COLLOQUIUM MATHEMATICUM

VOL. 136 2014 NO. 2

FULLY INERT SUBMODULES OF TORSION-FREE MODULES OVER THE RING OF p-ADIC INTEGERS

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Abstract. Fully inert submodules of torsion-free J_p-modules are investigated. It is proved that if the module considered is either free or complete, these submodules are exactly those which are commensurable with fully invariant submodules; examples are given of torsion-free J_p-modules for which this property fails.

1. Introduction. A subgroup H of an Abelian group G is *fully inert* if, given any endomorphism ϕ of G, H has finite index in $\phi H + H$. The class of fully inert subgroups includes finite subgroups, subgroups of finite index, and fully invariant subgroups. The motivation for studying fully inert subgroups comes from the investigation of the dynamical properties of an endomorphism of an Abelian group (see [4], [6] and [12]).

In two recent papers, [5] and [7], fully inert subgroups of divisible Abelian groups and of free Abelian groups have been characterized. The groups in these two classes have the pleasant property that their endomorphism rings contain many projections, which allows an easy characterization of their fully invariant subgroups and, consequently, facilitates a characterization of their fully inert subgroups. It is worth noting that divisible torsion-free groups of finite rank have plenty of fully inert subgroups which fail to be commensurable with fully invariant subgroups.

The definition of fully inert subgroup can be transferred to modules over the ring J_p of p-adic integers, since the torsion J_p-modules are exactly the p-groups. The goal of this paper is to investigate fully inert submodules of torsion-free J_p-modules. An advantage of dealing with these modules is that if they are reduced, then they are separable in the sense that their finitely generated submodules are contained in free summands of finite rank. Furthermore, their fully invariant submodules are, with one exception, precisely their own multiples (see Proposition 2.2 below).

Fully inert submodules of free and of complete J_p-modules are described in Sections 2 and 3, respectively. In both cases, they are exactly the sub-

2010 *Mathematics Subject Classification*: Primary 13C05; Secondary 13C10.
Key words and phrases: fully inert submodules, free J_p-modules, complete J_p-modules.

DOI: 10.4064/cm136-2-2

modules commensurable with the fully invariant submodules. They are also characterized by the fact that their cokernels are bounded p-groups with at most one infinite Ulm–Kaplansky invariant, plus an additional finiteness condition in case they are not of finite index in the containing module.

In the final Section 4, examples are provided of torsion-free J_p-modules containing fully inert submodules which fail to be commensurable with any fully invariant submodule.

For unexplained notions of Abelian group theory, we refer to [8] and [1].

2. Fully inert submodules of free J_p-modules. Let G be a J_p-module. A submodule H of G is said to be *fully inert* if, for any endomorphism ϕ of G, the J_p-module $(H+\phi H)/H$ is finite.

Let A, B be submodules of the J_p-module G. We say that A, B are *commensurable* if $(A+B)/A$ and $(A+B)/B$ are both finite. It is easily seen that commensurability is an equivalence relation (cf. [7, Lemma 2.3]).

The following easy fact was proved, for Abelian groups, in [5, Corollary 2.10], and the argument there is adaptable to J_p-modules. We give a direct proof, for the sake of completeness.

PROPOSITION 2.1. *Let G be a J_p-module, Z a fully invariant submodule of G, and H a submodule commensurable with Z. Then H is fully inert.*

Proof. By hypothesis, both $(H+Z)/Z$ and $(H+Z)/H$ are finitely generated torsion J_p-modules (i.e., p-groups), say $(H+Z)/Z = \langle a_i + Z : i \leq n\rangle$ and $(H+Z)/H = \langle b_i + H : i \leq n\rangle$ (we take the same number of generators to simplify the notation). Then $H \subseteq \langle a_i, Z : i \leq n\rangle$. Let ϕ be an arbitrary endomorphism of G. Since Z is fully invariant, we get $\phi H \subseteq \langle \phi(a_i), Z : i \leq n\rangle$. Now we note that every $\phi(a_i) + H$ is a torsion element of G/H; in fact, $p^m a_i \in Z$ and $p^m Z \subseteq H$ for some $m \geq 0$ sufficiently large, hence $p^{2m}\phi(a_i) \in p^m \phi Z \subseteq p^m Z \subseteq H$. From the relation

$$\begin{aligned}(H+\phi H)/H &\subseteq \langle a_i + H, \phi(a_i) + H, (Z+H)/H : i \leq n\rangle \\ &= \langle \phi(a_i) + H, b_i + H : i \leq n\rangle\end{aligned}$$

we conclude that $(H+\phi H)/H$ is a submodule of a finitely generated torsion J_p-module, hence it is finite. Since ϕ was arbitrary, H is fully inert. ∎

