system $\{\psi^m_{j,k} : 1 \le m \le M,\ j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ forms an orthonormal basis for $L^2(K)$. Similarly to $\mathbb{R}^n$, wavelets can be constructed from a multiresolution analysis which we define below (see [12]).

DEFINITION 2.5. Let K be a local field of characteristic $p > 0$, $\mathfrak{p}$ be a prime element of K and $u(n)$, $n \in \mathbb{N}_0$, be as defined in (2.2) and (2.3). A *multiresolution analysis* (MRA) of $L^2(K)$ is a sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(K)$ satisfying the following properties:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$;
(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(d) $f \in V_j$ if and only if $f(\mathfrak{p}^{-1}\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
(e) there is a function $\varphi \in V_0$, called the *scaling function*, such that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms an orthonormal basis for V_0.

Given such an MRA, as in the case of $\mathbb{R}^n$, we can find a set $\{\psi^l : 1 \le l \le q-1\}$ of basic wavelets consisising of $q-1$ functions. These functions are called the *wavelets associated with the scaling function* φ.

EXAMPLE 2.6. Let $\varphi = \chi_{\mathfrak{D}}$. Define $V_j = \overline{\operatorname{span}}\{\varphi\big(\mathfrak{p}^{-j}\cdot - u(k)\big) : k \in \mathbb{N}_0\}$. Then $\{V_j : j \in \mathbb{Z}\}$ forms an MRA of $L^2(K)$. This will be called the *Haar MRA*. Observe that

$$\varphi(x) = \sum_{k=0}^{q-1} \varphi(\mathfrak{p}^{-1}x - u(k)) = \sum_{k=0}^{q-1} q^{-1/2}\varphi_{1,k}(x).$$

Taking Fourier transform, we get

$$\hat{\varphi}(\xi) = q^{-1}\sum_{k=0}^{q-1} \overline{\chi_k(\mathfrak{p}\xi)}\,\hat{\varphi}(\mathfrak{p}\xi) = m_0(\mathfrak{p}\xi)\hat{\varphi}(\mathfrak{p}\xi),$$

where $m_0(\xi) = q^{-1}\sum_{k=0}^{q-1}\overline{\chi_k(\xi)}$.

A function f on K is said to be *integral-periodic* if $f(x+u(k)) = f(x)$ for every $k \in \mathbb{N}_0$. It can easily be verified that m_0 is integral-periodic. As in the case of $\mathbb{R}^n$, it can be shown that if we can find integral-periodic functions m_i, $1 \le i \le q-1$, such that the matrix

$$M(\xi) = [m_i(\xi + \mathfrak{p}u(j))]_{i,j=0}^{q-1}$$

is unitary for a.e. $\xi \in \mathfrak{D}$, then $\{\psi^1, \ldots, \psi^{q-1}\}$ forms a set of basic wavelets for $L^2(K)$, where

$$\hat{\psi}^i(\xi) = m_i(\mathfrak{p}\xi)\varphi(\mathfrak{p}\xi).$$

We define

$$\psi^i(x) = \sum_{j=0}^{q-1} a_{ij}\varphi_{1,j}(x), \qquad 1 \le i \le q-1, \tag{2.4}$$

where $A = (a_{ij})_{i,j=0}^{q-1}$ is an arbitrary unitary matrix such that $a_{0j} = q^{-1/2}$ for $0 \le j \le q-1$. Note that $\operatorname{supp}\psi^i \subseteq \mathfrak{D}$.

We claim that $\{\psi^1, \dots, \psi^{q-1}\}$ is a set of basic wavelets. Observe that we can write

$$\hat{\psi}^i(\xi) = m_i(\mathfrak{p}\xi)\varphi(\mathfrak{p}\xi),$$

where

$$m_i(\xi) = \sum_{j=0}^{q-1} a_{ij} q^{-1/2}\, \overline{\chi_j(\xi)}.$$

To prove the claim, we need to show that $M(\xi)$ is unitary for a.e. $\xi \in \mathfrak{D}$.

Let $E(\xi) = (q^{-1/2}\chi_0(\xi), q^{-1/2}\chi_1(\xi), \dots, q^{-1/2}\chi_{q-1}(\xi))$ and X_i denote the ith row vector of the unitary matrix A. Then $\{X_0, X_1, \dots, X_{q-1}\}$ is an orthonormal basis of $\mathbb{C}^q$ and $m_i(\xi) = \langle X_i, E(\xi)\rangle_{\mathbb{C}^q}$. Therefore, the inner product of the l_1th and l_2th columns of the matrix $M(\xi)$ is

$$\begin{aligned}
&\sum_{k=0}^{q-1} m_k(\xi + \mathfrak{p}u(l_1))\, \overline{m_k(\xi + \mathfrak{p}u(l_2))} \\
&\qquad = \sum_{k=0}^{q-1} \langle X_k, E(\xi + \mathfrak{p}u(l_1))\,\rangle \overline{\langle X_k, E(\xi + \mathfrak{p}u(l_1))\rangle} \\
&\qquad = \langle E(\xi + \mathfrak{p}u(l_1)), E(\xi + \mathfrak{p}u(l_1))\rangle \\
&\qquad = q^{-1} \sum_{k=0}^{q-1} \chi_k(\xi + \mathfrak{p}u(l_1))\, \overline{\chi_k(\xi + \mathfrak{p}u(l_2))} \\
&\qquad = q^{-1} \sum_{k=0}^{q-1} |\chi_k(\xi)|^2 \chi_k(\mathfrak{p}(u(l_1) - u(l_2)) \\
&\qquad = q^{-1} \sum_{k=0}^{q-1} \chi\big(u(k)\mathfrak{p}(u(l_1) - u(l_2))\big) = \delta_{l_1,l_2}.
\end{aligned}$$

The last equality follows from Proposition 2.4. This shows that the column vectors of $M(\xi)$ form an orthonormal basis of $\mathbb{C}^q$. Hence, $M(\xi)$ is unitary, which proves the claim.

