COLLOQUIUM MATHEMATICUM

VOL. 137 2014 NO. 1

A CONSTRUCTION OF THE HOM-YETTER–DRINFELD CATEGORY

BY

HAIYING LI and TIANSHUI MA (Xinxiang)

Abstract. In continuation of our recent work about smash product Hom-Hopf algebras [Colloq. Math. 134 (2014)], we introduce the Hom-Yetter–Drinfeld category ${}_H^H\mathbb{YD}$ via the Radford biproduct Hom-Hopf algebra, and prove that Hom-Yetter–Drinfeld modules can provide solutions of the Hom-Yang–Baxter equation and ${}_H^H\mathbb{YD}$ is a pre-braided tensor category, where (H,β,S) is a Hom-Hopf algebra. Furthermore, we show that $(A{}^\natural_\diamond H,\alpha\otimes\beta)$ is a Radford biproduct Hom-Hopf algebra if and only if (A,α) is a Hom-Hopf algebra in the category ${}_H^H\mathbb{YD}$. Finally, some examples and applications are given.

1. Introduction. The motivation to introduce Hom-type algebras comes from examples related to q-deformations of Witt and Virasoro algebras, which play an important role in physics, mainly in conformal field theory. Hom-structures (Lie algebras, algebras, coalgebras, Hopf algebras) have been intensively investigated in the literature recently: see [2, 3, 5, 8–11, 16–19, 24–32]. Hom-algebras are generalizations of algebras obtained by a twisting map, which have been introduced for the first time by Makhlouf and Silvestrov [18]. Here associativity is replaced by Hom-associativity; Hom-coassociativity for a Hom-coalgebra can be considered in a similar way.

Yau [24, 28] introduced and characterized the concept of module Hom-algebras as a twisted version of usual module algebras, and the dual version (i.e. comodule Hom-coalgebras) was studied by Zhang [31]. Based on Yau's definition of module Hom-algebras, Ma–Li–Yang [11] constructed smash product Hom-Hopf algebras $(A\natural H,\alpha\otimes\beta)$ generalizing Molnar's smash product (see [13]), gave the cobraided structure (in the sense of Yau's definition in [27]) on $(A\natural H,\alpha\otimes\beta)$, and also considered the case of twist tensor product Hom-Hopf algebras. Makhlouf and Panaite [16] defined and studied a class of Yetter–Drinfeld modules over Hom-bialgebras and derived the constructions of twistors, pseudotwistors, twisted tensor product and smash product in the Hom-case in [17].

Yetter–Drinfeld modules are known to be at the origin of a very vast family of solutions to the Yang–Baxter equation. Let H be a bialgebra, and

2010 *Mathematics Subject Classification*: 16T05, 81R50.

Key words and phrases: Hom-smash (co)product, Hom-Yetter–Drinfeld category, Radford biproduct, Hom-Yang–Baxter equation.

DOI: 10.4064/cm137-1-4

A a left H-module algebra and a left H-comodule coalgebra. Radford [20] gave a construction of a bialgebra (called a Radford biproduct bialgebra) by combining the smash product algebra $A\# H$ with the smash coproduct coalgebra $A\times H$. Majid [14,15] made the following conclusion: A is a bialgebra in the Yetter–Drinfeld category ${}_H^H\mathcal{YD}$ if and only if $A\star H$ is a Radford biproduct. The Radford biproduct plays an important role in the lifting method for the classification of finite-dimensional pointed Hopf algebras (see [1]).

In this paper, we introduce the Hom-Yetter–Drinfeld category ${}_H^H\mathbb{YD}$ via the Radford biproduct Hom-Hopf algebra, and prove that the Hom-Yetter–Drinfeld modules can provide solutions of the Hom-Yang–Baxter equation. Furthermore, we show that $(A_\diamond^\natural H, \alpha\otimes\beta)$ is a Radford biproduct Hom-Hopf algebra if and only if (A,α) is a Hom-Hopf algebra in the category ${}_H^H\mathbb{YD}$.

This article is organized as follows. In Section 2, we recall some definitions and results which will be used later. Let (H,β) be a Hom-bialgebra, and (A,α) a left (H,β)-module Hom-algebra and a left (H,β)-comodule Hom-coalgebra. In [11], the smash product Hom-algebra $(A\natural H, \alpha\otimes\beta)$ was constructed. In Section 3, we first define a smash coproduct Hom-coalgebra $(A\diamond H, \alpha\otimes\beta)$ (see Proposition 3.1), then derive necessary and sufficient conditions for $(A\natural H, \alpha\otimes\beta)$ and $(A\diamond H, \alpha\otimes\beta)$ to be a Hom-bialgebra, which is called the Radford biproduct Hom-bialgebra and denoted by $(A_\diamond^\natural H, \alpha\otimes\beta)$ (see Theorems 3.3, 3.6). In Section 4, we introduce the Hom-Yetter–Drinfeld category ${}_H^H\mathbb{YD}$ (see Definition 4.1,4.2), which is different from the one defined by Makhlouf and Panaite [16], the one defined by Chen and Zhang [5] and the one defined by Liu and Shen [9]. We also prove that Hom-Yetter–Drinfeld modules can provide solutions of the Hom-Yang–Baxter equation in the sense of Yau's definition in [26, 29, 30] (see Proposition 4.3) and that ${}_H^H\mathbb{YD}$ is a pre-braided tensor category (see Theorem 4.7). Furthermore, we deduce that $(A_\diamond^\natural H, \alpha\otimes\beta)$ is a Radford biproduct Hom-Hopf algebra if and only if (A,α) is a Hom-Hopf algebra in the category ${}_H^H\mathbb{YD}$ (see Theorem 4.8), which generalizes Majid's result [14, 15]. In the last section, some examples and applications are given.

Throughout this paper we freely use the Hopf algebra and coalgebra terminology introduced in [6, 21–23].

The authors have been informed by the Editor that paper [4] related to the subject of our paper is accepted for publication.

2. Preliminaries. Throughout this paper, we follow the definitions and terminology of [7, 11, 24, 26, 31], with all algebraic systems supposed to be over the field K. Given a K-space M, we write id_M for the identity map on M.

We now recall some useful definitions.

Definition 2.1. A *Hom-algebra* is a quadruple $(A, \mu, 1_A, \alpha)$ (abbr. (A, α)), where A is a K-linear space, $\mu : A \otimes A \to A$ is a K-linear map, $1_A \in A$ and α is an automorphism of A, such that

(HA1) $\alpha(aa') = \alpha(a)\alpha(a')$, $\alpha(1_A) = 1_A$,

(HA2) $\alpha(a)(a'a'') = (aa')\alpha(a'')$, $a1_A = 1_Aa = \alpha(a)$,

for all $a, a', a'' \in A$. Here we use the notation $\mu(a \otimes a') = aa'$.

Let (A, α) and (B, β) be two Hom-algebras. Then $(A \otimes B, \alpha \otimes \beta)$ is a Hom-algebra (called the tensor product Hom-algebra) with multiplication $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ and unit $1_A \otimes 1_B$.

Definition 2.2. A *Hom-coalgebra* is a quadruple $(C, \Delta, \varepsilon_C, \beta)$ (abbr. (C, β)), where C is a K-linear space, $\Delta : C \to C \otimes C$, $\varepsilon_C : C \to K$ are K-linear maps, and β is an automorphism of C, such that

(HC1) $\beta(c)_1 \otimes \beta(c)_2 = \beta(c_1) \otimes \beta(c_2)$, $\varepsilon_C \circ \beta = \varepsilon_C$,

(HC2) $\beta(c_1) \otimes c_{21} \otimes c_{22} = c_{11} \otimes c_{12} \otimes \beta(c_2)$, $\varepsilon_C(c_1)c_2 = c_1\varepsilon_C(c_2) = \beta(c)$,

for all $c \in A$. Here we use the notation $\Delta(c) = c_1 \otimes c_2$ (summation implicitly understood).

Let (C, α) and (D, β) be two Hom-coalgebras. Then $(C \otimes D, \alpha \otimes \beta)$ is a Hom-coalgebra (called the tensor product Hom-coalgebra) with comultiplication $\Delta(c \otimes d) = c_1 \otimes d_1 \otimes c_2 \otimes d_2$ and counit $\varepsilon_C \otimes \varepsilon_D$.

Definition 2.3. A *Hom-bialgebra* is a sextuple $(H, \mu, 1_H, \Delta, \varepsilon, \gamma)$ (abbr. (H, γ)), where $(H, \mu, 1_H, \gamma)$ is a Hom-algebra and $(H, \Delta, \varepsilon, \gamma)$ is a Hom-coalgebra, such that Δ and ε are morphisms of Hom-algebras, i.e.

$$\Delta(hh') = \Delta(h)\Delta(h'), \quad \Delta(1_H) = 1_H \otimes 1_H,$$
$$\varepsilon(hh') = \varepsilon(h)\varepsilon(h'), \quad \varepsilon(1_H) = 1.$$

Furthermore, if there exists a linear map $S : H \to H$ such that

$$S(h_1)h_2 = h_1S(h_2) = \varepsilon(h)1_H \quad \text{and} \quad S(\gamma(h)) = \gamma(S(h)),$$

then we call $(H, \mu, 1_H, \Delta, \varepsilon, \gamma, S)$ (abbr. (H, γ, S)) a *Hom-Hopf algebra*.

Let (H, γ) and (H', γ') be two Hom-bialgebras. A linear map $f : H \to H'$ is called a *Hom-bialgebra map* if $f \circ \gamma = \gamma' \circ f$ and at the same time f is a bialgebra map in the usual sense.

Definition 2.4 (see [24, 28]). Let (A, β) be a Hom-algebra. A *left* (A, β)-*Hom-module* is a triple $(M, \triangleright, \alpha)$, where M is a linear space, $\triangleright : A \otimes M \to M$ is a linear map, and α is an automorphism of M, such that

(HM1) $\alpha(a \triangleright m) = \beta(a) \triangleright \alpha(m)$,

(HM2) $\beta(a) \triangleright (a' \triangleright m) = (aa') \triangleright \alpha(m)$, $1_A \triangleright m = \alpha(m)$,

for all $a, a' \in A$ and $m \in M$.

