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ESTIMATES FOR THE ARCTANGENT FUNCTION RELATED TO SHAFER'S INEQUALITY

BY

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Abstract. The aim of this article is to give new refinements and sharpenings of Shafer's inequality involving the arctangent function. These are obtained by means of a change of variables, which makes the computations much easier than the classical approach.

1. Introduction and motivation. Shafer [9] presented the following inequality involving the arctangent function for $x > 0$:

$$\arctan x > \frac{3x}{1 + 2\sqrt{1 + x^2}}, \tag{1}$$

which has attracted the attention of many authors in recent years. Many refinements and extensions of this inequality, together with its analogues for arcsine and arccosine functions, have been provided (see, for example, [1], [2], [4]–[15], and the references therein). Moreover, it is proven in [3, p. 288] that, for every $x > 0$,

$$\arctan x < \frac{2x}{1 + \sqrt{1 + x^2}}. \tag{2}$$

Grinstein et al. [10] proved (1) by using the fact that the function $F(x)$ defined by

$$F(x) := \arctan x - \frac{3x}{1 + 2\sqrt{1 + x^2}}$$

is increasing for $x > 0$.

Qi [7]–[8] obtained some interesting results by using the behavior of the function $f_a(x)$ which is defined on $(0, \infty)$ by

$$f_a(x) = \frac{(a + \sqrt{1 + x^2}) \arctan x}{x},$$

where a is a real parameter.

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The results presented in this article are derived by means of the following change of variables:

$$x = \frac{2t}{1-t^2} \qquad (t \in (0,1)). \tag{3}$$

Clearly, as t increases from 0 to 1, x will increase from 0 to ∞. The importance of this change of variables is that $\sqrt{1+x^2}$ becomes a rational function of t:

$$\sqrt{1+x^2} = \frac{1+t^2}{1-t^2},$$

from which further computations will become fairly easy. For example, after making the change of variables (3), the inequality (1) becomes

$$G(t) > 0 \qquad (t \in (0,1)),$$

where

$$G(t) := \arctan\left(\frac{2t}{1-t^2}\right) - \frac{6t}{3+t^2}$$

and

$$G'(t) = \frac{8t^4}{(t^2+1)(t^2+3)^2} > 0.$$

On the other hand, the inequality (2) reads

$$H(t) < 0 \qquad (t \in (0,1)),$$

where

$$H(t) := \arctan\left(\frac{2t}{1-t^2}\right) - 2t$$

and

$$H'(t) = -\frac{2t^2}{t^2+1} < 0.$$

Many of the proofs of the results given in [4]–[15] can be simplified considerably by using the transformation (3). Our refinements and sharpenings of the inequalities (1) and (2) are presented in the following sections.

2. Refinements of Shafer's inequality. One improvement of Shafer's inequality (1) stems essentially from the fact that (1) gives good results near zero, since

$$\lim_{x\to 0}\left(\arctan x - \frac{3x}{1+2\sqrt{1+x^2}}\right) = 0, \tag{4}$$

which we write as

$$\arctan x \approx \frac{3x}{1+2\sqrt{1+x^2}} \qquad \text{as } x \to 0, \tag{5}$$

Related to (5), we present the following two-sided inequalities, which give good results near the origin.

THEOREM 1. *For every $x > 0$,*

$$\frac{3x}{1+2\sqrt{1+x^2}} + a(x) < \arctan x < \frac{3x}{1+2\sqrt{1+x^2}} + b(x), \tag{6}$$

where

$$a(x) = \frac{1}{180}x^5 - \frac{13}{1512}x^7 \quad \text{and} \quad b(x) = \frac{1}{180}x^5.$$

Another way to extend Shafer's inequality (1) starts from approximations near the origin of the form

$$\arctan x \approx \frac{3x + \omega(x)}{1+2\sqrt{1+x^2}},$$

where $\omega(x) \to 0$ as $x \to 0$. Our result is stated below.

THEOREM 2. *For every $x > 0$,*

$$\frac{3x + c(x)}{1+2\sqrt{1+x^2}} < \arctan x < \frac{3x + d(x)}{1+2\sqrt{1+x^2}}, \tag{7}$$

where

$$c(x) = \frac{1}{60}x^5 - \frac{17}{840}x^7 \quad \text{and} \quad d(x) = \frac{1}{60}x^5.$$

Evidently, (6) and (7) provide strong improvements of (1).

Now, observe that

$$\lim_{x\to\infty}\left(\arctan x - \frac{3x}{1+2\sqrt{1+x^2}}\right) = \frac{\pi - 3}{2}, \tag{8}$$

that is,

$$\arctan x \approx \frac{\pi-3}{2} + \frac{3x}{1+2\sqrt{1+x^2}} + \lambda(x),$$

where $\lambda(x) \to 0$ as $x \to \infty$. We present the following more precise result.