The wavelets constructed above are the analogue of the Haar wavelets on $\mathbb{R}^n$. We will call φ the *Haar scaling function* and the corresponding wavelets will be called the *Haar wavelets*. We would like to point out that the expression for the Haar wavelets given in [12] is not correct. In fact, they are not even orthogonal to each other.

An example of a unitary matrix A with a constant first row is the following. Let $a_{0j} = q^{-1/2}$ for $0 \le j \le q-1$. For $1 \le i \le q-1$, define

$$a_{ij} = \begin{cases} [(q-i)(q-i+1)]^{-1/2}, & j = 0, 1, \ldots, q-i-1, \\ -(q-i)[(q-i)(q-i+1)]^{-1/2}, & j = q-i, \\ 0, & j > q-i. \end{cases}$$

In the rest of the article, φ will denote the Haar scaling function and ψ^l, $1 \le l \le q-1$, the Haar wavelets.

3. Pointwise convergence of the wavelet expansions. We start with an elementary lemma on the properties of the Haar scaling function φ which will be useful in simplifying the expression for the projection operators defined later.

LEMMA 3.1. *Let φ be the Haar scaling function. Then:*

(a) $\sum_{k\in\mathbb{N}_0} \varphi(x-u(k))\varphi(y-u(k)) = \varphi(x-y)$ *for every* $x, y \in K$;

(b) $\int_K q^j \varphi(\mathfrak{p}^{-j}(x-y))\, dx = 1$.

Proof. (a) Recall that $\{u(n) + \mathfrak{D} : n \in \mathbb{N}_0\}$ is a disjoint collection. If $x, y \in u(k) + \mathfrak{D}$ for some k, then $\varphi\big(x - u(k)\big) = 1 = \varphi\big(y - u(k)\big)$ and $\varphi\big(x - u(l)\big) = 0 = \varphi\big(y - u(l)\big)$ for all $l \ne k$. So the left side equals 1. We have $x = u(k) + z_1$ and $y = u(k) + z_2$, where $z_1, z_2 \in \mathfrak{D}$. So $|x-y| = |z_1 - z_2| \le \max\{|z_1|, |z_2|\} \le 1$. Hence, the right side is also 1.

If $x \in u(k) + \mathfrak{D}$ and $y \in u(l) + \mathfrak{D}$, where $k \ne l$, then the left side is 0. Since $x = u(k) + z_1$ and $y = u(l) + z_2$ for some $z_1, z_2 \in \mathfrak{D}$, we have $x - y = \big(u(k) - u(l)\big) + (z_1 - z_2) = u(m) + z$ for some $m \in \mathbb{N}_0$ and $z \in \mathfrak{D}$, by Proposition 2.3. Since $u(k) \ne u(l)$, we have $u(m) \ne 0$. Hence $x - y \notin \mathfrak{D}$ since $\{u(k) + \mathfrak{D}\}$ is a disjoint collection. So the right side is also 0.

(b) Follows by a change of variables. ∎

Let $\{V_j : j \in \mathbb{Z}\}$ be the Haar MRA. We define the projection operators $P_j : L^2(K) \to V_j$ by

$$P_j f = \sum_{k\in\mathbb{N}_0} \langle f, \varphi_{j,k}\rangle \varphi_{j,k}.$$

We use Lemma 3.1 to get an integral representation of P_j as follows:

$$\begin{aligned} P_j f(x) &= \sum_{k\in\mathbb{N}_0} \Big\{ \int_K f(t) q^{j/2} \varphi(\mathfrak{p}^{-j} t - u(k))\, dt \Big\} q^{j/2} \varphi(\mathfrak{p}^{-j} x - u(k)) \\ &= \int_K q^j \Big\{ \sum_{k\in\mathbb{N}_0} \varphi(\mathfrak{p}^{-j} t - u(k) \varphi(\mathfrak{p}^{-j} x - u(k)) \Big\} f(t)\, dt \\ &= \int_K q^j \varphi(\mathfrak{p}^{-j}(t-x)) f(t)\, dt, \end{aligned}$$

by Lemma 3.1(a). The projection operators Q_j onto the wavelet subspaces $W_j = V_{j+1} \ominus V_j$ are defined by

$$Q_j f(x) = \sum_{l=1}^{q-1} \sum_{k\in\mathbb{N}_0} \langle f, \psi^l_{j,k}\rangle \psi^l_{j,k}(x).$$

Since $W_j = V_{j+1} \ominus V_j$, it is clear that $Q_j = P_{j+1} - P_j$. The following proposition shows that these operators are bounded on each L^p-space.

PROPOSITION 3.2. *Let $1 \le p \le \infty$. For all $f \in L^2(K) \cap L^p(K)$, we have*

(a) $\|P_j f\|_p \le \|f\|_p$.
(b) $\|Q_j f\|_p \le 2\|f\|_p$.

Proof. We observe that

$$|P_j f(x)| \le \int_K q^j |\varphi(\mathfrak{p}^{-j}(x-y))|\,|f(y)|\,dy \le \|f\|_\infty \int_K q^j \varphi(\mathfrak{p}^{-j}(x-y))\,dy = \|f\|_\infty.$$

Hence, $\|P_j f\|_\infty \le \|f\|_\infty$. Now,

$$\begin{aligned}\|P_j f\|_1 &= \int_K \Big|\int_K q^j \varphi(\mathfrak{p}^{-j}(x-y)) f(y)\,dy\Big|\,dx \\ &\le \int_K |f(y)| \Big(\int_K q^j \varphi(\mathfrak{p}^{-j}(x-y))\,dx\Big)\,dy = \int_K |f(y)|\,dy = \|f\|_1.\end{aligned}$$

Therefore, by the Riesz–Thorin interpolation theorem, $\|P_j f\|_p \le \|f\|_p$ for $1 \le p \le \infty$. Item (b) follows from (a) since $Q_j = P_{j+1} - P_j$. ∎

Thus, P_j and Q_j can be extended to $L^p(K)$ for $1 \le p \le \infty$. We now prove the convergence of $P_j f$ in L^p-norm.