The next result shows that the fully invariant submodules of a torsion-free J_p-module are the obvious ones; this result is well known—see, for example, [11, Exercise 72(b)].

PROPOSITION 2.2. *A nonzero submodule N of a torsion-free J_p-module M is fully invariant if and only if $N = p^k M$ for some nonnegative k or $N = D$, the maximal divisible submodule of M.*

The proof of the next proposition follows *verbatim* [7, proof of Proposition 2.4].

PROPOSITION 2.3. *For a nonzero submodule H of a free J_p-module A of finite rank n, the following are equivalent:*

(1) *H is fully inert in A;*
(2) *H has rank n;*
(3) *H has finite index in A;*
(4) *H is commensurable with some fully invariant submodule of A.*

Also the next result has its parallel in [7, Theorem 2.8].

THEOREM 2.4. *Let A be a free J_p-module of infinite rank, and let H be a submodule of A. Then H is fully inert in A if and only if H is commensurable with $p^k A$ for some $k \geq 0$.*

Proof. The proof goes as that of [7, Theorem 2.8], where the Cohen–Gluck theorem [2] is used. That theorem is valid for free modules over arbitrary PIDs, hence, in particular, for J_p-modules. Actually, the final part of that proof, where the case of two integers, none of which divides the other, is considered, is not needed, since J_p is a valuation domain. ∎

Now we characterize the fully inert submodules H of a free J_p-module A by means of the J_p-modules A/H and $(H + pA)/pA$.

THEOREM 2.5. *Let A be a free J_p-module of infinite rank, and let H be a submodule of A. Then H is fully inert in A if and only if A/H is a bounded p-group with at most one infinite Ulm–Kaplansky invariant and $(H + pA)/pA$ is finite in case A/H is infinite.*

Proof. Sufficiency. We prove that H is commensurable with $p^n A$ for some $n \geq 0$. If all the Ulm–Kaplansky invariants of the bounded p-group A/H are finite, then A/H is finite and H is commensurable with A itself, so it is fully inert by Proposition 2.1. If the nth Ulm–Kaplansky invariant of A/H is infinite, say equal to α, then there exists a submodule K such that $H \subseteq K \subseteq A$, K/H is finite, and $A/K \cong \bigoplus_\alpha \mathbb{Z}(p^n)$, hence $K \supseteq p^n A$. As commensurability is an equivalence relation and H is commensurable with K, it is enough to prove that K is commensurable with $p^n A$. By hypothesis, $(H + pA)/pA$ is finite, hence also $(K + pA)/pA$ is finite. The Cohen–Gluck theorem ensures that we can write $A = \bigoplus_\alpha e_\alpha J_p$ and $K = \bigoplus_\alpha p^{n_\alpha} e_\alpha J_p$; the isomorphism $A/K \cong \bigoplus_\alpha \mathbb{Z}(p^n)$ implies that each n_α equals either 0 or n. From the fact that $(K+pA)/pA$ is finite it follows that the set I of indices α such that $n_\alpha = 0$ is finite, therefore $K/p^n A$ is isomorphic to the direct sum of $|I|$ copies of $\mathbb{Z}(p^n)$, so it is finite, as desired.

