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A NOTE ON ARC-DISJOINT CYCLES IN TOURNAMENTS

BY

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Abstract. We prove that every vertex v of a tournament T belongs to at least

$$\max\{\min\{\delta^+(T), 2\delta^+(T) - d_T^+(v) + 1\}, \min\{\delta^-(T), 2\delta^-(T) - d_T^-(v) + 1\}\}$$

arc-disjoint cycles, where $\delta^+(T)$ (or $\delta^-(T)$) is the minimum out-degree (resp. minimum in-degree) of T, and $d_T^+(v)$ (or $d_T^-(v)$) is the out-degree (resp. in-degree) of v.

1. Introduction. Notation used in this paper is consistent with Bang-Jensen and Gutin [1]. Cycles are always directed. A *tournament* is an orientation of a complete graph. The *out-degree* (resp. *in-degree*) $d_T^+(v)$ (resp. $d_T^-(v)$) of a vertex v of a tournament T is the number of arcs with tail at v (resp. with head at v). We denote by $\delta^+(T)$ (resp. $\Delta^+(T)$) the minimum out-degree (resp. maximum out-degree) of T. Moreover, we denote by $\delta^-(T)$ (resp. $\Delta^-(T)$) the minimum in-degree (resp. maximum in-degree) of T.

Landau [2] proved that in every tournament T, if a vertex v has the minimum out-degree, then it belongs to $\delta^+(T)$ different 3-cycles. In this article, we prove that in every tournament T, every vertex v belongs to at least $C_T(v)$ arc-disjoint cycles, where $C_T(v)$ is equal to

$$\max\{\min\{\delta^+(T), 2\delta^+(T) - d_T^+(v) + 1\}, \min\{\delta^-(T), 2\delta^-(T) - d_T^-(v) + 1\}\}.$$

This implies that v belongs to at least $C_T(v)$ different 3-cycles. Moreover, if either $\Delta^+(T) \le 2\delta^+(T)$, or $\Delta^-(T) \le 2\delta^-(T)$, then every vertex of $T \ne K_1$ belongs to a 3-cycle.

Note that for every tournament T which has a vertex v such that the tournament $T - v$ is regular, the lower bound $C_T(v)$ is the best possible. Indeed, $d_T^+(v) + d_T^-(v) = 2\delta^+(T - v) + 1$. Thus, if $d_T^+(v) \le \delta^+(T - v)$, then

$$\min\{d_T^+(v), d_T^-(v)\} = d_T^+(v) = \delta^+(T) \le 2\delta^+(T) - d_T^+(v) + 1.$$

If $d_T^+(v) > \delta^+(T - v)$, then

$$\min\{d_T^+(v), d_T^-(v)\} = d_T^-(v) = 2\delta^+(T) - d_T^+(v) + 1 \le \delta^+(T).$$

Hence,

$$\min\{d_T^+(v), d_T^-(v)\} = \min\{\delta^+(T), 2\delta^+(T) - d_T^+(v) + 1\}.$$

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Similarly, from $d_T^+(v) + d_T^-(v) = 2\delta^-(T - v) + 1$, it follows that

$$\min\{d_T^+(v), d_T^-(v)\} = \min\{\delta^-(T), 2\delta^-(T) - d_T^-(v) + 1\}.$$

2. Arc-disjoint cycles through a vertex in a tournament. Let $T = (V, A)$ be a tournament with vertex set V and arc set A. For an arc $xy \in A$ the first vertex x is its *tail* and the second vertex y is its *head*. For a vertex v in T we use the following notation:

$$N^+(v) = \{u \in V \setminus \{v\} : vu \in A\}, \quad N^-(v) = \{u \in V \setminus \{v\} : uv \in A\}.$$

For a pair X, Y of vertex sets in T we define

$$(X, Y) = \{xy \in A : x \in X,\ y \in Y\}.$$

THEOREM 2.1. *Every vertex v of a tournament T belongs to at least*

$$\max\{\min\{\delta^+(T), 2\delta^+(T) - d_T^+(v) + 1\}, \min\{\delta^-(T), 2\delta^-(T) - d_T^-(v) + 1\}\}$$

arc-disjoint cycles.

Proof. For a vertex v of a tournament T, let $\Gamma = \{\gamma^1, \ldots, \gamma^m\}$ be a maximum family of arc-disjoint cycles through v. Let $\gamma^i = vv_1^i \ldots v_{n(i)}^i v$ for $i = 1, \ldots, m$. By Menger's theorem (see [1]) there exists a set Ω of m arcs covering all cycles containing the vertex v. Suppose that k is the number of arcs in Ω with head v. If $k > 0$, we can assume that the arc $v_{n(i)}^i v$ is in Ω for $1 \le i \le k$. Let us denote $K = \{v_1^i : 1 \le i \le k\}$, $L = \{v_1^i : k < i \le m\}$, $M = N^+(v) \setminus K \setminus L$, $X = \{v_{n(i)}^i : 1 \le i \le k\}$ (if $k = 0$ we set $K = X = \emptyset$), and $Y = N^-(v) \setminus X$.