THEOREM 3.3.

(a) *If $1 \le p < \infty$ and $f \in L^p(K)$, then $\lim_{j\to\infty} \|P_j f - f\|_p = 0$.*
(b) *$\lim_{j\to\infty} \|P_j f - f\|_\infty = 0$ for all bounded uniformly continuous functions f on K.*

Proof. (a) Suppose that $1 \le p < \infty$. Then

$$\begin{aligned}|P_j f(x) - f(x)| &= \Big|\int_K q^j \varphi(\mathfrak{p}^{-j}(x-y)) f(y)\,dy - f(x)\Big| \\ &= \Big|\int_K q^j \varphi(\mathfrak{p}^{-j}(x-y))(f(y) - f(x))\,dy\Big| \\ &\le \int_K q^j \varphi(\mathfrak{p}^{-j}(x-y))|f(y) - f(x)|\,dy \\ &= \int_K \varphi(t)|f(x - \mathfrak{p}^j t) - f(x)|\,dt.\end{aligned}$$

The second equality follows by Lemma 3.1(b). We now apply Minkowski's inequality for integrals to obtain

$$\begin{aligned}
(3.1)\qquad \|P_jf - f\|_p &\le \Big[\int_K\Big|\int_K \varphi(t)|f(x-\mathfrak{p}^jt) - f(x)|\,dt\Big|^p dx\Big]^{1/p} \\
&\le \Big[\int_K\Big(\int_K \varphi(t)|f(x-\mathfrak{p}^jt) - f(x)|\,dt\Big)^p dx\Big]^{1/p} \\
&\le \int_K\Big[\int_K |\varphi(t)|^p|f(x-\mathfrak{p}^jt) - f(x)|^p\,dx\Big]^{1/p} dt \\
&= \int_K \varphi(t)\|f(\cdot-\mathfrak{p}^jt) - f\|_p\,dt.
\end{aligned}$$

In a locally compact abelian group G, for $1\le p<\infty$, $\|f(\cdot+y) - f\|_p\to 0$ as $y\to 0$ in G (see [19, Theorem 1.1.5]). Since $|\mathfrak{p}^jt| = q^{-j}|t|\to 0$ as $j\to\infty$, the integrand in (3.1) tends to zero as $j\to\infty$. This integrand is dominated by $2\|f\|_p\varphi$, which is in $L^1(K)$. Hence, by the Lebesgue dominated convergence theorem, we obtain the result.

To prove (b), let $f\in L^\infty(K)$ be uniformly continuous. Then $w_j(t) = \sup_{x\in K}|f(x-\mathfrak{p}^jt) - f(x)|\to 0$ as $j\to\infty$ (uniformly in x). Therefore $|P_jf(x) - f(x)|\le\int_K\varphi(t)w_j(t)\,dt\to 0$ as $j\to\infty$ (uniformly in x), again by the dominated convergence theorem. ■

THEOREM 3.4.

(a) *If* $1\le p<\infty$ *and* $f\in L^p(K)$, *then* $\lim_{j\to-\infty}P_jf(x) = 0$ *for every* $x\in K$.
(b) *If* f *is locally integrable in* K, *then* $\lim_{j\to\infty}P_jf(x) = f(x)$ *for a.e.* $x\in K$.

Proof. (a) Let p' be the index conjugate to p. By Hölder's inequality, we have

$$\begin{aligned}
|P_jf(x)| &\le \int_K q^j\varphi(\mathfrak{p}^{-j}(x-y))|f(y)|\,dy \\
&\le q^j\Big(\int_K|\varphi(\mathfrak{p}^{-j}(x-y))|^{p'}\,dy\Big)^{1/p'}\Big(\int_K|f(y)|^p\,dy\Big)^{1/p} \\
&= q^jq^{-j/p'}\|f\|_p \quad \text{(by Lemma 3.1(b))} \\
&= q^{j/p}\|f\|_p\to 0 \quad \text{as } j\to-\infty.
\end{aligned}$$

(b) Let f be locally integrable in K. Then it follows from the Lebesgue differentiation theorem (see [20, Theorem 1.14, p. 29]) that for a.e. $x\in K$, we have

$$\lim_{j\to\infty} P_j f(x) = \lim_{j\to\infty} \int_K q^j \varphi(\mathfrak{p}^{-j}(x-y)) f(y)\,dy$$
$$= \lim_{j\to\infty} q^j \int_{|x-y|\le q^{-j}} f(y)\,dy = f(x). \ \blacksquare$$

4. Unconditionality of the Haar wavelets on $L^p(K)$. We first recall some basic facts on unconditional bases on a Banach space. We refer to Chapter 5 of [11] for the details.

Let X be a Banach space. A series $\sum_{n\in\mathbb{N}} x_n$ in X is said to *converge unconditionally* to y in X if $\sum_{n\in\mathbb{N}} x_{\sigma(n)}$ converges to y in X for every permutation σ of $\mathbb{N}$.

A sequence $\{x_n : n \in \mathbb{N}\}$ is called a *Schauder basis* of X if for each $x \in X$ there exists a unique sequence $\{\alpha_n(x) : n \in \mathbb{N}\}$ of scalars such that $x = \sum_{n\in\mathbb{N}} \alpha_n(x) x_n$ in X. If this convergence is unconditional, then the basis is said to be an *unconditional basis*.