Necessity. If A/H is finite, then it is trivially bounded and all its Ulm–Kaplansky invariants are finite. So, assume A/H infinite. By Theorem 2.4, H

is commensurable with p^kA for some $k \geq 1$. It follows that A/H is bounded, being a quotient of $A/(p^kA \cap H)$, which is bounded as an extension of the finite group $p^kA/(p^kA \cap H)$ by the bounded group A/p^kA. From the exact sequence

$$0 \to (p^kA + H)/H \to A/H \to A/(p^kA + H) \to 0$$

and from the finiteness of $(p^kA+H)/H$ it follows that A/H has exactly one infinite Ulm–Kaplansky invariant if and only if this happens for $A/(p^kA+H)$. From the exact sequence

$$0 \to (p^kA + H)/p^kA \to A/p^kA \to A/(p^kA + H) \to 0$$

and from the finiteness of $(p^kA + H)/p^kA$ it follows that $A/(p^kA + H)$ has exactly one infinite Ulm–Kaplansky invariant if and only if this happens for A/p^kA, which clearly holds, because A/p^kA is infinite and p^kA is commensurable with H. Finally, $(H + pA)/pA$ is finite, being an epic image of $(H + p^kA)/p^kA$, which is finite by the commensurability of H with p^kA. ∎

The next example shows that, in Theorem 2.5, the technical assumption that $(H + pA)/pA$ is finite cannot be avoided in general.

EXAMPLE 2.6. Let A be a free J_p-module of rank $\aleph_0$; then A/pA is a vector space of dimension $\aleph_0$. Take $H \supseteq pA$ such that both H/pA and A/H have infinite dimension. Then A/H is a bounded p-group with exactly one infinite Ulm–Kaplansky invariant, and H is commensurable neither with A nor with pA (so, *a fortiori*, not commensurable with p^nA, $n \geq 1$). Hence H is not fully inert, by Theorem 2.4.

3. Fully inert submodules of complete torsion-free J_p-modules. The main purpose of this section is to show that a fully inert submodule of a complete torsion-free J_p-module is commensurable with a fully invariant submodule. Thus we get a result analogous to those valid for free J_p-modules, proved in Section 2, as well as for free groups and torsion divisible groups, as proved in [7, Theorem 2.8] and [5, Theorem 5.3] respectively.

LEMMA 3.1. *Let X be a J_p-module that contains a bounded submodule Y such that X/Y is divisible. Let $m \geq 0$ be an integer such that $p^mY = 0$. Then $X = C \oplus D$, where C is bounded and $D = p^mX$ is the maximal divisible submodule of X, and C has the same annihilator as Y.*

Proof. Since, by assumption, $Y \subseteq X[p^m]$, we get

$$p^mX \cong \frac{X}{X[p^m]} \cong \frac{X/Y}{X[p^m]/Y}.$$

Then p^mX is divisible, being a homomorphic image of X/Y, and so $X = C \oplus p^mX$, where $C \cong X/p^mX$ is bounded. ∎

The next lemma, valid for arbitrary reduced torsion-free J_p-modules, has some independent interest.

LEMMA 3.2. *Let G be a reduced torsion-free J_p-module and H a fully inert submodule of G. Then G/H is torsion.*

Proof. Assume, for a contradiction, that $x+H \in G/H$ has infinite order. Then $\langle x\rangle \cap H = 0$, hence also $\langle p^n x\rangle \cap H = 0$ for all $n > 0$. Choose $y \in H$ and let $\langle y\rangle_*$ denote the purification of $J_p y$ in G. Since G is reduced, $\langle y\rangle_* = J_p z$, where $p^m z = y$, say. Then $\langle z\rangle$ is a pure submodule of G, and hence a summand, say $G = \langle z\rangle \oplus G_1$. Consider the endomorphism ψ of G defined by

$$\psi : z \mapsto x, \qquad G_1 \to 0.$$

Then

$$\frac{\psi(H)+H}{H} \supseteq \frac{\langle\psi(y)\rangle + H}{H} = \frac{\langle\psi(p^m z)\rangle + H}{H} = \frac{\langle p^m x\rangle + H}{H} \cong \langle p^m x\rangle.$$

It follows that $(\psi(H)+H)/H$ is infinite, which is impossible. ■

From now on, $A = \bigoplus_{\alpha<\lambda} J_p e_\alpha$ denotes a free J_p-module, where λ is a cardinal number, and $\hat{A}$ denotes the p-adic completion of A. We want to characterize the fully inert submodules of $\hat{A}$.