Let $(M, \triangleright_M, \alpha_M)$ and $(N, \triangleright_N, \alpha_N)$ be two left (A, β)-Hom-modules. Then a linear morphism $f : M \to N$ is called a *morphism of left* (A, β)-*Hom-modules* if $f(h \triangleright_M m) = h \triangleright_N f(m)$ and $\alpha_M \circ f = f \circ \alpha_N$.

REMARKS. (1) It is obvious that (A, μ, β) is a left (A, β)-Hom-module.

(2) When $\beta = \mathrm{id}_A$ and $\alpha = \mathrm{id}_M$, a left (A, β)-Hom-module is the usual left A-module.

DEFINITION 2.5 (see [24,28]). Let (H, β) be a Hom-bialgebra and (A, α) a Hom-algebra. If $(A, \triangleright, \alpha)$ is a left (H, β)-Hom-module and for all $h \in H$ and $a, a' \in A$,

(HMA1) $\beta^2(h) \triangleright (aa') = (h_1 \triangleright a)(h_2 \triangleright a')$,

(HMA2) $h \triangleright 1_A = \varepsilon_H(h) 1_A$,

then $(A, \triangleright, \alpha)$ is called an (H, β)-*module Hom-algebra.*

REMARKS. (1) When $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_H$, an (H, β)-module Hom-algebra is the usual H-module algebra.

(2) Similar to the case of Hopf algebras, Yau [24, 28] concluded that (HMA1) is satisfied if and only if μ_A is a morphism of H-modules for suitable H-module structures on $A \otimes A$ and A.

(3) The smash product Hom-Hopf algebra $(A \natural H, \alpha \otimes \beta)$ is different from the one defined by Chen, Wang and Zhang [3], since here the construction of $(A \natural B, \alpha \otimes \beta)$ is based on the concept of the module Hom-algebra introduced by Yau [24, 28], while two of conditions [3, (6.1), (6.2)] are the same as in the case of Hopf algebra.

DEFINITION 2.6 (see [31]). Let (C, β) be a Hom-coalgebra. A *left* (C, β)-*Hom-comodule* is a triple (M, ρ, α), where M is a linear space, $\rho : M \to C \otimes M$ (write $\rho(m) = m_{-1} \otimes m_0$, $\forall m \in M$) is a linear map, and α is an automorphism of M, such that

(HCM1) $\alpha(m)_{-1} \otimes \alpha(m)_0 = \beta(m_{-1}) \otimes \alpha(m_0)$,

(HCM2) $\beta(m_{-1}) \otimes m_{0-1} \otimes m_{00} = m_{-11} \otimes m_{-12} \otimes \alpha(m_0)$,
$\varepsilon_C(m_{-1}) m_0 = \alpha(m)$,

for all $m \in M$.

Let (M, ρ^M, α_M) and (N, ρ^N, α_N) be two left (C, β)-Hom-comodules. Then a linear map $f : M \to N$ is called a *map of left* (C, β)-*Hom-comodules* if $f(m)_{-1} \otimes f(m)_0 = m_{-1} \otimes f(m_0)$ and $\alpha_M \circ f = f \circ \alpha_N$.

REMARKS. (1) It is obvious that (C, Δ_C, β) is a left (C, β)-Hom-comodule.

(2) When $\beta = \mathrm{id}_A$ and $\alpha = \mathrm{id}_M$, a left (C, β)-Hom-comodule is the usual left C-comodule.

DEFINITION 2.7 (see [31]). Let (H,β) be a Hom-bialgebra and (C,α) a Hom-coalgebra. If (C,ρ,α) is a left (H,β)-Hom-comodule and for all $c\in C$,

(HCMC1) $\beta^2(c_{-1})\otimes c_{01}\otimes c_{02}=c_{1-1}c_{2-1}\otimes c_{10}\otimes c_{20}$,
(HCMC2) $c_{-1}\varepsilon_C(c_0)=1_H\varepsilon_C(c)$,

then (C,ρ,α) is called an (H,β)-*comodule Hom-coalgebra*.

REMARKS. (1) When $\alpha=\mathrm{id}_A$ and $\beta=\mathrm{id}_H$, an (H,β)-comodule Hom-coalgebra is the usual H-comodule coalgebra.

(2) Similar to the case of Hopf algebras, Zhang [31] concluded that (HCMC1) is satisfied if and only if Δ_C is a morphism of H-comodules for suitable H-comodule structures on $C\otimes C$ and C.

DEFINITION 2.8 (see [11]). Let (H,β) be a Hom-bialgebra and (C,α) a Hom-coalgebra. If $(C,\triangleright,\alpha)$ is a left (H,β)-Hom-module and for all $h\in H$ and $c\in A$,

(HMC1) $(h\triangleright c)_1\otimes(h\triangleright c)_2=(h_1\triangleright c_1)\otimes(h_2\triangleright c_2)$,
(HMC2) $\varepsilon_C(h\triangleright c)=\varepsilon_H(h)\varepsilon_C(c)$,

then $(C,\triangleright,\alpha)$ is called an (H,β)-*module Hom-coalgebra*.

REMARK. When $\alpha=\mathrm{id}_C$ and $\beta=\mathrm{id}_H$, an (H,β)-module Hom-coalgebra is the usual H-module coalgebra.

DEFINITION 2.9 (see [25]). Let (H,β) be a Hom-bialgebra and (A,α) a Hom-algebra. If (A,ρ,α) is a left (H,β)-Hom-comodule and for all $a,a'\in A$,

(HCMA1) $\rho(aa')=a_{-1}a'_{-1}\otimes a_0a'_0$,
(HCMA2) $\rho(1_A)=1_H\otimes 1_A$,

then (A,ρ,α) is called an (H,β)-*comodule Hom-algebra*.

REMARK. When $\alpha=\mathrm{id}_A$ and $\beta=\mathrm{id}_H$, an (H,β)-comodule Hom-algebra is the usual H-comodule algebra.

DEFINITION 2.10 (see [11]). Let (H,β) be a Hom-bialgebra and $(A,\triangleright,\alpha)$ an (H,β)-module Hom-algebra. Then $(A\natural H,\alpha\otimes\beta)$ ($A\natural H=A\otimes H$ as a linear space) with multiplication

$$(a\otimes h)(a'\otimes h')=a(h_1\triangleright\alpha^{-1}(a'))\otimes\beta^{-1}(h_2)h',$$

where $a,a'\in A,\ h,h'\in H$, and with unit $1_A\otimes 1_H$, is a Hom-algebra; we call it a *smash product Hom-algebra*.

REMARK. Here the multiplication of a smash product Hom-algebra is different from the one defined by Makhlouf and Panaite in [17, Theorem 3.1].

DEFINITION 2.11 (see [1, 15, 16]). Let H be a bialgebra and M a linear space which is a left H-module with action $\triangleright: H\otimes M\to M$, $h\otimes m\mapsto h\triangleright m$, and a left H-comodule with coaction $\rho: M\to H\otimes M$, $\rho(m)=m_{-1}\otimes m_0$.

Then M is called a (left-left) *Yetter–Drinfeld module* over H if the following compatibility condition holds, for all $h \in H$ and $m \in M$:

(YD) $h_1 m_{-1} \otimes (h_2 \triangleright m_0) = (h_1 \triangleright m)_{-1} h_2 \otimes (h_1 \triangleright m)_0$.

When H is a Hopf algebra, then (YD) is equivalent to

(YD)′ $h_1 m_{-1} S_H(h_3) \otimes (h_2 \triangleright m_0) = (h \triangleright m)_{-1} \otimes (h \triangleright m)_0$.

3. Radford biproduct Hom-Hopf algebra. In this section, we mainly generalize the Radford biproduct bialgebra of [20, Theorem 1] to the Hom-setting.

Dual to Definition 2.10, we have:

PROPOSITION 3.1. *Let (H, β) be a Hom-bialgebra and (C, ρ, α) an (H, β)-comodule Hom-coalgebra. Then $(C \diamond H, \alpha \otimes \beta)$ ($C \diamond H = C \otimes H$ as a linear space) with comultiplication*

$$\Delta_{C\diamond H}(c \otimes h) = c_1 \otimes c_{2-1}\beta^{-1}(h_1) \otimes \alpha^{-1}(c_{20}) \otimes h_2,$$

where $c \in C$, $h \in H$, and with counit $\varepsilon_C \otimes \varepsilon_H$, is a Hom-coalgebra; we call it a smash coproduct Hom-coalgebra.

In fact, dual to [11, Theorem 3.1], we have

PROPOSITION 3.2. *Let $(C, \Delta_C, \varepsilon_C, \alpha)$ and $(H, \Delta_H, \varepsilon_H, \beta)$ be two Hom-coalgebras, and $T : C \otimes H \to H \otimes C$ (write $T(c \otimes h) = h_T \otimes c_T$, $\forall c \in C$, $h \in H$) a linear map such that for all $c \in C$ and $h \in H$,*

$$\alpha(c)_T \otimes \beta(h)_T = \alpha(c_T) \otimes \beta(h_T).$$

Then $(C \diamond_T H, \alpha \otimes \beta)$ ($C \diamond_T H = C \otimes H$ as a linear space) with comultiplication

$$\Delta_{C\diamond_T H}(c \otimes h) = c_1 \otimes \beta^{-1}(h_1)_T \otimes \alpha^{-1}(c_{2T}) \otimes h_2,$$

and with counit $\varepsilon_C \otimes \varepsilon_H$, becomes a Hom-coalgebra if and only if the following conditions hold:

(C1) $\varepsilon_H(h_T)c_T = \varepsilon_H(h)\alpha(c)$, $h_T \varepsilon_C(c_T) = \beta(h)\varepsilon_C(c)$,
(C2) $h_{T1} \otimes h_{T2} \otimes \alpha(c_T) = \beta(\beta^{-1}(h_1)_T) \otimes h_{2t} \otimes c_{Tt}$,
(C3) $\beta(h_T) \otimes \alpha(c)_{T1} \otimes \alpha(c)_{T2} = h_{Tt} \otimes \alpha(c_1)_t \otimes \alpha(c_{2T})$,

where $c \in C, h \in H$ and t is a copy of T.