Let $\{x_n : n \in \mathbb{N}\}$ be a Schauder basis for a Banach space X, so that for every $x \in X$, there is a unique sequence $\{f_n(x) : n \in \mathbb{N}\}$ of scalars such that $x = \sum_{n\in\mathbb{N}} f_n(x) x_n$. Given a sequence $\beta = \{\beta_n : n \in \mathbb{N}\}$ of scalars, we define the operator $T_\beta : X \to X$ by

$$T_\beta(x) = \sum_{n\in\mathbb{N}} \beta_n f_n(x) x_n.$$

There are many ways to verify whether a basis is unconditional. We will use the following criterion. We refer to [11] for its proof.

THEOREM 4.1. *For a basis $\{x_n : n \in \mathbb{N}\}$ of a Banach space X, the following statements are equivalent:*

(a) *$\{x_n : n \in \mathbb{N}\}$ is an unconditional basis for X.*
(b) *There exists a constant $C > 0$ such that $\|T_\beta(x)\| \le C\|x\|$ for all $\beta = \{\beta_n : n \in \mathbb{N}\}$ with $|\beta_n| \le 1$ for all n.*
(c) *There exists a constant $C > 0$ such that $\|T_\beta(x)\| \le C\|x\|$ for all $\beta = \{\beta_n : n \in \mathbb{N}\}$ with $\beta_n = \pm 1$ for all n.*
(d) *There exists a constant $C > 0$ such that $\|T_\beta(x)\| \le C\|x\|$ for all finitely nonzero sequences $\beta = \{\beta_n : n \in \mathbb{N}\}$ with $\beta_n = 1$ or 0 for all n.*

Note that $L^p(K)\cap L^2(K)$ is dense in $L^p(K)$ for $1 < p < \infty$ and the system of Haar wavelets is an orthonormal basis for $L^2(K)$. Hence, it follows from the above theorem that to show that $\{\psi^l_{j,k} : 1 \le l \le q-1,\ j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ forms an unconditional basis for $L^p(K)$, it is enough to show the uniform boundedness of the operators T_β on $L^p(K)$, where

$$T_\beta f = \sum_{l=1}^{q-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \beta^l_{j,k} \langle f, \psi^l_{j,k}\rangle \psi^l_{j,k}$$

for all sequences $\beta = \{\beta^l_{j,k}\}$ with the property that $\beta^l_{j,k} = 1$ for a finite number of indices and $\beta^l_{j,k} = 0$ for the remaining indices.

In other words, $\{\psi^l_{j,k} : 1 \le l \le q-1,\ j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ forms an unconditional basis for $L^p(K)$ if and only if the operators $T_F : L^p(K) \to L^p(K)$ defined by

$$T_F f = \sum_{(l,j,k)\in F} \langle f, \psi^l_{j,k}\rangle \psi^l_{j,k} \tag{4.1}$$

are uniformly bounded on $L^p(K)$ where F varies over finite subsets of $\{1, \dots, q-1\} \times \mathbb{Z} \times \mathbb{N}_0$.

The main ingredient needed to show this is a local field version of Calderón–Zygmund decomposition proved by Phillips [17]. We refer to [20, p. 148] for the proof of the following theorem.

Theorem 4.2 (Calderón–Zygmund decomposition for local fields). *Given $f \in L^1(K)$ with $f \ge 0$ and $\alpha > 0$, there exists a countable collection of mutually disjoint spheres $\{S_k\}$ and a decomposition $f = g + b$, with $b = \sum_k b_k$, such that*

(a) $|\Omega| = \sum_k |S_k| < \frac{1}{\alpha}\|f\|_1$, *where* $\Omega = \cup_k S_k$;
(b) $|f(x)| \le \alpha$ *for a.e.* $x \notin \Omega$;
(c) $g(x) = f(x)$, $x \notin \Omega$;
(d) $|g(x)| \le q\alpha$ *for a.e.* $x \in \Omega$;
(e) $b(x) = 0$, $x \notin \Omega$;
(f) $\operatorname{supp} b_k \subset S_k$ *and* $\int_{S_k} b_k(x)\,dx = \int_{S_k} b(x)\,dx = 0$ *for all* k.

The proof of Calderón–Zygmund decomposition also implies the following useful facts:

(1) $b \in L^1(K)$ and $\|b\|_1 \le 2\|f\|_1$;
(2) $g \in L^1(K)$ and $\|g\|_1 = \|f\|_1$;
(3) $g \in L^\infty(K)$ and $\|g\|_\infty \le q\alpha$;
(4) $g \in L^2(K)$ and $\|g\|_2 \le (q\alpha\|f\|_1)^{1/2}$.

We are now ready to state and prove the main result of this section.

Theorem 4.3. *Let $1 < p < \infty$ and ψ^l, $1 \le l \le q-1$, be the Haar wavelets. Then $\{\psi^l_{j,k} : 1 \le l \le q-1,\ j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ forms an unconditional basis for $L^p(K)$.*

Proof. As we discussed earlier, to prove this theorem it is enough to show that the operators T_F are uniformly bounded on each $L^p(K)$, $1 < p < \infty$, where T_F is defined in (4.1).

We first show that T_F is of weak type $(1,1)$. That is, we have to show that there exists a constant $C > 0$ such that

$$|\{x \in K : |T_F f(x)| > \alpha\}| \le \frac{C}{\alpha}\|f\|_1 \quad \text{for all } \alpha > 0 \text{ and } f \in L^1(K).$$

Without loss of generality, we can assume that $f \ge 0$. Applying Calderón–Zygmund decomposition, we have

$$\begin{aligned}|\{x \in K : |T_F f(x)| > \alpha\}| &\le |\{x \in K : |T_F g(x)| > \alpha/2\}| \\ &\quad + |\{x \in K : |T_F b(x)| > \alpha/2\}| \\ &= \mathrm{I} + \mathrm{II}, \quad \text{say.}\end{aligned}$$

We first estimate I. We have

$$\begin{aligned}\mathrm{I} = \int\limits_{\{x\in K:\, |T_F g(x)|>\alpha/2\}} dx &\le \frac{4}{\alpha^2}\int\limits_K |T_F g(x)|^2\,dx \\ &\le \frac{4}{\alpha^2}\int\limits_K |g(x)|^2\,dx \le \frac{4q}{\alpha}\|f\|_1,\end{aligned} \tag{4.2}$$

since $\|g\|_2^2 \le q\alpha\|f\|_1$.