LEMMA 3.3. *Let H be a fully inert submodule of $\hat{A}$. Then $(A+H)/H$ is bounded.*

Proof. Since $\hat{A}$ is reduced, we know from Lemma 3.2 that $\hat{A}/H$ is a torsion group. Our statement is true when λ is finite, since in that case $\hat{A} = A$, and a finitely generated torsion J_p-module is clearly bounded. Hence, in the remainder of the proof, we may safely assume that $\lambda \geq \omega$.

We suppose, for a contradiction, that $(A+H)/H$ is unbounded. Now, for $n < \omega$, we pick nonzero elements $y_n = a_n e_n \in H$ for suitable positive integers a_n (which, under the present circumstances, are powers of p). This is possible since $\hat{A}/H$ is torsion. For $\alpha < \lambda$, we denote by $o(e_\alpha + H)$ its order in $(A+H)/H$. We construct by induction a strictly ascending chain $\{m_n\}_{n<\omega}$ of positive integers such that

(i) $$m_{n+1} > a_{n+1} m_n,$$

and a sequence $\{f(n)\}_{n\geq 1}$ of ordinals $< \lambda$ such that

(ii) $$o(e_{f(n)} + H) = m_n.$$

We start with any $m_0 > 0$. Since $(A+H)/H$ is unbounded, we readily see that there exists $f(1) < \lambda$ such that $o(e_{f(1)}+H) > a_0 m_0$. Of course, here we define $m_1 = o(e_{f(1)}+H)$. Moreover, once m_{n-1} is defined, we may choose $f(n) < \lambda$ such that $o(e_{f(n)} + H) = m_n > a_n m_{n-1}$, again since $(A+H)/H$ is unbounded, and we are done by induction.

Now we consider the endomorphism ϕ of A that extends the assignments

$$\phi : e_n \mapsto e_{f(n)}, \quad n \geq 1, \quad e_\alpha \mapsto 0, \quad \alpha \neq n.$$

Let $\hat{\phi} \in \mathrm{End}_{J_p}(\hat{A})$ be the extension of ϕ to the completion of A. Note that $\hat{\phi}(y_n) = a_n e_{f(n)} \in \hat{\phi}(H)$ for all $n \geq 1$, and

$$o(a_n e_{f(n)} + H) \geq m_n/a_n > m_{n-1}.$$

Since $\lim_n m_n = \infty$, it follows that $(\hat{\phi}H + H)/H$ contains elements of arbitrarily large orders, and therefore $(\hat{\phi}H + H)/H$ is unbounded, hence infinite, a contradiction. ∎

As we will see in Theorem 4.1, the preceding lemma is no longer true if we replace $\hat{A}$ with an arbitrary submodule of $\hat{A}$ containing A.

PROPOSITION 3.4. *Let H be fully inert in $\hat{A}$. Then $\hat{A}/H$ is reduced. As a consequence, $\hat{A}/H$ is bounded and $\hat{A} = A + H$*

Proof. By Lemma 3.3, $(A+H)/H$ is a bounded submodule of $\hat{A}/H$ such that the quotient is divisible, being isomorphic to a quotient of $\hat{A}/A$. Thus we are in a position to apply Lemma 3.1, concluding that $\hat{A}/H = C \oplus D$, where D is divisible, and C is bounded, with the same annihilator as that of $(A+H)/H$, say $p^k J_p$. We conclude that $\hat{A}/H$ is bounded if it is reduced. Moreover, if $\hat{A}/H$ is bounded, say $p^k(\hat{A}/H) = 0$, then necessarily $\hat{A} = A+H$, since, by density, $\hat{A} = A + p^k\hat{A}$, and $p^k\hat{A} \subseteq H$. Thus it suffices to show that $\hat{A}/H$ is reduced.