First we prove that

$$|(K \cup X \cup M, Y)| \le |(L, K \cup X \cup M)|. \tag{1}$$

Assume that an arc wy belongs to $(K \cup X \cup M, Y)$. Notice that $yv \notin \Omega$. If $w \in K \cup M$, then the arc wy of the cycle $vwyv$ belongs to $\Omega \setminus \{v_{n(i)}^i v : i \le k\}$. If $w \in X$, then $w = v_{n(i)}^i$ for some $i \le k$. Hence, the arc $wy = v_{n(i)}^i y$ of the cycle $vv_1^i \ldots v_{n(i)}^i yv$ belongs to $\Omega \setminus \{v_{n(i)}^i v : i \le k\}$. Thus, wy is an arc of the cycle γ^i, for some $i > k$. Suppose that v_l^i is the first vertex of the cycle γ^i which does not belong to L. Notice that wy is the only arc of γ^i which belongs to Ω, because Ω and Γ have the same number of elements. Hence, the vertex v_l^i does not belong to Y. Otherwise, the cycle $vv_1^i \ldots v_{l-1}^i v_l^i v$ would not be covered by Ω. Thus the edge $v_{l-1}^i v_l^i$ of the cycle γ^i belongs to $(L, K \cup X \cup M)$. Accordingly, to every arc in $(K \cup X \cup M, Y)$ we can assign an arc in $(L, K \cup X \cup M)$ such that the two arcs belong to the same cycle γ^i, for some $i > k$. The above assignment is injective, because Ω and Γ have the same number of elements, and Γ is a family of arc-disjoint cycles. Hence, (1) holds.

By (1) we obtain

$$\begin{aligned}|K\cup X\cup M|(|V|-1) &= |(V\setminus L, K\cup X\cup M)| + |(L, K\cup X\cup M)| \\ &\quad + |(K\cup X\cup M, V)| \\ &\geq |(V\setminus L, K\cup X\cup M)| + |(K\cup X\cup M, Y)| \\ &\quad + |(K\cup X\cup M, V)| \\ &= |(K\cup X\cup M, K\cup X\cup M)| \\ &\quad + |(\{v\}, K\cup X\cup M)| + |(Y, K\cup X\cup M)| \\ &\quad + |(K\cup X\cup M, Y)| + |(K\cup X\cup M, V)| \\ &\geq |K\cup X\cup M|\cdot\frac{|K\cup X\cup M|-1}{2} + |K| + |M| \\ &\quad + |K\cup X\cup M||Y| + (|K|+|X|+M|)\delta^+(T).\end{aligned}$$

Since $|V|-1 = d_T^+(v) + |X| + |Y|$ and $|K| = |X|$, we have

$$(2|K|+|M|)d_T^+(v) \geq (2|K|+|M|)\frac{|M|}{2} + \frac{|M|}{2} + (2|K|+M|)\delta^+(T).$$

Thus, either $|M| = 0$, or $d_T^+(v) > |M|/2 + \delta^+(T)$. Hence, either $|M| = 0$, or

$$d_T^+(v) - |M| > 2\delta^+(T) - d_T^+(v).$$

Accordingly, the vertex v belongs to at least

$$\min\{\delta^+(T), 2\delta^+(T) - d_T^+(v) + 1\}$$

arc-disjoint cycles. By considering the tournament obtained from T by reversing the directions of the arcs of A, we conclude in a similar fashion that the vertex v belongs to at least $\min\{\delta^-(T), 2\delta^-(T) - d_T^-(v) + 1\}$ arc-disjoint cycles. ■

Remark 1. There exists a regular tournament R with a vertex which does not belong to $\delta^+(R)$ arc-disjoint 3-cycles. For example, let R be the

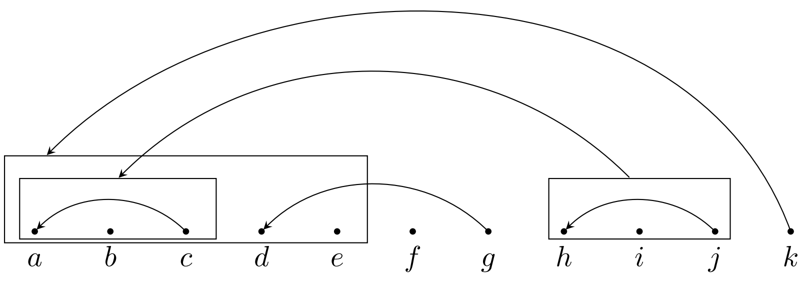

Fig. 1. A regular tournament R. $(\{h,i,j\},\{a,b,c\}) \cup (\{k\},\{a,b,c,d,e\}) \cup \{ca, gd, jh\}$ is the set of all backward arcs with respect to the ordering a,b,c,d,e,f,g,h,i,j,k of vertices in R.

tournament in Fig. 1. Let kv_1v_2 be a 3-cycle through the vertex k. Notice that, if $v_1 \in \{a, b, c\}$, then $v_2 \in \{f, g\}$. Hence, the vertex k does not belong to $\delta^+(R)$ arc-disjoint 3-cycles.

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