To estimate II, we first show that $T_F b(x) = 0$ for all $x \notin \Omega = \bigcup_k S_k$. For such an x, we have

$$\begin{aligned}T_F b_i(x) &= \sum_{(l,j,k)\in F} \langle b_i, \psi^l_{j,k}\rangle \psi^l_{j,k}(x) = \sum_{(l,j,k)\in F} \Big[\int\limits_K b_i(y)\psi^l_{j,k}(y)\,dy\Big]\psi^l_{j,k}(x) \\ &= \sum_{(l,j,k)\in F} \Big[\int\limits_{S_i} b_i(y)\psi^l_{j,k}(y)\,dy\Big]\psi^l_{j,k}(x).\end{aligned}$$

Note that $\operatorname{supp} b_i \subseteq S_i = h_i + \mathfrak{P}^{m_i} = h_i + \mathfrak{p}^{m_i}\mathfrak{D}$ for some integer m_i. Since $\operatorname{supp}\psi^l \subseteq \mathfrak{D}$, it follows that $\operatorname{supp}\psi^l_{j,k} \subseteq \mathfrak{p}^j(u(k)+\mathfrak{D}) = S_{j,k}$, say. Observe that $|S_i| = q^{-m_i}$ and $|S_{j,k}| = q^{-j}$.

We deduce from the ultrametric inequality that in a local field, any two spheres are either disjoint or one sphere contains the other. In view of this we consider three cases.

Case 1. If $S_i \cap S_{j,k} = \emptyset$, then $T_F b_i(x) = 0$. So, $T_F b(x) = \sum_i T_F b_i(x) = 0$.

Case 2. If $S_{j,k} \subset S_i$, then $x \notin S_{j,k}$ since $x \notin S_i$. Hence, $\psi^l_{j,k}(x) = 0$. So $T_F b(x) = 0$.

Case 3. If $S_i \subset S_{j,k}$, then we can assume that $S_i \ne S_{j,k}$ (otherwise we can apply Case 2 to conclude $\psi^l_{j,k}(x) = 0$). We claim that $\psi^l_{j,k}$ is constant on S_i. Note that

$$\psi^l_{j,k}(x) = q^{j/2} \sum_{m=0}^{q-1} a_{lm} q^{1/2} \varphi\big(\mathfrak{p}^{-1}(\mathfrak{p}^{-j}x - u(k)) - u(m)\big)$$
$$= q^{(j+1)/2} \sum_{m=0}^{q-1} a_{lm} \chi_{E_m}(x),$$

where $E_m = \mathfrak{p}^{j+1}\mathfrak{D} + \mathfrak{p}^j u(k) + \mathfrak{p}^{j+1}u(m)$. To prove the claim, it is enough to show that S_i is contained entirely in E_m for some $m = 1, \dots, q-1$. We assume that this is not the case and get a contradiction.

First, we note that since $S_i \subset S_{j,k}$ and $S_i \neq S_{j,k}$, it follows that $|S_i| < |S_{j,k}|$. That is, $q^{-m_i} < q^{-j}$.

Now, suppose that there exist $y_1, y_2 \in S_i$ such that $y_1 \in E_m$ and $y_2 \in E_n$ so that $y_1 = \mathfrak{p}^{j+1}d_1 + \mathfrak{p}^j u(k) + \mathfrak{p}^{j+1}u(m)$ and $y_2 = \mathfrak{p}^{j+1}d_2 + \mathfrak{p}^j u(k) + \mathfrak{p}^{j+1}u(n)$ for some $d_1, d_2 \in \mathfrak{D}$. Hence,

$$y_1 - y_2 = \mathfrak{p}^{j+1}(d_1 - d_2) + \mathfrak{p}^{j+1}(u(m) - u(n)) = \mathfrak{p}^{j+1}(d_1 - d_2) + \mathfrak{p}^{j+1}u(r)$$

for some integer $r > 0$. Now, $|\mathfrak{p}^{j+1}(d_1 - d_2)| = |\mathfrak{p}^{j+1}|\,|d_1 - d_2| \le q^{-j-1}$ and $|\mathfrak{p}^{j+1}u(r)| = q^{-j-1}|u(r)| \ge q^{-j}$ since $|u(r)| \ge q$ if $r \ge 1$ (see Proposition 2.3(a)). By (2.1), it follows that

$$|y_1 - y_2| = \max\{|\mathfrak{p}^{j+1}(d_1 - d_2)|, |\mathfrak{p}^{j+1}u(m)|\} \ge q^{-j}.$$

On the other hand, since $y_1, y_2 \in S_i$, we can write $y_1 = h_i + \mathfrak{p}^{m_i}v_1$ and $y_2 = h_i + \mathfrak{p}^{m_i}v_2$ for some $v_1, v_2 \in \mathfrak{D}$. Hence $|y_1 - y_2| = |\mathfrak{p}^{m_i}(v_1 - v_2)| \le q^{-m_i}$. From these two calculations we obtain $q^{-j} \le q^{-m_i}$. This contradicts the fact that $q^{-m_i} < q^{-j}$. So the claim that $\psi^l_{j,k}$ is constant on S_i is proved. Hence,

$$\int_{S_i} b_i(y)\psi^l_{j,k}(y)\,dy = C \int_{S_i} b_i(y)\,dy = 0,$$

by Theorem 4.2(f). This shows that $T_F b(x) = 0$ for all $x \notin \Omega$. By Theorem 4.2(a), we have

$$\text{II} = |\{x \in K : |T_F b(x)| > \alpha/2\}| \le |\Omega| \le \tfrac{1}{\alpha}\|f\|_1. \tag{4.3}$$

Combining (4.2) and (4.3), we get $|\{x \in K : |T_F f(x)| > \alpha\}| \le \frac{C}{\alpha}\|f\|_1$, where $C = 4q + 1$. Note that C is independent of F. This completes the proof of the fact that T_F is of weak type $(1,1)$.