We call this Hom-coalgebra a *T-smash coproduct Hom-coalgebra.*

REMARKS. (1) Letting $T(c \otimes h) = c_{-1}h \otimes c_0$ in $C \diamond_T H$, we get the smash coproduct Hom-coalgebra $C \diamond H$.

(2) Here the comultiplication of a T-smash coproduct Hom-coalgebra is slightly different from the one defined by Zheng [32]. And the conditions (C1)–(C3) are simpler than the ones in [32].

THEOREM 3.3. *Let (H,β) be a Hom-bialgebra, (A,α) a left (H,β)-module Hom-algebra with module structure $\triangleright : H\otimes A\to A$ and a left (H,β)-comodule Hom-coalgebra with comodule structure $\rho : A\to H\otimes A$. Then the following are equivalent:*

- *$(A{}^{\natural}_{\diamond}H, \mu_{A\natural H}, 1_A\otimes 1_H, \Delta_{A\diamond H}, \varepsilon_A\otimes\varepsilon_H, \alpha\otimes\beta)$ is a Hom-bialgebra, where $(A\natural H, \alpha\otimes\beta)$ is a smash product Hom-algebra and $(A\diamond H, \alpha\otimes\beta)$ is a smash coproduct Hom-coalgebra.*
- *The following conditions hold (for all $a,b\in A$ and $h\in H$):*

  (R1) *(A,ρ,α) is an (H,β)-comodule Hom-algebra,*
  (R2) *$(A,\triangleright,\alpha)$ is an (H,β)-module Hom-coalgebra,*
  (R3) *ε_A is a Hom-algebra map and $\Delta_A(1_A)=1_A\otimes 1_A$,*
  (R4) *$\Delta_A(ab)=a_1(\beta^2(a_{2-1})\triangleright\alpha^{-1}(b_1))\otimes\alpha^{-1}(a_{20})b_2$,*
  (R5) *$h_1\beta(a_{-1})\otimes(\beta^3(h_2)\triangleright a_0)=(\beta^2(h_1)\triangleright a)_{-1}h_2\otimes(\beta^2(h_1)\triangleright a)_0$.*

In this case, we call this Hom-bialgebra a *Radford biproduct Hom-bialgebra* and denote it by $(A{}^{\natural}_{\diamond}H, \alpha\otimes\beta)$.

Proof. ($\Leftarrow$) It is easy to prove that $\varepsilon_{A^{\natural}_{\diamond}H}=\varepsilon_A\otimes\varepsilon_H$ is a morphism of Hom-algebras. Next we check $\Delta_{A^{\natural}_{\diamond}H}=\Delta_{A\diamond H}$ is a morphism of Hom-algebras as follows. For all $a,b\in A$ and $h,g\in H$, we have

$$
\begin{aligned}
&\Delta_{A^{\natural}_{\diamond}H}((a\otimes h)(b\otimes g))\\
&= (a(h_1\triangleright\alpha^{-1}(b)))_1\otimes(a(h_1\triangleright\alpha^{-1}(b)))_{2-1}\beta^{-1}((\beta^{-1}(h_2)g)_1)\\
&\quad\otimes\alpha^{-1}((a(h_1\triangleright\alpha^{-1}(b)))_{20})\otimes(\beta^{-1}(h_2)g)_2\\
&\overset{(HA1),(HC1)}{=} (a(h_1\triangleright\alpha^{-1}(b)))_1\otimes(a(h_1\triangleright\alpha^{-1}(b)))_{2-1}(\beta^{-2}(h_{21})\beta^{-1}(g_1))\\
&\quad\otimes\alpha^{-1}((a(h_1\triangleright\alpha^{-1}(b)))_{20})\otimes\beta^{-1}(h_{22})g_2\\
&\overset{(R4)}{=} a_1(\beta^2(a_{2-1})\triangleright\alpha^{-1}((h_1\triangleright\alpha^{-1}(b))_1))\\
&\quad\otimes(\alpha^{-1}(a_{20})(h_1\triangleright\alpha^{-1}(b))_2)_{-1}(\beta^{-2}(h_{21})\beta^{-1}(g_1))\\
&\quad\otimes\alpha^{-1}((\alpha^{-1}(a_{20})(h_1\triangleright\alpha^{-1}(b))_2)_0)\otimes\beta^{-1}(h_{22})g_2\\
&\overset{(HCA1)}{=} a_1(\beta^2(a_{2-1})\triangleright\alpha^{-1}((h_1\triangleright\alpha^{-1}(b))_1))\\
&\quad\otimes(\alpha^{-1}(a_{20})_{-1}(h_1\triangleright\alpha^{-1}(b))_{2-1})(\beta^{-2}(h_{21})\beta^{-1}(g_1))\\
&\quad\otimes\alpha^{-1}(\alpha^{-1}(a_{20})_0)\alpha^{-1}((h_1\triangleright\alpha^{-1}(b))_{20})\otimes\beta^{-1}(h_{22})g_2\\
&\overset{(HMC1)}{=} a_1(\beta^2(a_{2-1})\triangleright\alpha^{-1}(h_{11}\triangleright\alpha^{-1}(b_1)))\\
&\quad\otimes(\alpha^{-1}(a_{20})_{-1}(h_{12}\triangleright\alpha^{-1}(b_2))_{-1})(\beta^{-2}(h_{21})\beta^{-1}(g_1))\\
&\quad\otimes\alpha^{-1}(\alpha^{-1}(a_{20})_0)\alpha^{-1}((h_{12}\triangleright\alpha^{-1}(b_2))_0)\otimes\beta^{-1}(h_{22})g_2\\
&\overset{(HA2)}{=} a_1(\beta^2(a_{2-1})\triangleright\alpha^{-1}(h_{11}\triangleright\alpha^{-1}(b_1)))\\
&\quad\otimes(\alpha^{-1}(a_{20})_{-1}\beta^{-1}((h_{12}\triangleright\alpha^{-1}(b_2))_{-1}(\beta^{-2}(h_{21})))g_1\\
&\quad\otimes\alpha^{-1}(\alpha^{-1}(a_{20})_0)\alpha^{-1}((h_{12}\triangleright\alpha^{-1}(b_2))_0)\otimes\beta^{-1}(h_{22})g_2
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(HC2)}{=} && a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}(\beta(h_1) \triangleright \alpha^{-1}(b_1)))\\
& && \otimes (\alpha^{-1}(a_{20})_{-1}\beta^{-1}((\beta^{-1}(h_{211}) \triangleright \alpha^{-1}(b_2))_{-1}\beta^{-3}(h_{212})))g_1\\
& && \otimes \alpha^{-1}(\alpha^{-1}(a_{20})_0)\alpha^{-1}((\beta^{-1}(h_{211}) \triangleright \alpha^{-1}(b_2))_0) \otimes \beta^{-1}(h_{22})g_2\\
&\overset{(HC1)}{=} && a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}(\beta(h_1) \triangleright \alpha^{-1}(b_1)))\\
& && \otimes (\alpha^{-1}(a_{20})_{-1}\beta^{-1}((\beta^2(\beta^{-3}(h_{21})_1) \triangleright \alpha^{-1}(b_2))_{-1}\beta^{-3}(h_{21})_2))g_1\\
& && \otimes \alpha^{-1}(\alpha^{-1}(a_{20})_0)\alpha^{-1}((\beta^2(\beta^{-3}(h_{21})_1) \triangleright \alpha^{-1}(b_2))_0) \otimes \beta^{-1}(h_{22})g_2\\
&\overset{(R5)}{=} && a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}(\beta(h_1) \triangleright \alpha^{-1}(b_1)))\\
& && \otimes (\alpha^{-1}(a_{20})_{-1}\beta^{-1}(\beta^{-3}(h_{21})_1)\beta(\alpha^{-1}(b_2)_{-1})))g_1\\
& && \otimes \alpha^{-1}(\alpha^{-1}(a_{20})_0)\alpha^{-1}(\beta^3(\beta^{-3}(h_{21})_2) \triangleright \alpha^{-1}(b_2)_0)\\
&\overset{(HCM1),(HC1)}{=} && a_1(\beta^2(a_{2-1}) \triangleright \alpha^{-1}(\beta(h_1) \triangleright \alpha^{-1}(b_1)))\\
& && \otimes (\beta^{-1}(a_{20-1})\beta^{-1}(\beta^{-3}(h_{211})b_{2-1}))g_1\\
& && \otimes \alpha^{-2}(a_{200})\alpha^{-1}(h_{212} \triangleright \alpha^{-1}(b_{20})) \otimes \beta^{-1}(h_{22})g_2\\
&\overset{(HCM2)}{=} && a_1(\beta(a_{2-11}) \triangleright \alpha^{-1}(\beta(h_1) \triangleright \alpha^{-1}(b_1)))\\
& && \otimes (\beta^{-1}(a_{2-12})\beta^{-1}(\beta^{-3}(h_{211})b_{2-1}))g_1\\
& && \otimes \alpha^{-1}(a_{20})\alpha^{-1}(h_{212} \triangleright \alpha^{-1}(b_{20})) \otimes \beta^{-1}(h_{22})g_2\\
&\overset{(HA2)}{=} && a_1(\beta(a_{2-11}) \triangleright \alpha^{-1}(\beta(h_1) \triangleright \alpha^{-1}(b_1)))\\
& && \otimes (\beta^{-1}(a_{2-12})\beta^{-3}(h_{211}))(b_{2-1}\beta^{-1}(g_1))\\
& && \otimes \alpha^{-1}(a_{20})\alpha^{-1}(h_{212} \triangleright \alpha^{-1}(b_{20})) \otimes \beta^{-1}(h_{22})g_2\\
&\overset{(HC2)}{=} && a_1(\beta(a_{2-11}) \triangleright \alpha^{-1}(h_{11} \triangleright \alpha^{-1}(b_1)))\\
& && \otimes (\beta^{-1}(a_{2-12})\beta^{-2}(h_{12}))(b_{2-1}\beta^{-1}(g_1))\\
& && \otimes \alpha^{-1}(a_{20})\alpha^{-1}(\beta(h_{21}) \triangleright \alpha^{-1}(b_{20})) \otimes \beta^{-1}(h_{22})g_2\\
&\overset{(HM1)}{=} && a_1(\beta(a_{2-11}) \triangleright (\beta^{-1}(h_{11}) \triangleright \alpha^{-2}(b_1)))\\
& && \otimes (\beta^{-1}(a_{2-12})\beta^{-2}(h_{12}))(b_{2-1}\beta^{-1}(g_1))\\
& && \otimes \alpha^{-1}(a_{20})(h_{21} \triangleright \alpha^{-2}(b_{20})) \otimes \beta^{-1}(h_{22})g_2\\
&\overset{(HM2)}{=} && a_1((a_{2-11}\beta^{-1}(h_{11})) \triangleright \alpha^{-1}(b_1))\\
& && \otimes (\beta^{-1}(a_{2-12})\beta^{-2}(h_{12}))(b_{2-1}\beta^{-1}(g_1))\\
& && \otimes\alpha^{-1}(a_{20})(h_{21} \triangleright \alpha^{-2}(b_{20})) \otimes \beta^{-1}(h_{22})g_2\\
&\overset{(HA1)}{=} && a_1((a_{2-1}\beta^{-1}(h_1))_1 \triangleright \alpha^{-1}(b_1))\\
& && \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1))_2)(b_{2-1}\beta^{-1}(g_1))\\
& && \otimes \alpha^{-1}(a_{20})(h_{21} \triangleright \alpha^{-2}(b_{20})) \otimes \beta^{-1}(h_{22})g_2\\
&= && (a_1 \otimes a_{2-1}\beta^{-1}(h_1) \otimes \alpha^{-1}(a_{20}) \otimes h_2)\\
& && \qquad \times (b_1 \otimes b_{2-1}\beta^{-1}(h_1) \otimes \alpha^{-1}(b_{20}) \otimes h_2)\\
&= && \Delta_{A^\natural_\diamond H}(a \otimes h)\Delta_{A^\natural_\diamond H}(b \otimes g),
\end{aligned}
$$