THEOREM 3. *For every $x > 0$,*

$$\begin{aligned} \frac{\pi-3}{2} + \frac{3x}{1+2\sqrt{1+x^2}} + u(x) &< \arctan x \\ &< \frac{\pi-3}{2} + \frac{3x}{1+2\sqrt{1+x^2}} + v(x), \end{aligned} \tag{9}$$

where

$$u(x) = -\frac{1}{4x} \quad \text{and} \quad v(x) = -\frac{1}{4x} + \frac{3}{8x^2}.$$

REMARK 1. Evidently, since $u(x) \to 0$ and $v(x) \to 0$ as $x \to \infty$, the assertion (9) considerably improves Shafer's inequality (1) for large values of x.

To prove Theorems 1–3 above, we need the following three lemmas.

Lemma 1. *Let*

$$w(t) := 1431 - 904t + 2077t^2 - 1504t^3 + 753t^4 - 216t^5 + 27t^6.$$

Then $w(t) > 0$ *for every* $t \in [0, 1]$.

Proof. It suffices to prove that $w_1(t) > w_2(t)$, where

$$w_1(t) = 1431 + 2077t^2 + 753t^4 + 27t^6,$$
$$w_2(t) = 904t + 1504t^3 + 216t^5.$$

As $w_1(0) = 1431$ and $w_2(3/4) = \frac{174\,561}{128} = 1363.757\ldots$, for every $t \in [0, 3/4]$ we have

$$w_1(t) \geq w_1(0) > w_2(3/4) \geq w_2(t).$$

As $w_1(\frac{3}{4}) = \frac{11\,642\,355}{4096} = 2842.371\ldots$ and $w_2(1) = 2624$, for every $t \in [3/4, 1]$ we have

$$w_1(t) \geq w_1(3/4) > w_2(1) \geq w_2(t). \blacksquare$$

Let us call the points $0 < 3/4 < 1$ in the proof of Lemma 1 *breaking points*. We are in a position to state the next two lemmas.

Lemma 2. *Let*

$$z(t) := 3753 - 2743t^2 + 3864t^4 - 2240t^6 + 735t^8 - 105t^{10}.$$

Then $z(t) > 0$ *for every* $t \in [0, 1]$.

Lemma 3. *Let* $s(t) := 102 - 133t^2 + 155t^4 - 75t^6 + 15t^8$. *Then* $s(t) > 0$ *for every* $t \in [0, 1]$.

After separating the terms of the same sign in $z(t)$ and $s(t)$, the proofs of Lemmas 2 and 3 are similar to the proof of Lemma 1, using the breaking points $0 < 4/5 < 1$ and $0 < 3/4 < 9/10 < 1$, respectively.

Proof of Theorem 1. In view of the transformation (3), we have to prove that $f(t) > 0$ and $g(t) < 0$, for $t \in (0, 1)$, where

$$f(t) := \arctan\left(\frac{2t}{1-t^2}\right) - \frac{6t}{3+t^2} - \frac{1}{180}\left(\frac{2t}{1-t^2}\right)^5 + \frac{13}{1512}\left(\frac{2t}{1-t^2}\right)^7,$$
$$g(t) := \arctan\left(\frac{2t}{1-t^2}\right) - \frac{6t}{3+t^2} - \frac{1}{180}\left(\frac{2t}{1-t^2}\right)^5.$$

Indeed, we have

$$f'(t) = \frac{8t^8 w(t^2)}{27(t^2+1)(t^2+3)^2(t^2-1)^8} > 0$$

with $w(t)$ given in Lemma 1, and

$$g'(t) = -\frac{8t^6(3-t^2)(3t^4-8t^2+13)(3t^4-t^2+2)}{9(t^2+1)(t^2+3)^2(t^2-1)^6} < 0,$$

which shows that $f(t)$ is strictly increasing and $g(t)$ is strictly decreasing on $(0,1)$. As $f(0) = g(0) = 0$, the assertion follows readily. ■

Proof of Theorem 2. By virtue of (3), we have to prove that $h(t) > 0$ and $k(t) < 0$ on $(0,1)$, where

$$h(t) := \arctan\left(\frac{2t}{1-t^2}\right) - \frac{3\left(\frac{2t}{1-t^2}\right) + \frac{1}{60}\left(\frac{2t}{1-t^2}\right)^5 - \frac{17}{840}\left(\frac{2t}{1-t^2}\right)^7}{1+2\left(\frac{1+t^2}{1-t^2}\right)},$$

$$k(t) := \arctan\left(\frac{2t}{1-t^2}\right) - \frac{3\left(\frac{2t}{1-t^2}\right) + \frac{1}{60}\left(\frac{2t}{1-t^2}\right)^5}{1+2\left(\frac{1+t^2}{1-t^2}\right)}.$$

We have

$$h'(t) = \frac{8t^8 z(t)}{105(t^2+1)(t^2+3)^2(t^2-1)^7} > 0,$$

where $z(t)$ is given by Lemma 2, and

$$k'(t) = -\frac{8t^6 s(t)}{15(t^2+1)(t^2+3)^2(t^2-1)^5} < 0,$$

where $s(t)$ is given by Lemma 3. As $h(0) = k(0) = 0$, the conclusion follows. ■