Since $\{\psi^l_{j,k} : 1 \le l \le q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ forms an orthonormal basis for $L^2(K)$, we have $\|T_F f\|_2 \le \|f\|_2$, $f \in L^2(K)$. So T_F is of strong type $(2,2)$. Hence, by the Marcinkiewicz interpolation theorem, it follows that $\|T_F f\|_p \le C\|f\|_p$, $f \in L^p(K)$, for $1 < p \le 2$. Since the operator T_F is its own adjoint, it follows from duality that $\|T_F f\|_p \le C\|f\|_p$, $f \in L^p(K)$, for $2 \le p < \infty$. This completes the proof of Theorem 4.3. ∎

We now state a lemma which is just the quantitative information contained in the unconditionality of a basis for an L^p-space. The proof follows from Khinchin's inequality (see, e.g., Corollary 7.11 in [24]).

LEMMA 4.4. *Let $\{f_n : n \in \mathbb{N}\}$ be an unconditional basis for $L^p(X, d\mu)$, where μ is a σ-finite positive measure on X. Then there exist constants c_1 and c_2 with $0 < c_1 \le c_2 < \infty$ such that*

$$c_1 \Big\| \sum_{n\in\mathbb{N}} a_n f_n \Big\|_p \le \Big\| \Big(\sum_{n\in\mathbb{N}} |a_n|^2 |f_n|^2 \Big)^{1/2} \Big\|_p \le c_2 \Big\| \sum_{n\in\mathbb{N}} a_n f_n \Big\|_p$$

for every scalar sequence $\{a_n : n \in \mathbb{N}\}$.

Applying this lemma to the unconditional basis $\{\psi^l_{j,k} : 1 \le l \le q-1,\ j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ of $L^p(K)$, we get the following result.

THEOREM 4.5. *For each p, $1 < p < \infty$, there exist constants c_p and C_p with $0 < c_p \le C_p < \infty$ such that*

$$c_p \|f\|_p \le \Big\| \Big(\sum_{l=1}^{q-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle f, \psi^l_{j,k}\rangle|^2 |\psi^l_{j,k}|^2 \Big)^{1/2} \Big\|_p \le C_p \|f\|_p$$

for all $f \in L^p(K)$.

The above theorem tells us that

$$f \in L^p(K) \quad \text{if and only if} \quad \Big(\sum_{l=1}^{q-1} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle f, \psi^l_{j,k}\rangle|^2 |\psi^l_{j,k}|^2 \Big)^{1/2} \in L^p(K).$$

So it can be viewed as a characterization of $L^p(K)$ in terms of the wavelet coefficients $\langle f, \psi^l_{j,k}\rangle$.

5. Greedy basis property in $L^p(K)$. In this section we will show that the normalized Haar wavelets form a greedy basis of $L^p(K)$ for $1 < p < \infty$. The concept of greedy basis originated from the study of best m-term approximation in Banach spaces.

Let X be a Banach space and $\mathcal{B} = \{x_n : n \in \mathbb{N}\}$ be a normalized Schauder basis of X, so that for each $x \in X$, there is a unique sequence $\{c_n(x) : n \in \mathbb{N}\}$ of scalars such that $x = \sum_{n\in\mathbb{N}} c_n(x) x_n$.

The *best m-term approximation* of $x \in X$ with respect to the basis $\mathcal{B}$ is defined to be

$$\sigma_m(x) := \inf \Big\| x - \sum_{k\in\Lambda} c_k x_k \Big\|_X,$$

where the infimum is taken over scalars c_k and sets of indices Λ with cardinality m.

A computationally efficient method to obtain m-term approximations is the so-called greedy algorithm. For $x \in X$, let ρ be a permutation of $\mathbb{N}$ such that

$$|c_{\rho(1)}| \geq |c_{\rho(2)}| \geq \cdots .$$

The *mth greedy approximation* of x with respect to the basis $\mathcal{B}$ corresponding to the permutation ρ is defined to be

$$G_m(x,\rho) := \sum_{n=1}^{m} c_{\rho(n)}(x) x_{\rho(n)}.$$

It is clear that $\sigma_m(x) \leq \|x - G_m(x,\rho)\|_X$. A basis $\mathcal{B}$ is called a *greedy basis* if the reverse inequality holds up to a constant, that is, there exists a permutation ρ of $\mathbb{N}$ such that

$$\|x - G_m(x,\rho)\|_X \leq C\sigma_m(x)$$

for some constant C independent of x and m.

We will need the notion of another type of basis. A normalized basis $\{x_n : n \in \mathbb{N}\}$ in a Banach space X is called a *democratic basis* of X if there exists a constant C such that

$$\Big\|\sum_{n\in A} x_n\Big\|_X \leq C\Big\|\sum_{n\in B} x_n\Big\|_X$$

for any finite subsets A and B with the same cardinality.

It was proved by Konyagin and Temlyakov in [13] that a basis in a Banach space is greedy if and only if it is unconditional and democratic. In the same paper, it was also shown that the last two properties are independent of each other. For a detailed account of this subject we refer to the book [22].

Temlyakov [21] proved that the Haar basis (and any wavelet system L^p-equivalent to it) is greedy in the Lebesgue spaces $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. On the other hand, Wojtaszczyk [26] proved that the Haar system is not greedy in rearrangement invariant spaces other than L^p.