and $\Delta_{A^\natural_\diamond H}(1_A \otimes 1_H) = 1_A \otimes 1_H \otimes 1_A \otimes 1_H$ can be proved directly.

(⇒) We only verify that conditions (R4) and (R5) hold; the others hold similarly. As $\Delta_{A^{\natural}_{\diamond}H} = \Delta_{A\diamond H}$ is a morphism of Hom-algebras, for all $a, b \in A$ and $h, g \in H$ we have

$$\begin{aligned} a_1((a_{2-1}\beta^{-1}(h_1))_1 \triangleright \alpha^{-1}(b_1)) \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1))_2)(b_{2-1}\beta^{-1}(g_1)) \\ \otimes \alpha^{-1}(a_{20})(h_{21} \triangleright \alpha^{-2}(b_{20})) \otimes \beta^{-1}(h_{22})g_2 \\ = (a(h_1 \triangleright \alpha^{-1}(b)))_1 \otimes (a(h_1 \triangleright \alpha^{-1}(b)))_{2-1}\beta^{-1}((\beta^{-1}(h_2)g)_1) \\ \otimes \alpha^{-1}((a(h_1 \triangleright \alpha^{-1}(b)))_{20}) \otimes (\beta^{-1}(h_2)g)_2. \end{aligned}$$

Applying $\mathrm{id}_A \otimes \varepsilon_H \otimes \mathrm{id}_A \otimes \varepsilon_H$ to the above equation and setting $h = g = 1_H$ we get (HB). (HYD) can be obtained by applying $\varepsilon_A \otimes \mathrm{id}_H \otimes \mathrm{id}_A \otimes \varepsilon_H$ to the above equation and setting $a = 1_A$ and $g = 1_H$. ∎

Remarks. If $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_H$, then we get the well-known Radford biproduct bialgebra of [20, Theorem 1].

(2) Theorem 3.3 is different from the one defined by Liu and Shen [9], because the Hom-smash product there is based on the concept of module Hom-algebra in [3] and ours is based on Yau's [24, 28].

Corollary 3.4 (see [11]). *Let $(A, \alpha), (H, \beta)$ be two Hom-bialgebras, and $(A, \triangleright, \alpha)$ an (H, β)-module Hom-algebra. Then the smash product Hom-algebra $(A \natural H, \alpha \otimes \beta)$ endowed with the tensor product Hom-coalgebra structure becomes a Hom-bialgebra if and only if $(A, \triangleright, \alpha)$ is an (H, β)-module Hom-coalgebra and*

$$h_1 \otimes h_2 \triangleright a = h_2 \otimes h_1 \triangleright a.$$

Proof. Let the comodule action ρ be trivial, i.e. $\rho(a) = 1_H \otimes \alpha(a)$ in Theorem 3.3. ∎

Corollary 3.5. *Let $(C, \alpha), (H, \beta)$ be two Hom-bialgebras, and (C, ρ, α) an (H, β)-comodule Hom-coalgebra. Then the smash coproduct Hom-coalgebra $(C \diamond H, \alpha \otimes \beta)$ endowed with the tensor product Hom-algebra structure becomes a Hom-bialgebra if and only if (C, ρ, α) is an (H, β)-comodule Hom-algebra and*

$$hc_{-1} \otimes c_0 = c_{-1}h \otimes c_0.$$

Proof. Let the module action $\triangleright$ be trivial, i.e. $h \triangleright c = \varepsilon_H(h)\alpha(c)$ in Theorem 3.3. ∎

Theorem 3.6. *Let (H, β, S_H) be a Hom-Hopf algebra, and (A, α) be a Hom-algebra and a Hom-coalgebra. Assume that $(A^{\natural}_{\diamond}H, \alpha \otimes \beta)$ is a Radford biproduct Hom-bialgebra defined as above, and $S_A : A \to A$ is a linear map such that $S_A(a_1)a_2 = a_1 S_A(a_2) = \varepsilon_A(a)1_A$ and $\alpha \circ S_A = S_A \circ \alpha$. Then $(A^{\natural}_{\diamond}H, \alpha \otimes \beta, S_{A^{\natural}_{\diamond}H})$ is a Hom-Hopf algebra, where*

$$S_{A^{\natural}_{\diamond}H}(a \otimes h) = (S_H(a_{-1}\beta^{-1}(h))_1 \triangleright S_A(\alpha^{-2}(a_0))) \otimes \beta^{-1}(S_H(a_{-1}\beta^{-1}(h))_2).$$

Proof. We can compute that $(A^{\natural}_{\diamond}H, \alpha\otimes\beta, S_{A^{\natural}_{\diamond}H})$ is a Hom-Hopf algebra as follows. For all $a\in A$ and $h\in H$, we have

$$
\begin{aligned}
&(S_{A^{\natural}_{\diamond}H} * \mathrm{id}_{A^{\natural}_{\diamond}H})(a\otimes h)\\
&= (S_H(a_{1-1}\beta^{-1}(a_{2-1}\beta^{-1}(h_1)))_1 \triangleright S_A(\alpha^{-2}(a_{10})))\\
&\qquad \times (\beta^{-1}(S_H(a_{1-1}\beta^{-1}(a_{2-1}\beta^{-1}(h_1)))_2)_1 \triangleright \alpha^{-2}(a_{20}))\\
&\quad \otimes \beta^{-1}(\beta^{-1}(S_H(a_{1-1}\beta^{-1}(a_{2-1}\beta^{-1}(h_1)))_2)_2)h_2\\
&\overset{\text{(HA1),(HA2)}}{=} (S_H(\beta^{-1}(a_{1-1}a_{2-1})\beta^{-1}(h_1))_1 \triangleright S_A(\alpha^{-2}(a_{10})))\\
&\qquad \times (\beta^{-1}(S_H(\beta^{-1}(a_{1-1}a_{2-1})\beta^{-1}(h_1))_2)_1 \triangleright \alpha^{-2}(a_{20}))\\
&\quad \otimes \beta^{-1}(\beta^{-1}(S_H(\beta^{-1}(a_{1-1}a_{2-1})\beta^{-1}(h_1))_2)_2)h_2\\
&\overset{\text{(HCMC1)}}{=} (S_H(\beta(a_{-1})\beta^{-1}(h_1))_1 \triangleright S_A(\alpha^{-2}(a_{01})))\\
&\qquad \times (\beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_2)_1 \triangleright \alpha^{-2}(a_{02}))\\
&\quad \otimes \beta^{-1}(\beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_2)_2)h_2\\
&\overset{\text{(HC1),(HC2)}}{=} (\beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_{11}) \triangleright S_A(\alpha^{-2}(a_{01})))\\
&\qquad \times (\beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_{12}) \triangleright \alpha^{-2}(a_{02}))\\
&\quad \otimes \beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_2)h_2\\
&\overset{\text{(HC1),(HMA1)}}{=} (\beta(S_H(\beta(a_{-1})\beta^{-1}(h_1))_1) \triangleright (S_A(\alpha^{-2}(a_{01}))\alpha^{-2}(a_{02}))\\
&\quad \otimes \beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_2)h_2\\
&\overset{\text{(HA1)}}{=} (\beta(S_H(\beta(a_{-1})\beta^{-1}(h_1))_1) \triangleright 1_A\varepsilon_A(a_0))\\
&\quad \otimes \beta^{-1}(S_H(\beta(a_{-1})\beta^{-1}(h_1))_2)h_2\\
&\overset{\text{(HCMC2)}}{=} (\beta(S_H(h_1)_1) \triangleright 1_A\varepsilon_A(a)) \otimes \beta^{-1}(S_H(h_1)_2)h_2\\
&\overset{\text{(HMA2)}}{=} 1_A\varepsilon_A(a) \otimes S_H(h_1)h_2 = (1_A\otimes 1_H)\varepsilon_A(a)\varepsilon_H(h)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathrm{id}_{A^{\natural}_{\diamond}H} * S_{A^{\natural}_{\diamond}H})(a\otimes h)\\
&= a_1((a_{2-1}\beta^{-1}(h_1))_1 \triangleright \alpha^{-1}(S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2))_1\\
&\qquad \triangleright S_A(\alpha^{-2}(\alpha^{-1}(a_{20})_0))))\\
&\quad \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1))_2)\beta^{-1}(S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2))_2)\\
&\overset{\text{(HM1)}}{=} a_1((a_{2-1}\beta^{-1}(h_1))_1 \triangleright (\beta^{-1}(S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2))_1)\\
&\qquad \triangleright S_A(\alpha^{-3}(\alpha^{-1}(a_{20})_0))))\\
&\quad \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1))_2)\beta^{-1}(S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2))_2)\\
&\overset{\text{(HM2),(HA1)}}{=} a_1(\beta^{-1}((a_{2-1}\beta^{-1}(h_1))_1 S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2))_1)\\
&\qquad \triangleright S_A(\alpha^{-2}(\alpha^{-1}(a_{20})_0)))\\
&\quad \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1))_2 S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2))_2)
\end{aligned}
$$