Assume, for a contradiction, that $\hat{A}/H$ contains a copy of $\mathbb{Z}(p^\infty)$ (recall that $\hat{A}/H$ is torsion), say $G/H \cong \mathbb{Z}(p^\infty)$ where $G \subseteq \hat{A}$. Then $G = H + \langle x_i : i < \omega\rangle$, where the x_i are suitable elements of $\hat{A}$ that satisfy the relations

$$px_{i+1} - x_i \in H, \quad i < \omega.$$

Now define a map $\phi : A \to \hat{A}$ by

$$\phi : e_i \mapsto x_{i+k}, \quad i < \omega, \quad e_\alpha \mapsto 0, \quad \omega \leq \alpha < \lambda.$$

Then ϕ extends to an endomorphism $\hat{\phi}$ of $\hat{A}$. Since $p^kA \subseteq H$, we get $\hat{\phi}(p^k e_i) = p^k x_{i+k} \in \hat{\phi}H$. Since $p^k x_{i+k} \equiv x_i$ modulo H, it follows that $H + \hat{\phi}H \supseteq H + \langle x_i : i < \omega\rangle = G$, and therefore $(H + \hat{\phi}H)/H$ is infinite, since it contains $G/H \cong \mathbb{Z}(p^\infty)$, a contradiction. ∎

We prove now that every fully inert submodule of a complete torsion-free J_p-module belongs to the equivalence class of a fully invariant subgroup with respect to the commensurability relation. Firstly we need the following lemma, of independent interest.

LEMMA 3.5. *Let H be a fully inert submodule of $\hat{A}$. Then $H \cap A$ is fully inert in A.*

Proof. Take any endomorphism $\phi : A \to A$; our aim is to show that $(H \cap A + \phi(H \cap A))/H \cap A$ is finite. Consider the unique extension $\hat{\phi} \in \mathrm{End}_{J_p}(\hat{A})$ of ϕ; obviously, $\phi(H \cap A) \subseteq \hat{\phi}H$. Since H is fully inert in $\hat{A}$, the J_p-module $(H + \hat{\phi}H)/H$ is finite, hence obviously so is $(H + \phi(H \cap A))/H$. Noticing that

$$\begin{aligned}\frac{H \cap A + \phi(H \cap A)}{H \cap A} &= \frac{A \cap (H \cap A + \phi(H \cap A))}{A \cap ((H \cap A + \phi(H \cap A)) \cap H} \\ &\cong \frac{A \cap (H \cap A + \phi(H \cap A)) + H}{H} = \frac{H + \phi(H \cap A)}{H}\end{aligned}$$

we get the desired conclusion. ∎

THEOREM 3.6. *A submodule H of a complete torsion-free J_p-module $\hat{A}$ is fully inert if and only if it is commensurable with $p^n\hat{A}$ for some $n \geq 0$.*

Proof. Sufficiency is provided by Proposition 2.1. Let H be a fully inert submodule of $\hat{A}$. Since $\hat{A} = A + H$ by Proposition 3.4, we have the canonical isomorphism $\hat{A}/H \cong A/A \cap H$. In view of Lemma 3.5, we know that $A \cap H$ is fully inert in A. Then, by Theorem 2.4, $A \cap H$ is commensurable with p^nA for a suitable $n \geq 0$. We verify that H is commensurable with $p^n\hat{A}$.

One finiteness condition is readily checked. Indeed,

$$\frac{H + p^n\hat{A}}{H} = \frac{H + p^n(A + H)}{H} \cong \frac{p^nA}{H \cap p^nA} \cong \frac{H \cap A + p^nA}{H \cap A},$$

hence the first J_p-module in the above relation is finite, since $H \cap A$ commensurable with p^nA yields $(H \cap A + p^nA)/(H \cap A)$ finite.

Now we want to show that $(H \cap A + p^nA)/p^nA$ finite implies that $(H + p^n\hat{A})/p^n\hat{A} = (H + p^nA)/p^n\hat{A}$ is also finite. Note that, by the modular property, $(H \cap A + p^nA)/p^nA = (A/p^nA) \cap (H + p^nA)/p^nA$.