We now consider the Haar wavelets $\psi^l_{j,k}$, $1 \leq l \leq q-1$, $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$, defined in Example 2.6. Let $h^l_{j,k} = \psi^l_{j,k}/\|\psi^l_{j,k}\|_p$ be the Haar wavelets normalized in $L^p(K)$.

THEOREM 5.1. *For each p, $1 < p < \infty$, the normalized Haar wavelets $\{h^l_{j,k} : 1 \leq l \leq q-1,\ j \in \mathbb{Z},\ k \in \mathbb{N}_0\}$ form a greedy basis of $L^p(K)$.*

Proof. Since we have already proved that the Haar wavelets form an unconditional basis of $L^p(K)$, we just have to show that the system of normalized Haar wavelets is a democratic basis of $L^p(K)$.

Suppose that A and B are two finite subsets of $\{1,\ldots,q-1\} \times \mathbb{Z} \times \mathbb{N}_0$ such that $|A| = |B|$. Let

$$f = \sum_{(l,j,k)\in A} h^l_{j,k} \quad \text{and} \quad g = \sum_{(l,j,k)\in B} h^l_{j,k}.$$

Note that if $(l,j,k)\notin A$, then $\langle f,\psi^l_{j,k}\rangle=0$. If $(l,j,k)\in A$, then

$$\langle f,\psi^l_{j,k}\rangle\psi^l_{j,k}=\langle h^l_{j,k},\psi^l_{j,k}\rangle\psi^l_{j,k}=\|\psi^l_{j,k}\|_p^{-1}\psi^l_{j,k}=h^l_{j,k}.$$

Hence, by Theorem 4.5, we have

$$c_p\|f\|_p\le\Big\|\Big(\sum_{(l,j,k)\in A}|h^l_{j,k}|^2\Big)^{1/2}\Big\|_p\le C_p\|f\|_p. \tag{5.1}$$

Similarly,

$$c_p\|g\|_p\le\Big\|\Big(\sum_{(l,j,k)\in B}|h^l_{j,k}|^2\Big)^{1/2}\Big\|_p\le C_p\|g\|_p. \tag{5.2}$$

Let $\alpha_l=\sum_{m=0}^{q-1}|a_{lm}|^p$. Then a simple calculation shows that $\|\psi^l_{jk}\|_p=q^{(j+1)(1/2-1/p)}\alpha_l^{1/p}$. Hence,

$$h^l_{j,k}=\left(\frac{q^{j+1}}{\alpha_l}\right)^{1/p}\sum_{m=0}^{q-1}a_{lm}\chi_{E_{mjk}},$$

where $E_{mjk}=\mathfrak{p}^{j+1}(\mathfrak{D}+\mathfrak{p}^{-1}u(k)+u(m))$ so that supp $h^l_{j,k}=\bigcup_{m=0}^{q-1}E_{mjk}$ $=C_{jk}$, say. Note that C_{jk} is contained in the sphere $\mathfrak{P}^j+\mathfrak{p}^ju(k)$. Observe that for each $x\in K$, $|h^l_{j,k}(x)|$ is either 0 or $(q^{j+1}/\alpha_l)^{1/p}|a_{lm}|$ for some $m\in\{0,1,\dots,q-1\}$.

Let $\mathcal{C}=\bigcup_{(l,j,k)\in A}C_{jk}$. For $x\in\mathcal{C}$, let

$$\max_{(l,j,k)\in A}|h^l_{j,k}(x)|=\left(\frac{q^{j(x)+1}}{\alpha_{l(x)}}\right)^{1/p}|a_{l(x),m(x)}|.$$

Since two spheres in a local field are either disjoint or one contains the other, it follows that C_{j_1,k_1} and C_{j_2,k_2} are disjoint if $j_1=j_2$ as they are of same size. So the above maximum occurs for a unique $j(x)$ such that $(l,j(x),k)\in A$. Hence, for any other $(l,j,k)\in A$, $|h^l_{j,k}(x)|^p$ is either 0 or of the form $(q^{r+1}/\alpha_l)|a_{lm}|^p$ for some l and r, where $r<j(x)$. Moreover, each such positive value of $|h^l_{j,k}(x)|$ can be obtained at most once. Since the indices l and m vary over finite sets, we can find a constant C independent of l and m such that

$$\frac{q^{r+1}}{\alpha_l}|a_{lm}|^p\le C\frac{q^{r+1}}{\alpha_{l(x)}}|a_{l(x),m(x)}|^p.$$

For any $x\in\mathcal{C}$, we now have

$$\begin{aligned}\sum_{(l,j,k)\in A}|h^l_{j,k}(x)|^p&\le C\frac{q^{j(x)+1}}{\alpha_{l(x)}}|a_{l(x),m(x)}|^p(1+q^{-1}+q^{-2}+q^{-3}+\cdots)\\&=C\frac{q}{q-1}\max_{(l,j,k)\in A}|h^l_{j,k}(x)|^p.\end{aligned}$$

Hence,

$$\Big(\sum_{(l,j,k)\in A}|h^l_{j,k}(x)|^2\Big)^{p/2}\geq \max_{(l,j,k)\in A}|h^l_{j,k}(x)|^p\geq \frac{q-1}{Cq}\sum_{(l,j,k)\in A}|h^l_{j,k}(x)|^p.$$