$$\begin{aligned}
&\overset{\text{(HC1)}}{=} a_1(\beta^{-1}((a_{2-1}\beta^{-1}(h_1))S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2)))_1 \\
&\qquad \triangleright S_A(\alpha^{-2}(\alpha^{-1}(a_{20})_0))) \\
&\quad \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1))S_H(\alpha^{-1}(a_{20})_{-1}\beta^{-1}(h_2)))_2 \\
&\overset{\text{(HCM1)}}{=} a_1(\beta^{-1}((a_{2-1}\beta^{-1}(h_1))S_H(\beta^{-1}(a_{20-1})\beta^{-1}(h_2)))_1 \triangleright S_A(\alpha^{-3}(a_{200}))) \\
&\quad \otimes \beta^{-1}((a_{2-1}\beta^{-1}(h_1))S_H(\beta^{-1}(a_{20-1})\beta^{-1}(h_2)))_2 \\
&\overset{\text{(HCM2)}}{=} a_1(\beta^{-1}((\beta^{-1}(a_{2-11})\beta^{-1}(h_1))S_H(\beta^{-1}(a_{2-12})\beta^{-1}(h_2)))_1 \\
&\qquad \triangleright S_A(\alpha^{-2}(a_{20}))) \\
&\quad \otimes \beta^{-1}((\beta^{-1}(a_{2-11})\beta^{-1}(h_1))S_H(\beta^{-1}(a_{2-12})\beta^{-1}(h_2)))_2 \\
&\overset{\text{(HC1)}}{=} a_1((1_H \triangleright S_A(\alpha^{-2}(a_{20})))\varepsilon_H(a_{2-1}) \otimes 1_H\varepsilon_H(h) \\
&\overset{\text{(HCM2)}}{=} a_1(1_H \triangleright S_A(\alpha^{-1}(a_2))) \otimes 1_H\varepsilon_H(h) \\
&\overset{\text{(HM2)}}{=} a_1S_A(a_2) \otimes 1_H\varepsilon_H(h) \\
&= (1_A \otimes 1_H)\varepsilon_A(a)\varepsilon_H(h),
\end{aligned}$$

while

$$\begin{aligned}
&S_{A\natural_\diamond H}(\alpha(a) \otimes \beta(h)) \\
&\quad = (S_H(\alpha(a)_{-1}h)_1 \triangleright S_A(\alpha^{-2}(\alpha(a)_0))) \otimes \beta^{-1}(S_H(\alpha(a)_{-1}h)_2) \\
&\quad \overset{\text{(HCM1)}}{=} (S_H(\beta(a_{-1})h)_1 \triangleright S_A(\alpha^{-1}(a_0))) \otimes \beta^{-1}(S_H(\beta(a_{-1})h)_2) \\
&\quad = (\alpha \otimes \beta)(S_{A\natural_\diamond H}(a \otimes h)),
\end{aligned}$$

finishing the proof. ■

Corollary 3.7 (see [11]). *Let $(A, \alpha, S_A), (H, \beta, S_H)$ be two Hom-Hopf algebras, and $(A \natural H, \alpha \otimes \beta)$ a smash product Hom-bialgebra. Then $(A \natural H, \alpha \otimes \beta, S_{A\natural H})$ is a Hom-Hopf algebra, where*

$$S_{A\natural H}(a \otimes h) = (S_H(h)_1 \triangleright \alpha^{-1}(S_A(a))) \otimes \beta^{-1}(S_H(h)_2).$$

Proof. Let the comodule action ρ be trivial, i.e. $\rho(a) = 1_H \otimes \alpha(a)$ in Theorem 3.6. ■

Corollary 3.8. *Let $(C, \alpha, S_C), (H, \beta, S_H)$ be two Hom-Hopf algebras, and $(C \diamond H, \alpha \otimes \beta)$ a smash coproduct Hom-bialgebra. Then $(C \diamond H, \alpha \otimes \beta, S_{C\diamond H})$ is a Hom-Hopf algebra, where*

$$S_{C\diamond H}(c \otimes h) = S_C(\alpha^{-1}(c_{(0)})) \otimes S_H(c_{(-1)}\beta^{-1}(h)).$$

Proof. Let the module action $\triangleright$ be trivial, i.e. $h \triangleright c = \varepsilon_H(h)\alpha(c)$ in Theorem 3.6. ■

4. Hom-Yetter–Drinfeld category. In this section, we give the definition of a Hom-Yetter–Drinfeld module and also prove that the category

${}^H_H\mathbb{YD}$ of Hom-Yetter–Drinfeld modules is a pre-braided tensor category. Furthermore, we show that $(A\natural_\diamond H, \alpha\otimes\beta)$ is a Radford biproduct Hom-bialgebra if and only if (A,α) is a Hom-bialgebra in the category ${}^H_H\mathbb{YD}$.

DEFINITION 4.1. Let (H,β) be a Hom-bialgebra, $(M,\triangleright_M,\alpha_M)$ a left (H,β)-module with action $\triangleright_M : H\otimes M \to M$, $h\otimes m\mapsto h\triangleright_M m$, and (M,ρ^M,α_M) a left (H,β)-comodule with coaction $\rho^M : M\to H\otimes M, m\mapsto m_{-1}\otimes m_0$. Then we call $(M,\triangleright_M,\rho^M,\alpha_M)$ a (left-left) *Hom-Yetter–Drinfeld module* over (H,β) if

(HYD) $h_1\beta(m_{-1})\otimes(\beta^3(h_2)\triangleright_M m_0) = (\beta^2(h_1)\triangleright_M m)_{-1}h_2\otimes(\beta^2(h_1)\triangleright_M m)_0$

for all $h\in H$ and $m\in M$.

REMARKS. (1) The compatibility condition (HYD) is different from condition (2.1) in [16, Definition 2.1], condition (3.1) in [5, Definition 3.1] and condition (4.1) in [9, Definition 4.1].

(2) When $\beta = \mathrm{id}_H$, condition (HYD) is exactly condition (YD).

(3) Let (H,β) be a Hom-bialgebra and K a field. Then (K,id_K) is a (left-left) Hom-Yetter–Drinfeld module over (H,β) with the module and comodule actions defined as follows: $H\otimes K\to K$, $h\otimes k\mapsto\varepsilon(h)k$ and $K\to H\otimes K$, $k\mapsto 1_H\otimes k$.

(4) When (H,β,S_H) is a Hom-Hopf algebra, then the condition (HYD) is equivalent to

(HYD)′ $(\beta^4(h)\triangleright_M m)_{-1}\otimes(\beta^4(h)\triangleright_M m)_0$
$$= \beta^{-2}(h_{11}\beta(m_{-1}))S_H(h_2)\otimes(\beta^3(h_{12})\triangleright_M m_0).$$

Proof. $(\Rightarrow)$ We have

$$\begin{aligned}
&\beta^{-2}(h_{11}\beta(m_{-1}))S(h_2)\otimes(\beta^3(h_{12})\triangleright m_0)\\
&\overset{\text{(HYD)}}{=} \beta^{-2}((\beta^2(h_{11}\triangleright m))_{-1}h_{12})S(h_2)\otimes(\beta^2(h_{11}\triangleright m))_0\\
&\overset{\text{(HA1),(HA2)}}{=} \beta^{-1}((\beta^2(h_{11}\triangleright m))_{-1})(\beta^{-2}(h_{12})\beta^{-1}(S(h_2)))\otimes(\beta^2(h_{11}\triangleright m))_0\\
&\overset{\text{(HC2)}}{=} \beta^{-1}((\beta^2(h_1\triangleright m))_{-1})(\beta^{-2}(h_{21})\beta^{-2}(S(h_{22})))\otimes(\beta^2(h_1\triangleright m))_0\\
&\overset{\text{(HA1)}}{=} \beta^{-1}((\beta^2(h_1\triangleright m))_{-1})(\beta^{-2}(h_{21}S(h_{22})))\otimes(\beta^2(h_1\triangleright m))_0\\
&\overset{\text{(HA2),(HC2)}}{=} (\beta^4(h)\triangleright m)_{-1}\otimes(\beta^4(h)\triangleright m)_0.
\end{aligned}$$

$(\Leftarrow)$ We have

$$\begin{aligned}
&(\beta^2(h_1)\triangleright m)_{-1}h_2\otimes(\beta^2(h_1)\triangleright m)_0\\
&\overset{\text{(HYD)}'}{=} (\beta^{-2}(\beta^{-2}(h_1)_{11}\beta(m_{-1}))S(\beta^{-2}(h_1)_2))h_2\otimes(\beta^3(\beta^{-2}(h_1)_{12})\triangleright m_0)\\
&\overset{\text{(HC1)}}{=} (\beta^{-2}(\beta^{-2}(h_{111})\beta(m_{-1}))S(\beta^{-2}(h_{12})))h_2\otimes(\beta(h_{112})\triangleright m_0)\\
&\overset{\text{(HC2),(HC1)}}{=} (\beta^{-2}(\beta^{-1}(h_{11})\beta(m_{-1}))S(\beta^{-2}(h_{21})))\beta^{-1}(h_{22})\otimes(\beta^2(h_{12})\triangleright m_0)
\end{aligned}$$

$$\overset{(\mathrm{HA2}),(\mathrm{HA1})}{=} (\beta^{-1}(\beta^{-1}(h_{11})\beta(m_{-1}))(\beta^{-2}S(h_{21})h_{22}) \otimes (\beta^2(h_{12}) \rhd m_0)$$
$$= (\beta^{-1}(\beta^{-1}(h_{11})\beta(m_{-1}))1_H\varepsilon_H(h_2) \otimes (\beta^2(h_{12}) \rhd m_0)$$
$$\overset{(\mathrm{HC1}),(\mathrm{HC2}),(\mathrm{HA1})}{=} h_1\beta(m_{-1}) \otimes (\beta^3(h_2) \rhd m_0).$$

Here we use $\rhd$, S instead of $\rhd_M$, S_H, respectively. ∎

Definition 4.2. Let (H,β) be a Hom-bialgebra. We denote by ${}^H_H\mathbb{YD}$ the category whose objects are all Hom-Yetter–Drinfeld modules $(M, \rhd_M, \rho^M, \alpha_M)$ over (H,β); the morphisms are morphisms of left (H,β)-modules and left (H,β)-comodules.