Firstly, from $\hat{A} = A + p^n\hat{A}$ we derive $\hat{A}/p^nA = p^n\hat{A}/p^nA \oplus A/p^nA$, where the sum is direct since $p^n\hat{A}/p^nA \cong \hat{A}/A$ is torsion-free divisible and A/p^nA is bounded. The modular property yields the equalities

$$\begin{aligned}\frac{H + p^nA}{p^nA} &= \frac{H + p^nA}{p^nA} \cap \frac{\hat{A}}{p^nA} = \frac{H + p^nA}{p^nA} \cap \left(\frac{p^n\hat{A}}{p^nA} \oplus \frac{A}{p^nA}\right) \\ &= \frac{p^n\hat{A}}{p^nA} \oplus \left(\frac{H + p^nA}{p^nA} \cap \frac{A}{p^nA}\right).\end{aligned}$$

We derive that

$$\frac{H + p^nA}{p^n\hat{A}} \cong \frac{H + p^nA}{p^nA} \cap \frac{A}{p^nA}$$

where the second member of the congruence is known to be finite. The desired conclusion follows. ∎

From Theorem 3.6 we derive a characterization similar to that of Theorem 2.5 for fully inert submodules of free J_p-modules.

In the next easy lemma we emphasize a crucial isomorphism.

LEMMA 3.7. *Let A be a free J_p-module, $\hat{A}$ its completion, H a submodule of $\hat{A}$ such that $\hat{A} = A + H$. Then $(H + p\hat{A})/p\hat{A} \cong (H \cap A + pA)/pA$.*

Proof. Note that from $\hat{A} = A + H$ it also follows $\hat{A} = A + p\hat{A} = A + pH$. Since $pA = A \cap p\hat{A}$, we get the isomorphisms

$$\frac{H \cap A + pA}{pA} \cong \frac{H \cap A}{H \cap pA} = \frac{H \cap A}{H \cap A \cap p\hat{A}} \cong \frac{H \cap A + p\hat{A}}{p\hat{A}}.$$

To reach our conclusion, it remains to verify that $H \cap A + p\hat{A} = H + p\hat{A}$. It suffices to show that $H \subseteq H \cap A + pH$, and this inclusion readily follows from $H \subseteq A + pH$. ∎

THEOREM 3.8. *Let $\hat{A}$ be the completion of a free J_p-module A of infinite rank, and let H be a submodule of $\hat{A}$. Then H is fully inert in $\hat{A}$ if and only if $\hat{A}/H$ is a bounded p-group with at most one infinite Ulm–Kaplansky invariant and $(H + p\hat{A})/p\hat{A}$ is finite when $\hat{A}/H$ is infinite.*

Proof. Necessity. From Proposition 3.4 we know that $\hat{A} = A + H$, so $\hat{A}/H \cong A/(A \cap H)$. From Lemma 3.5 we get $A \cap H$ fully inert in A, so we are in a position to apply Theorem 2.5. Then $A/(A \cap H)$ is bounded, and $(A \cap H + pA)/pA$ is finite, whence, by Lemma 3.7, $(H + p\hat{A})/p\hat{A}$ is also finite.

Sufficiency. If $\hat{A}/H$ is finite, then H is obviously fully inert. Assume that $\hat{A}/H$ is a bounded J_p-module (equivalently, a p-group) with exactly one infinite Ulm–Kaplansky invariant. Since $\hat{A}/H$ is bounded, we get $H \supseteq p^k\hat{A}$ for some $k \geq 0$, hence $\hat{A} = A + p^k\hat{A} = A + H$. Moreover, since exactly one Ulm–Kaplansky invariant is infinite, we may write $\hat{A}/H = F \oplus \bigoplus_\alpha \mathbb{Z}(p^n)$ for some infinite cardinal α and a positive integer n, where F is a finite direct sum of cyclic p-groups not isomorphic to $\mathbb{Z}(p^n)$.

Let $H \subseteq K \subseteq \hat{A}$ be such that $K/H = F$. Note that $A + K = \hat{A}$ and $\hat{A}/K = \bigoplus_\alpha \mathbb{Z}(p^n)$. Since commensurability is an equivalence relation, to reach the desired conclusion it suffices to show that K is commensurable with the fully invariant submodule $p^n\hat{A}$ of $\hat{A}$. As $A + K = \hat{A}$, we have the isomorphism $\hat{A}/K \cong A/(A \cap K)$. Moreover, since $(H + p\hat{A})/p\hat{A}$ and K/H are finite, we get $(K + p\hat{A})/p\hat{A}$ finite, and therefore also $(K \cap A + pA)/pA$ is finite, by Lemma 3.7. Thus, from Theorem 2.5, it follows that $K \cap A$ is fully inert in A.

