COLLOQUIUM MATHEMATICUM

VOL. 136 2014 NO. 2

ON $A^2 \pm nB^4 + C^4 = D^8$

BY

SUSIL KUMAR JENA (Odisha)

Abstract. We prove that for each $n \in \mathbb{N}_+$ the Diophantine equation $A^2 \pm nB^4 + C^4 = D^8$ has infinitely many primitive integer solutions, i.e. solutions satisfying $\gcd(A, B, C, D) = 1$.

1. Introduction. Dem'yanenko [3] did not know that his parametric solution to the Diophantine equation

$$x^4 - y^4 = z^4 + t^2$$

would be used by Noam Elkies [5] to disprove Euler's conjecture [4] for fourth powers that at least four integral fourth powers are required to sum to an integral fourth power, except for the trivial case $y^4 = y^4$. In this paper, we give a parametric solution for the family of Diophantine equations

$$A^2 \pm nB^4 + C^4 = D^8 \tag{1.1}$$

where n is any non-zero integer. The parametric solution is based on an identity which is given in the next section.

The papers [1, 2] by Bremner and Ulas contain similar material concerning Diophantine equations of the form $a(x^p - y^q) = b(z^r - w^s)$, where the exponents satisfy the identity $1/p+1/q+1/r+1/s = 1$. In [1] they prove that the equation $x^8 - y^8 = -z^4 + w^2$ has infinitely many non-trivial primitive solutions. Moreover, in [2] they prove that for each $a, b \in \mathbb{Z} \setminus \{0\}$ each of the Diophantine equations $a(x^2 - y^4) = b(z^8 - w^8)$, $a(x^2 - y^8) = b(z^4 - w^8)$ has infinitely many non-trivial solutions in co-prime polynomials. These results serve as a good motivation for the research presented in this paper.

2. Constructing the basic identity. We give a basic identity which will be used to get the main results of this paper. The identity is

$$((8m+1)^4 - 2^7m)^2 + m(4(8m-1))^4 + (32m)^4 = (8m+1)^8 \tag{2.1}$$

2010 *Mathematics Subject Classification*: Primary 11D41; Secondary 11D72.

Key words and phrases: Diophantine equation $A^2+nB^4+C^4 = D^8$, Diophantine equation $A^2 - nB^4 + C^4 = D^8$, Euler's conjecture.

DOI: 10.4064/cm136-2-6

where m is any non-zero real number. The proof of (2.1) is very simple. For any two non-zero real numbers a and b we have

$$(a+b)^4-(a-b)^4=8ab(a^2+b^2). \tag{2.2}$$

Take $a=8m$ and $b=1$ in (2.2) to get

$$(8m+1)^4-(8m-1)^4=2^6m(2^6m^2+1)=2^{12}m^3+2^6m. \tag{2.3}$$

Changing sides of (2.3) and rearranging, we get

$$2^6m+(8m-1)^4+2^{12}m^3=(8m+1)^4. \tag{2.4}$$

Multiplying (2.4) by 2^8m and performing simple manipulations we get

$$2^{14}m^2-2^8m(8m+1)^4+m(4(8m-1))^4+(32m)^4=0. \tag{2.5}$$

Adding $(8m+1)^8$ to both the sides of (2.5) and using the identity

$$(8m+1)^8-2^8m(8m+1)^4+2^{14}m^2=((8m+1)^4-2^7m)^2$$

we get (2.1).

3. The main result.

The main result of this paper is the following theorem.

THEOREM 3.1. *For any given $n\in\mathbb{Z}\setminus\{0\}$ the Diophantine equation $A^2+nB^4+C^4=D^8$ has infinitely many non-trivial primitive integer solutions (A,B,C,D) satisfying* $\gcd(A,D)=\gcd(B,D)=\gcd(C,D)=1$.

Proof. The proof is based on the identity

$$((8np^4+1)^4-2^7np^4)^2+n(4p(8np^4-1))^4+(32np^4)^4=(8np^4+1)^8, \tag{3.1}$$

which is obtained by putting $m=np^4$ in (2.1). Comparing the Diophantine equation

$$A^2+nB^4+C^4=D^8 \tag{3.2}$$

with the identity (3.1), we get a polynomial solution of (3.2) given by $(A,B,C,D)=((8np^4+1)^4-2^7np^4,4p(8np^4-1),32np^4,8np^4+1)$. Now, for any given $n\in\mathbb{Z}\setminus\{0\}$, p can take infinitely many integral values so that we get infinitely many non-trivial primitive integer solutions (A,B,C,D) . Moreover, we can easily check that

$$\gcd(A,D)=\gcd((8np^4+1)^4-2^7np^4,8np^4+1)=1,$$
$$\gcd(B,D)=\gcd(4p(8np^4-1),8np^4+1)=1,$$
$$\gcd(C,D)=(32np^4,8np^4+1)=1.$$

Thus, the proof is complete. ■

COROLLARY 3.2. *Each of the Diophantine equations $A^2+B^4+C^4=D^8$, $A^2-B^4+C^4=D^8$ has infinitely many non-trivial primitive integer solutions (A,B,C,D) satisfying* $\gcd(A,D)=\gcd(B,D)=\gcd(C,D)=1$.

Proof. This follows from Theorem 3.1 when $n = \pm 1$. ∎

4. Some remarks. At present, we do not know of any implications of the results of this paper, but just as Noam Elkies [5] disproved Euler's conjecture [4] for fourth powers by extending the work of Dem'yanenko [3], we hope that the results of this paper can be extended to go further in finding the non-trivial polynomial solutions of the Diophantine equation $A^4 \pm B^4 + C^4 = D^8$ in an elementary way.

The referee has rightly remarked that a natural question arises whether it is possible to extend our result and prove the existence of infinitely many primitive polynomial solutions of (1.1). This question remains open.

Acknowledgements. I express my gratitude to the anonymous referee for drawing my attention to the papers [1, 2] by Bremner and Ulas. In fact, the last paragraph in the introductory section is due to the referee. I am grateful to my family for supporting me in pursuing my dream.

REFERENCES

[1] A. Bremner and M. Ulas, *On $x^p \pm y^q \pm z^r \pm w^s = 0$, with $1/p + 1/q + 1/r + 1/s = 1$*, Int. J. Number Theory 7 (2011), 2081–2090.

[2] A. Bremner and M. Ulas, *On certain diophantine equations of diagonal type*, J. Number Theory 136 (2014), 46–64.

[3] V. A. Dem'yanenko, *L. Euler's conjecture*, Acta Arith. 25 (1973/74), 127–135 (in Russian).

[4] L. E. Dickson, *History of the Theory of Numbers*, Vol. 2, Carnegie Institute of Washington, 1923, 682–682.

[5] N. Elkies, *On $A^4 + B^4 + C^4 = D^4$*, Math. Comp. 51 (1988), 825–835.

Susil Kumar Jena
Department of Electronics & Telecommunication Engineering
KIIT University, Bhubaneswar 751024
Odisha, India
E-mail: susil_kumar@yahoo.co.uk

Received 12 January 2014;
revised 3 June 2014 (6128)