In the following, we give a solution of the Hom-Yang–Baxter equation introduced and studied by Yau [26, 29, 30].

Proposition 4.3. *Let (H,β) be a Hom-bialgebra and $(M, \rhd_M, \rho^M, \alpha_M)$, $(N, \rhd_N, \rho^N, \alpha_N) \in {}^H_H\mathbb{YD}$. Define the linear map*

$$\tau_{M,N} : M \otimes N \to N \otimes M, \qquad m \otimes n \mapsto \beta^3(m_{-1}) \rhd_N n \otimes m_0,$$

for $m \in M$ and $n \in N$. Then $\tau_{M,N} \circ (\alpha_M \otimes \alpha_N) = (\alpha_N \otimes \alpha_M) \circ \tau_{M,N}$, and if $(P, \rhd_P, \rho^P, \alpha_P) \in {}^H_H\mathbb{YD}$, the maps $\tau_{\_,\_}$ satisfy the Hom-Yang–Baxter equation

$$(\alpha_P \otimes \tau_{M,N}) \circ (\tau_{M,P} \otimes \alpha_N) \circ (\alpha_M \otimes \tau_{N,P}) = (\tau_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes \tau_{M,P}) \circ (\tau_{M,N} \otimes \alpha_P).$$

Proof. We only check the second equality; the first one is easy. For all $m \in M$, $n \in N$ and $p \in P$, we have

$$(\alpha_P \otimes \tau_{M,N}) \circ (\tau_{M,P} \otimes \alpha_N) \circ (\alpha_M \otimes \tau_{N,P})(m \otimes n \otimes p)$$
$$= (\beta^3(\alpha_M(m)_{-1}) \rhd_P (\beta^3(n_{-1}) \rhd_P p)) \otimes \beta^3(\alpha_M(m)_{0-1}) \rhd_N \alpha_N(n_0) \otimes \alpha_M(m)_{00}$$
$$\overset{(\mathrm{HM1})}{=} (\beta^4(\alpha_M(m)_{-1}) \rhd_P (\beta^4(n_{-1}) \rhd_P \alpha_P(p))) \otimes \beta^3(\alpha_M(m)_{0-1}) \rhd_N \alpha_N(n_0) \otimes \alpha_M(m)_{00}$$
$$\overset{(\mathrm{HCM1})}{=} (\beta^5(m_{-1}) \rhd_P (\beta^4(n_{-1}) \rhd_P \alpha_P(p))) \otimes \beta^4(m_{0-1}) \rhd_N \alpha_N(n_0) \otimes \alpha_M(m_{00})$$
$$\overset{(\mathrm{HCM2})}{=} (\beta^4(m_{-11}) \rhd_P (\beta^4(n_{-1}) \rhd_P \alpha_P(p))) \otimes \beta^4(m_{-12}) \rhd_N \alpha_N(n_0) \otimes \alpha_M^2(m_0)$$
$$\overset{(\mathrm{HM2})}{=} ((\beta^3(m_{-11})\beta^4(n_{-1})) \rhd_P \alpha_P^2(p)) \otimes \beta^4(m_{-12}) \rhd_N \alpha_N(n_0) \otimes \alpha_M^2(m_0)$$
$$\overset{(\mathrm{HCM1})}{=} ((\beta^3(m_{-11}\alpha_N(n)_{-1})) \rhd_P \alpha_P^2(p)) \otimes \beta^4(m_{-12}) \rhd_N \alpha_N(n)_0 \otimes \alpha_M^2(m_0)$$
$$\overset{(\mathrm{HA1})}{=} (\beta^2(\beta(m_{-11})\beta(\alpha_N(n)_{-1}))) \rhd_P \alpha_P^2(p)) \otimes \beta^3(\beta(m_{-12})) \rhd_N \alpha_N(n)_0 \otimes \alpha_M^2(m_0)$$

$$
\begin{aligned}
&\overset{\text{(HC1)}}{=} (\beta^2(\beta(m_{-1})_1\beta(\alpha_N(n)_{-1}))) \rhd_P \alpha_P^2(p)) \otimes \beta^3(\beta(m_{-1})_2) \rhd_N \alpha_N(n)_0 \\
&\quad \otimes \alpha_M^2(m_0) \\
&\overset{\text{(HYD)}}{=} (\beta^2((\beta^2(\beta(m_{-1})_1) \rhd_N \alpha_N(n))_{-1}\beta(m_{-1})_2) \rhd_P \alpha_P^2(p)) \\
&\quad \otimes \beta^2(\beta(m_{-1})_1) \rhd_N \alpha_N(n))_0 \otimes \alpha_M^2(m_0) \\
&\overset{\text{(HA1),(HC1)}}{=} ((\beta^2((\beta^3(m_{-11}) \rhd_N \alpha_N(n))_{-1})\beta^3(m_{-12})) \rhd_P \alpha_P^2(p)) \\
&\quad \otimes (\beta^3(m_{-11}) \rhd_N \alpha_N(n))_0 \otimes \alpha_M^2(m_0) \\
&\overset{\text{(HCM2)}}{=} ((\beta^2((\beta^4(m_{-1}) \rhd_N \alpha_N(n))_{-1})\beta^3(m_{0-1})) \rhd_P \alpha_P^2(p)) \\
&\quad \otimes (\beta^4(m_{-1}) \rhd_N \alpha_N(n))_0 \otimes \alpha_M(m_{00}) \\
&\overset{\text{(HM2)}}{=} (\beta^3((\beta^4(m_{-1}) \rhd_N \alpha_N(n))_{-1}) \rhd_P (\beta^3(m_{0-1}) \rhd_P \alpha_P(p))) \\
&\quad \otimes (\beta^4(m_{-1}) \rhd_N \alpha_N(n))_0 \otimes \alpha_M(m_{00}) \\
&\overset{\text{(HM1)}}{=} (\beta^3(\alpha_N(\beta^3(m_{-1}) \rhd_N n)_{-1}) \rhd_P (\beta^3(m_{0-1}) \rhd_P \alpha_P(p))) \\
&\quad \otimes \alpha_N(\beta^3(m_{-1}) \rhd_N n)_0 \otimes \alpha_M(m_{00}) \\
&= (\tau_{N,P} \otimes \alpha_M) \circ (\alpha_N \otimes \tau_{M,P}) \circ (\tau_{M,N} \otimes \alpha_P)(m \otimes n \otimes p). \ \blacksquare
\end{aligned}
$$

LEMMA 4.4. *Let (H,β) be a Hom-bialgebra and $(M, \rhd_M, \rho^M, \alpha_M)$, $(N, \rhd_N, \rho^N, \alpha_N) \in {}^H_H\mathbb{YD}$. Define the linear maps*

$$\rhd_{M\otimes N} : H \otimes M \otimes N \to M \otimes N, \quad h \otimes m \otimes n \mapsto (h_1 \rhd_M m) \otimes (h_2 \rhd_N n),$$

and

$$\rho^{M\otimes N} : M \otimes N \to H \otimes M \otimes N, \quad m \otimes n \mapsto \beta^{-2}(m_{-1}n_{-1}) \otimes m_0 \otimes n_0,$$

for $h \in H$, $m \in M$ and $n \in N$. Then $(M \otimes N, \rhd_{M\otimes N}, \rho^{M\otimes N}, \alpha_M \otimes \alpha_N)$ is a Hom-Yetter–Drinfeld module.

Proof. It is easy to check that $(M \otimes N, \rhd_{M\otimes N}, \alpha_M \otimes \alpha_N)$ is an (H,β)-Hom-module and $(M \otimes N, \rho^{M\otimes N}, \alpha_M \otimes \alpha_N)$ is an (H,β)-Hom-comodule. Since for $h \in H$, $m \in M$ and $n \in N$, we have

$$
\begin{aligned}
&(\beta^2(h_1) \rhd_{M\otimes N} (m \otimes n))_{-1}h_2 \otimes (\beta^2(h_1) \rhd_{M\otimes N} (m \otimes n))_0 \\
&\quad = ((\beta^2(h_1)_1 \rhd_M m) \otimes (\beta^2(h_1)_2 \rhd_N n))_{-1}h_2 \\
&\qquad \otimes ((\beta^2(h_1)_1 \rhd_M m) \otimes (\beta^2(h_1)_2 \rhd_N n))_0 \\
&\quad = \beta^{-2}(((\beta^2(h_1)_1 \rhd_M m)_{-1}(\beta^2(h_1)_2 \rhd_N n)_{-1})\beta^2(h_2)) \\
&\qquad \otimes (\beta^2(h_1)_1 \rhd_M m)_0 \otimes (\beta^2(h_1)_2 \rhd_N n)_0 \\
&\quad \overset{\text{(HA1),(HA2)}}{=} \beta^{-2}(\beta((\beta^2(h_{11}) \rhd_M m)_{-1})((\beta^2(h_{12}) \rhd_N n)_{-1}\beta(h_2))) \\
&\qquad \otimes (\beta^2(h_{11}) \rhd_M m)_0 \otimes (\beta^2(h_{12}) \rhd_N n)_0 \\
&\quad \overset{\text{(HC2)}}{=} \beta^{-2}(\beta((\beta^3(h_1) \rhd_M m)_{-1})((\beta^2(h_{21}) \rhd_N n)_{-1}h_{22})) \\
&\qquad \otimes (\beta^3(h_1) \rhd_M m)_0 \otimes (\beta^2(h_{21}) \rhd_N n)_0
\end{aligned}
$$