Therefore,

$$\Big[\int_K\Big(\sum_{(l,j,k)\in A}|h^l_{j,k}(x)|^2\Big)^{p/2}dx\Big]^{1/p}\geq\Big(\frac{q-1}{Cq}\Big)^{1/p}\Big(\sum_{(l,j,k)\in A}\int_K|h^l_{j,k}(x)|^p\,dx\Big)^{1/p}$$
$$=\Big(\frac{q-1}{Cq}\Big)^{1/p}|A|^{1/p},$$

since $\|h^l_{j,k}\|_p=1$. Substituting in (5.1), we get

$$\Big(\frac{Cq}{q-1}\Big)^{1/p}C_p\|f\|_p\geq|A|^{1/p}. \tag{5.3}$$

Arguing as above, for each $x\in\mathcal{C}$, we have

$$\sum_{(l,j,k)\in B}|h^l_{j,k}(x)|^2\leq C(q)\max_{(l,j,k)\in B}|h^l_{j,k}(x)|^2.$$

Hence,

$$\Big(\sum_{(l,j,k)\in B}|h^l_{j,k}(x)|^2\Big)^{p/2}\leq C(q)^{p/2}\max_{(l,j,k)\in B}|h^l_{j,k}(x)|^p$$
$$\leq C(q)^{p/2}\sum_{(l,j,k)\in B}|h^l_{j,k}(x)|^p.$$

Therefore,

$$\Big[\int_K\Big(\sum_{(l,j,k)\in B}|h^l_{j,k}(x)|^2\Big)^{p/2}dx\Big]^{1/p}\leq\Big[\int_K C(q)^{p/2}\sum_{(l,j,k)\in B}|h^l_{j,k}(x)|^p\,dx\Big]^{1/p}$$
$$=C(q)^{p/2}|B|^{1/p}.$$

Substituting in (5.2), we get

$$\|g\|_p\leq c_p^{-1}C(q)^{p/2}|B|^{1/p}. \tag{5.4}$$

Since $|A|=|B|$, from (5.3) and (5.4), we have $\|g\|_p\leq K_p\|f\|_p$. This completes the proof of Theorem 5.1. ∎

REFERENCES

[1] B. Behera and Q. Jahan, *Wavelet packets and wavelet frame packets on local fields of positive characteristic*, J. Math. Anal. Appl. 395 (2012), 1–14.

[2] B. Behera and Q. Jahan, *Multiresolution analysis on local fields and characterization of scaling functions*, Adv. Pure Appl. Math. 3 (2012), 181–202.

[3] B. Behera and Q. Jahan, *Biorthogonal wavelets on local fields of positive characteristic*, Comm. Math. Anal. 15 (2013), 52–75.

[4] B. Behera and Q. Jahan, *Affine, quasi-affine and coaffine frames on local fields of positive characteristic*, preprint, 2013.

[5] B. Behera and Q. Jahan, *Characterization of wavelets and MRA wavelets on local fields of positive characteristic*, Collect. Math., to appear.

[6] J. Benedetto and R. Benedetto, *A wavelet theory for local fields and related groups*, J. Geom. Anal. 14 (2004), 423–456.

[7] N. Chuong and D. Duong, *Wavelet bases in the Lebesgue spaces on the field of p-adic numbers*, p-Adic Numbers Ultrametric Anal. Appl. 5 (2013), 106–121.

[8] S. Dahlke, *Multiresolution analysis and wavelets on locally compact abelian groups*, in: Wavelets, Images, and Surface Fitting, P. J. Laurent et al. (eds.), A K Peters, 1994, 141–156.

[9] G. Gripenberg, *Wavelet bases in $L_p(\mathbb{R})$*, Studia Math. 106 (1993), 175–187.

[10] D. Han, D. R. Larson, M. Papadakis and Th. Stavropoulos, *Multiresolution analyses of abstract Hilbert spaces and wandering subspaces*, in: Contemp. Math. 247, Amer. Math. Soc., 1999, 259–284.

[11] E. Hernández and G. Weiss, *A first course on wavelets*, CRC Press, 1996.

[12] H. Jiang, D. Li and N. Jin, *Multiresolution analysis on local fields*, J. Math. Anal. Appl. 294 (2004), 523–532.

[13] S. Konyagin and V. Temlyakov, *A remark on greedy approximation in Banach spaces*, East J. Approx. 5 (1999), 365–379.

[14] W. Lang, *Wavelet analysis on the Cantor dyadic group*, Houston J. Math. 24 (1998), 533–544.

[15] P. Lemarié, *Base d'ondelettes sur les groupes de Lie stratifiés*, Bull. Soc. Math. France 117 (1989), 211–232.

[16] Y. Meyer, *Wavelets and Operators*, Cambridge Univ. Press, 1992.

[17] K. Phillips, *Hilbert transforms for the p-adic and p-series fields*, Pacific J. Math. 23 (1967), 329–347.

[18] D. Ramakrishnan and R. Valenza, *Fourier Analysis on Number Fields*, Springer, 1999.

[19] W. Rudin, *Fourier Analysis on Groups*, Interscience, 1962.

[20] M. H. Taibleson, *Fourier Analysis on Local Fields*, Princeton Univ. Press, 1975.

[21] V. Temlyakov, *The best m-term approximation and greedy algorithms*, Adv. Comput. Math. 8 (1998), 249–265.

[22] V. Temlyakov, *Greedy Approximation*, Cambridge Univ. Press, 2011.

[23] P. Wojtaszczyk, *Banach Spaces for Analysts*, Cambridge Univ. Press, 1991.

[24] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, Cambridge Univ. Press, 1997.

[25] P. Wojtaszczyk, *Wavelets as unconditional bases in $L_p(\mathbb{R})$*, J. Fourier Anal. Appl. 5 (1999), 73–85.

[26] P. Wojtaszczyk, *Greediness of the Haar system in rearrangement invariant spaces*, in: Approximation and Probability, T. Figiel and A. Kamont (eds.), Banach Center Publ. 72, Inst. Math., Polish Acad. Sci., Warszawa, 2006, 385–395.

[27] S. Zheng, *Riesz type kernels over the ring of integers of a local field*, J. Math. Anal. Appl. 208 (1997), 528–552.

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Received 12 December 2013 (6105)