The Cohen–Gluck theorem ensures that $A \cong \bigoplus_\alpha J_p e_\alpha$, and $K \cap A \cong \bigoplus_\alpha J_p p^{n_\alpha} e_\alpha$, where n_α is either n or zero. Arguing as in the proof of Theorem

2.5, we see that only finitely many n_α are zero, say

$$K \cap A \cong \bigoplus_I J_p \oplus \bigoplus_\beta J_p p^{n_\beta} e_\beta = \bigoplus_I J_p \oplus p^n A_1,$$

where I is a finite set of indices and $n_\beta = n$ for every β. Note that $\hat{A} = \bigoplus_I J_p \oplus \hat{A}_1$; moreover, if we denote by $\hat{C}$ the completion of $K \cap A$ in its p-adic topology, we have $\hat{C} = \bigoplus_I J_p \oplus p^n \hat{A}_1$. Then we readily see that $\hat{C}$ is commensurable with $p^n \hat{A}$.

As $\hat{A}/K$ is reduced, [9, Corollary 3.9.2] ensures that K is complete in its p-adic topology, which coincides with the topology induced by the p-adic topology of $\hat{A}$, because $\hat{A}/K$ is bounded. Since $K/(A \cap K) \cong \hat{A}/A$ is divisible, $K \cap A$ is dense in K, and therefore $K = \hat{K}$ coincides with the completion $\hat{C}$ of $K \cap A$. We conclude that K is commensurable with $p^n \hat{A}$, as desired. ■

4. Fully inert submodules not commensurable with fully invariant submodules. The purpose of this final section is to provide examples of torsion-free J_p-modules that admit fully inert submodules not commensurable with any fully invariant submodule. These examples are furnished by theorems on realization of commutative rings as endomorphism rings of J_p-modules—see, for example, [10] or [3].

THEOREM 4.1. *There exist torsion-free J_p-modules X that contain fully inert submodules noncommensurable with $p^n X$, for all $n \geq 0$.*

Proof. We start with a free J_p-module $A = \bigoplus_{i<\omega} J_p e_i$ of countable rank, and consider its p-adic completion $\hat{A}$. It follows from [10, Theorem 4.1] or [3, Theorem 7.10] that there exists a J_p-module X, with $A \subseteq X \subseteq \hat{A}$, such that

$$\mathrm{End}_{J_p}(X) = J_p.1_X \oplus E_0(X),$$

where E_0 is the ideal of endomorphisms ϕ such that ϕX is free of finite rank.

We first verify that every submodule H of X such that X/H is torsion, is necessarily fully inert. Take any endomorphism f of X; then $f = r + \phi$, with $r \in J_p$ and $\phi \in E_0(X)$. Since $H + f(H) = H + \phi H$, it suffices to show that $(H + \phi H)/H$ is finite. Indeed, $\phi H/H \cap \phi H$ is a finitely generated torsion J_p-module, since ϕH is free of finite rank, and X/H is torsion, by assumption. It follows that $\phi H/H \cap \phi H \cong (H + \phi H)/H$ is finite.

Now we observe that X/pX is a $\mathbb{Z}(p)$-vector space of infinite dimension. So we may choose a submodule H containing pX and such that H/pX and X/H both have infinite dimension over $\mathbb{Z}(p)$. It readily follows that H is commensurable neither with X nor with pX; *a fortiori*, H is not commensurable with $p^n X$, $n \geq 2$, since $|H/p^n X| \geq |H/pX|$. As X/H is certainly torsion, H is fully inert in X but it is not commensurable with any fully invariant submodule of X. ■

Acknowledgements. Research supported by "Progetti di Eccellenza 2011/12" of Fondazione CARIPARO.

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Received 12 December 2013;
revised 27 February 2014 (6108)