$$\begin{aligned}
&\overset{\text{(HYD)}}{=} \beta^{-2}(\beta((\beta^3(h_1) \triangleright_M m)_{-1})(h_{21}\beta(n_{-1}))) \\
&\quad \otimes (\beta^3(h_1) \triangleright_M m)_0 \otimes (\beta^3(h_{22}) \triangleright_N n_0) \\
&\overset{\text{(HA2)}}{=} \beta^{-2}(((\beta^3(h_1) \triangleright_M m)_{-1}h_{21})\beta^2(n_{-1})) \otimes (\beta^3(h_1) \triangleright_M m)_0 \\
&\quad \otimes (\beta^3(h_{22}) \triangleright_N n_0) \\
&\overset{\text{(HC2)}}{=} \beta^{-2}(((\beta^2(h_{11}) \triangleright_M m)_{-1}h_{12})\beta^2(n_{-1})) \\
&\quad \otimes (\beta^2(h_{11}) \triangleright_M m)_0 \otimes (\beta^4(h_2) \triangleright_N n_0) \\
&\overset{\text{(HYD)}}{=} \beta^{-2}((h_{11}\beta(m_{-1}))\beta^2(n_{-1})) \otimes (\beta^3(h_{12}) \triangleright_M m_0) \otimes (\beta^4(h_2) \triangleright_N n_0) \\
&\overset{\text{(HA1)}}{=} (\beta^{-2}(h_{11})\beta^{-1}(m_{-1}))n_{-1} \otimes (\beta^3(h_{12}) \triangleright_M m_0) \otimes (\beta^4(h_2) \triangleright_N n_0) \\
&\overset{\text{(HA2)}}{=} \beta^{-1}(h_{11})(\beta^{-1}(m_{-1})\beta^{-1}(n_{-1})) \otimes (\beta^3(h_{12}) \triangleright_M m_0) \\
&\quad \otimes (\beta^4(h_2) \triangleright_N n_0) \\
&\overset{\text{(HC2)}}{=} h_1(\beta^{-1}(m_{-1})\beta^{-1}(n_{-1})) \otimes (\beta^3(h_{21}) \triangleright_M m_0) \otimes (\beta^3(h_{22}) \triangleright_N n_0) \\
&\overset{\text{(HC1),(HA1)}}{=} h_1\beta(\beta^{-2}(m_{-1}n_{-1})) \otimes (\beta^3(h_2)_1 \triangleright_M m_0) \otimes (\beta^3(h_2)_2 \triangleright_N n_0) \\
&= h_1\beta((m \otimes n)_{-1}) \otimes (\beta^3(h_2) \triangleright_{M\otimes N} (m \otimes n)_0),
\end{aligned}$$

condition (HYD) holds. Therefore $(M \otimes N, \triangleright_{M\otimes N}, \rho^{M\otimes N}, \alpha_M \otimes \alpha_N)$ is a Hom-Yetter–Drinfeld module. ■

LEMMA 4.5. *Let (H,β) be a Hom-bialgebra and $(M, \triangleright_M, \rho^M, \alpha_M)$, $(N,\triangleright_N, \rho^N, \alpha_N)$, $(P, \triangleright_P, \rho^P, \alpha_P) \in {}^H_H\mathbb{YD}$. With notation as above, define the linear map*

$$a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P),$$
$$(m \otimes n) \otimes p \mapsto \alpha_M^{-1}(m) \otimes (n \otimes \alpha_P(p)),$$

for $m \in M$, $n \in N$ and $p \in P$. Then $a_{M,N,P}$ is an isomorphism of left (H,β)-Hom-modules and left (H,β)-Hom-comodules.

Proof. Same as the proof of [16, Proposition 3.2]. ■

LEMMA 4.6. *Let (H,β) be a Hom-bialgebra and $(M, \triangleright_M, \rho^M, \alpha_M)$, $(N, \triangleright_N, \rho^N, \alpha_N) \in {}^H_H\mathbb{YD}$. Define the linear map*

$$c_{M,N} : M \otimes N \to N \otimes M, \quad m \otimes n \mapsto (\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n)) \otimes \alpha_M^{-1}(m_0),$$

where $m \in M$ and $n \in N$. Then $c_{M,N}$ is a morphism of left (H,β)-Hom-modules and left (H,β)-Hom-comodules.

Proof. For all $h \in H$, $m \in M$ and $n \in N$, firstly,

$$\begin{aligned}
&(\alpha_N \otimes \alpha_M) \circ c_{M,N}(m \otimes n) \\
&\quad = \alpha_N(\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n)) \otimes m_0 \\
&\quad \overset{\text{(HM1)}}{=} (\beta^3(m_{-1}) \triangleright_N n) \otimes m_0 \\
&\quad \overset{\text{(HCM1)}}{=} (\beta^2(\alpha_M(m)_{-1}) \triangleright_N \alpha_N^{-1}(\alpha_N(n)) \otimes \alpha_M^{-1}(\alpha_M(m)_0) \\
&\quad = c_{M,N} \circ (\alpha_M \otimes \alpha_N)(m \otimes n),
\end{aligned}$$

secondly,

$$
\begin{aligned}
c_{M,N}(h \triangleright_{M\otimes N} (m\otimes n)) &= c_{M,N}((h_1 \triangleright_M m)\otimes (h_2 \triangleright_N n))\\
&= (\beta^2((h_1 \triangleright_M m)_{-1}) \triangleright_N \alpha_N^{-1}(h_2 \triangleright_N n)) \otimes \alpha_M^{-1}((h_1 \triangleright_M m)_0)\\
&\overset{(HM1)}{=} (\beta^2((h_1 \triangleright_M m)_{-1}) \triangleright_N (\beta^{-1}(h_2) \triangleright_N \alpha_N^{-1}(n))) \otimes \alpha_M^{-1}((h_1 \triangleright_M m)_0)\\
&\overset{(HM2)}{=} ((\beta((h_1 \triangleright_M m)_{-1})\beta^{-1}(h_2)) \triangleright_N n) \otimes \alpha_M^{-1}((h_1 \triangleright_M m)_0)\\
&\overset{(HA1)}{=} (\beta((h_1 \triangleright_M m)_{-1}\beta^{-2}(h_2)) \triangleright_N n) \otimes \alpha_M^{-1}((h_1 \triangleright_M m)_0)\\
&\overset{(HYD)}{=} (\beta(\beta^{-2}(h)_1\beta(m_{-1})) \triangleright_N n) \otimes \alpha_M^{-1}(\beta^3(\beta^{-2}(h)_2) \triangleright_M m_0)\\
&\overset{(HC1)}{=} (\beta(\beta^{-2}(h_1)\beta(m_{-1})) \triangleright_N n) \otimes \alpha_M^{-1}(\beta^3(\beta^{-2}(h_2)) \triangleright_M m_0)\\
&= ((\beta^{-1}(h_1)\beta^2(m_{-1})) \triangleright_N n) \otimes \alpha_M^{-1}(\beta(h_2) \triangleright_M m_0)\\
&\overset{(HM1)}{=} ((\beta^{-1}(h_1)\beta^2(m_{-1})) \triangleright_N n) \otimes (h_2 \triangleright_M \alpha_M^{-1}(m_0))\\
&\overset{(HM2)}{=} (h_1 \triangleright_N (\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n))) \otimes (h_2 \triangleright_M \alpha_M^{-1}(m_0))\\
&= h \triangleright_{N\otimes M} ((\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n)) \otimes \alpha_M^{-1}(m_0))\\
&= h \triangleright_{N\otimes M} c_{M,N}(m\otimes n);
\end{aligned}
$$

finally,

$$
\begin{aligned}
&(\rho^{N\otimes M} \circ c_{M,N})(m\otimes n))\\
&= \beta^{-2}((\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n))_{-1}\alpha_M^{-1}(m_0)_{-1})\\
&\quad \otimes (\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n))_0 \otimes \alpha_M^{-1}(m_0)_0\\
&\overset{(HCM1)}{=} \beta^{-2}((\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n))_{-1}\beta^{-1}(m_{0-1}))\\
&\quad \otimes (\beta^2(m_{-1}) \triangleright_N \alpha_N^{-1}(n))_0 \otimes \alpha_M^{-1}(m_{00})\\
&\overset{(HCM2)}{=} \beta^{-2}((\beta(m_{-11}) \triangleright_N \alpha_N^{-1}(n))_{-1}\beta^{-1}(m_{-12}))\\
&\quad \otimes (\beta(m_{-11}) \triangleright_N \alpha_N^{-1}(n))_0 \otimes m_0\\
&\overset{(HC1)}{=} \beta^{-2}((\beta^2(\beta^{-1}(m_{-1})_1) \triangleright_N \alpha_N^{-1}(n))_{-1}\beta^{-1}(m_{-1})_2)\\
&\quad \otimes (\beta^2(\beta^{-1}(m_{-1})_1) \triangleright_N \alpha_N^{-1}(n))_0 \otimes m_0\\
&\overset{(HYD)}{=} \beta^{-2}(\beta^{-1}(m_{-1})_1\beta(\alpha_N^{-1}(n)_{-1})) \otimes (\beta^3(\beta^{-1}(m_{-1})_2) \triangleright_N \alpha_N^{-1}(n)_0) \otimes m_0\\
&\overset{(HC1),(HA1)}{=} \beta^{-3}(m_{-11})\beta^{-1}(\alpha_N^{-1}(n)_{-1}) \otimes (\beta^{-2}(m_{-12}) \triangleright_N \alpha_N^{-1}(n)_0) \otimes m_0\\
&\overset{(HCM1)}{=} \beta^{-3}(m_{-11})\beta^{-2}(n_{-1}) \otimes (\beta^{-2}(m_{-12}) \triangleright_N \alpha_N^{-1}(n_0)) \otimes m_0\\
&\overset{(HCM2)}{=} \beta^{-2}(m_{-1})\beta^{-2}(n_{-1}) \otimes (\beta^{-2}(m_{0-1}) \triangleright_N \alpha_N^{-1}(n_0)) \otimes \alpha_M^{-1}(m_{00})\\
&\overset{(HA1)}{=} \beta^{-2}(m_{-1}n_{-1}) \otimes (\beta^{-2}(m_{0-1}) \triangleright_N \alpha_N^{-1}(n_0)) \otimes \alpha_M^{-1}(m_{00})\\
&= (\mathrm{id}\otimes c_{M,N})(\beta^{-2}(m_{-1}n_{-1}) \otimes m_0 \otimes n_0)\\
&= (\mathrm{id}\otimes c_{M,N}) \circ \rho^{M\otimes N}(m\otimes n).
\end{aligned}
$$

Thus $c_{M,N}$ is a morphism of left (H,β)-Hom-modules and left (H,β)-Hom-comodules. ∎

REMARK. The pre-braiding $(c_{M,N})$ differs from the one in [16, Proposition 3.3].