Proof of Theorem 3. By using (3), we have to prove that $p(t) > 0$ and $q(t) < 0$ when $t \in (0,1)$, where

$$p(t) := \arctan\left(\frac{2t}{1-t^2}\right) - \frac{\pi-3}{2} - \frac{6t}{3+t^2} + \frac{1}{4\left(\frac{2t}{1-t^2}\right)},$$

$$q(t) := \arctan\left(\frac{2t}{1-t^2}\right) - \frac{\pi-3}{2} - \frac{6t}{3+t^2} + \frac{1}{4\left(\frac{2t}{1-t^2}\right)} - \frac{3}{8\left(\frac{2t}{1-t^2}\right)^2}.$$

We find that

$$p'(t) = -\frac{(1-t^2)(-t^3+7t^2+3t+3)(t^3+7t^2-3t+3)}{8t^2(t^2+1)(t^2+3)^2} < 0,$$

$$q'(t) = \frac{(t-1)^2(t+1)(t^4+4t^3+18t^2+12t+9)(-3t^3+7t^2-3t+3)}{16t^3(t^2+1)(t^2+3)^2} > 0,$$

which imply the result as $\lim_{t\to 1_-} p(t) = \lim_{t\to 1_-} q(t) = 0$. ■

3. The dual of Shafer's inequality. In this section, we discuss the dual inequality (2) of Shafer's inequality (1). We begin by proposing approximations near the origin of the form given by

$$\arctan x \approx \frac{2x}{1+\sqrt{1+x^2}} + \varepsilon(x)$$

with $\varepsilon(x) \to 0$ as $x \to 0$. For large values of x, approximations of the form given by

$$\arctan x \approx \left(\frac{\pi}{2} - 2\right) + \frac{2x}{1+\sqrt{1+x^2}} + \upsilon(x)$$

with $\upsilon(x) \to 0$ as $x \to \infty$ should be considered. More precisely, we state the following inequalities which give good results near the origin and at infinity, respectively.

THEOREM 4. *For every $x > 0$,*

$$-\frac{1}{12}x^3 < \arctan x - \frac{2x}{1+\sqrt{1+x^2}} < -\frac{1}{12}x^3 + \frac{3}{40}x^5. \tag{10}$$

THEOREM 5. *For every $x > 0$,*

$$\left(\frac{\pi}{2} - 2\right) + \frac{1}{x} - \frac{1}{x^2} < \arctan x - \frac{2x}{1+\sqrt{1+x^2}} < \left(\frac{\pi}{2} - 2\right) + \frac{1}{x}. \tag{11}$$

REMARK 2. It is easily seen that (10) and (11) provide significant improvements of (2).

Proof of Theorem 4. Using (3), we have to prove that $\rho(t) > 0$ and $\theta(t) < 0$ on $(0,1)$, where

$$\rho(t) := \arctan\left(\frac{2t}{1-t^2}\right) - 2t + \frac{1}{12}\left(\frac{2t}{1-t^2}\right)^3,$$
$$\theta(t) := \arctan\left(\frac{2t}{1-t^2}\right) - 2t + \frac{1}{12}\left(\frac{2t}{1-t^2}\right)^3 - \frac{3}{40}\left(\frac{2t}{1-t^2}\right)^5.$$

Indeed, we have

$$\rho'(t) = \frac{2t^4(3-t^2)(2-t^2+t^4)}{(1+t^2)(1-t^2)^4} > 0,$$
$$\theta'(t) = -\frac{2t^6(29-14t^2+14t^4-6t^6+t^8)}{(1+t^2)(1-t^2)^6} < 0.$$

Since $\rho(0) = \theta(0) = 0$, the result follows. ∎

Proof of Theorem 5. Making use of (3), we have to prove that $\phi(t) > 0$ and $\eta(t) < 0$ on $(0,1)$, where

$$\phi(t) := \left[\arctan\left(\frac{2t}{1-t^2}\right) - 2t\right] - \left(\frac{\pi}{2} - 2\right) - \frac{1-t^2}{2t} + \left(\frac{1-t^2}{2t}\right)^2,$$

$$\eta(t) := \left[\arctan\left(\frac{2t}{1-t^2}\right) - 2t\right] - \left(\frac{\pi}{2} - 2\right) - \frac{1-t^2}{2t}.$$

It is easily observed that

$$\phi'(t) = -\frac{(1+t)(1-t)^2(1+2t^2-t^3)}{2t^3(1+t^2)} < 0,$$

$$\eta'(t) = \frac{(1-t^2)(1+3t^2)}{2t^2(1+t^2)} > 0.$$

Since $\lim_{t\to 1_-} \phi(t) = \lim_{t\to 1_-} \eta(t) = 0$, the proof is complete. ∎

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