THEOREM 4.7. *Let* (H,β) *be a Hom-bialgebra. Then the Hom-Yetter–Drinfeld category* ${}_H^H\mathbb{YD}$ *is a pre-braided tensor category, with tensor product, associativity constraints, and pre-braiding defined in Lemmas* 4.4, 4.5 *and* 4.6, *respectively, and with the unit* $I=(K,\mathrm{id}_K)$.

Proof. The proof of the pentagon axiom for $a_{M,N,P}$ coincides with the proof of [16, Theorem 3.4]. Next we prove the hexagonal relation for $c_{M,N}$. Let $(M,\triangleright_M,\rho^M,\alpha_M),(N,\triangleright_N,\rho^N,\alpha_N),(P,\triangleright_P,\rho^P,\alpha_P)\in{}_H^H\mathbb{YD}$. Then for all $m\in M$, $n\in N$ and $p\in P$, we have

$$
\begin{aligned}
&((\mathrm{id}_N\otimes c_{M,P})\circ(a_{N,M,P})\circ(c_{M,N}\otimes\mathrm{id}_P))((m\otimes n)\otimes p)\\
&\quad= \alpha_N^{-1}(\beta^2(m_{-1})\triangleright_N\alpha_N^{-1}(n))\otimes((\beta^2(\alpha_M^{-1}(m_0)_{-1})\triangleright_P p)\\
&\qquad\otimes\alpha_M^{-1}(\alpha_M^{-1}(m_0)_0))\\
&\quad\overset{(\mathrm{HCM1})}{=} \alpha_N^{-1}(\beta^2(m_{-1})\triangleright_N\alpha_N^{-1}(n))\otimes((\beta(m_{0-1})\triangleright_P p)\otimes\alpha_M^{-2}(m_{00}))\\
&\quad\overset{(\mathrm{HCM2})}{=} \alpha_N^{-1}(\beta(m_{-11})\triangleright_N\alpha_N^{-1}(n))\otimes((\beta(m_{-12})\triangleright_P p)\otimes\alpha_M^{-1}(m_0))\\
&\quad\overset{(\mathrm{HC1})}{=} \alpha_N^{-1}(\beta(m_{-1})_1\triangleright_N\alpha_N^{-1}(n))\otimes((\beta(m_{-1})_2\triangleright_P p)\otimes\alpha_M^{-1}(m_0))\\
&\quad\overset{(\mathrm{HCM1})}{=} \alpha_N^{-1}(\beta^2(\alpha_M^{-1}(m)_{-1})_1\triangleright_N\alpha_N^{-1}(n))\\
&\qquad\otimes((\beta^2(\alpha_M^{-1}(m)_{-1})_2\triangleright_P p)\otimes\alpha_M^{-1}(m)_0)\\
&\quad= (a_{N,P,M}\circ c_{M,N\otimes P}\circ a_{M,N,P})((m\otimes n)\otimes p),
\end{aligned}
$$

and

$$
\begin{aligned}
&((c_{M,P}\otimes\mathrm{id}_N)\circ(a^{-1}_{N,M,P})\circ(\mathrm{id}_M\otimes c_{N,P}))(m\otimes(n\otimes p))\\
&\quad= ((\beta^2(\alpha_M(m)_{-1})\triangleright_P\alpha_P^{-1}(\beta^2(n_{-1})\triangleright_P\alpha_P^{-1}(p)))\otimes\alpha_M^{-1}(\alpha_M(m)_0))\\
&\qquad\otimes\alpha_N^{-2}(n_0)\\
&\quad\overset{(\mathrm{HM1})}{=} ((\beta^2(\alpha_M(m)_{-1})\triangleright_P(\beta(n_{-1})\triangleright_P\alpha_P^{-2}(p)))\otimes\alpha_M^{-1}(\alpha_M(m)_0))\\
&\qquad\otimes\alpha_N^{-2}(n_0)\\
&\quad\overset{(\mathrm{HM2})}{=} (((\beta(\alpha_M(m)_{-1})\beta(n_{-1}))\triangleright_P\alpha_P^{-1}(p))\otimes\alpha_M^{-1}(\alpha_M(m)_0))\otimes\alpha_N^{-2}(n_0)\\
&\quad\overset{(\mathrm{HM1}),(\mathrm{HA1})}{=} (\alpha_P((\alpha_M(m)_{-1}n_{-1}))\triangleright_P\alpha_P^{-2}(p))\otimes\alpha_M^{-1}(\alpha_M(m)_0))\otimes\alpha_N^{-2}(n_0)\\
&\quad= (a^{-1}_{P,M,N}\circ c_{M\otimes N,P}\circ a^{-1}_{M,N,P})(m\otimes(n\otimes p)),
\end{aligned}
$$

finishing the proof. ∎

By Theorems 3.3, 3.6 and 4.7, we can get the main result in this paper.

Theorem 4.8. *Let (H,β) be a Hom-bialgebra, (A,α) a left (H,β)-module Hom-algebra and a left (H,β)-comodule Hom-coalgebra. Then $(A{}^{\natural}_{\diamond}H, \mu_{A\natural H}, 1_A\otimes 1_H, \Delta_{A\diamond H}, \varepsilon_A\otimes\varepsilon_H, \alpha\otimes\beta)$ is a Radford biproduct Hom-bialgebra if and only if (A,α) is a Hom-bialgebra in the Hom-Yetter–Drinfeld category ${}^H_H\mathbb{YD}$.*

Proof. This is obvious if we compare conditions (R4) and (R5) in Theorem 3.3 with condition (HYD) in Definition 4.1 and the definition of the pre-braiding $c_{M,N}$ in Lemma 4.6, respectively. ∎

Remarks. (1) If $\alpha = \mathrm{id}_A$ and $\beta = \mathrm{id}_H$ in Theorem 4.8, then we get Majid's conclusion about the usual Radford biproduct and Yetter–Drinfeld category.

(2) $(A{}^{\natural}_{\diamond}H, \mu_{A\natural H}, 1_A\otimes 1_H, \Delta_{A\diamond H}, \varepsilon_A\otimes\varepsilon_H, \alpha\otimes\beta, S_{A{}^{\natural}_{\diamond}H})$ is a Radford biproduct Hom-Hopf algebra if and only if (A,α,S_A) is a Hom-Hopf algebra in the Hom-Yetter–Drinfeld category ${}^H_H\mathbb{YD}$.

5. Applications. In this section, we give some applications of the above results.

Example 5.1. Let $K\mathbb{Z}_2 = K\{1,a\}$ be a Hopf group algebra (see [23]). Then $(K\mathbb{Z}_2, \mathrm{id}_{K\mathbb{Z}_2})$ is a Hom-Hopf algebra.

Let $T_{2,-1} = K\{1,g,x,y \mid g^2 = 1,\, x^2 = 0,\, y = gx,\, gy = -gy = x\}$ be Taft's Hopf algebra (see [13]). Its coalgebra structure and antipode are given by

$$\Delta(g) = g\otimes g,\quad \Delta(x) = x\otimes g + 1\otimes x,\quad \Delta(y) = y\otimes 1 + g\otimes y,$$
$$\varepsilon(g) = 1,\quad \varepsilon(x) = 0,\quad \varepsilon(y) = 0,$$

and

$$S(g) = g,\quad S(x) = y,\quad S(y) = -x.$$

Define a linear map α: $T_{2,-1}\to T_{2,-1}$ by

$$\alpha(1) = 1,\quad \alpha(g) = g,\quad \alpha(x) = kx,\quad \alpha(y) = ky$$

where $0\neq k\in K$. Then α is an automorphism of Hopf algebras.

So we get a Hom-Hopf algebra $H_\alpha = (T_{2,-1}, \alpha\circ\mu_{T_{2,-1}}, 1_{T_{2,-1}}, \Delta_{T_{2,-1}}\circ\alpha, \varepsilon_{T_{2,-1}}, \alpha)$ (see [19]). By a direct computation we get:

Lemma 5.1.1. *With the notations above, define a module action $\triangleright$: $K\mathbb{Z}_2\otimes H_\alpha\to H_\alpha$ by*

$$1_{K\mathbb{Z}_2}\triangleright 1_{H_\alpha} = 1_{H_\alpha},\quad 1_{K\mathbb{Z}_2}\triangleright g = g,$$
$$1_{K\mathbb{Z}_2}\triangleright x = kx,\quad 1_{K\mathbb{Z}_2}\triangleright y = ky,$$
$$a\triangleright 1_{H_\alpha} = 1_{H_\alpha},\quad a\triangleright g = g,$$
$$a\triangleright x = kx,\quad a\triangleright y = ky,$$

Then $(H_\alpha, \rhd, \alpha)$ *is a* $(K\mathbb{Z}_2, \mathrm{id}_{K\mathbb{Z}_2})$*-module Hom-algebra. Therefore,* $(H_\alpha\natural K\mathbb{Z}_2, \alpha\otimes\mathrm{id}_{K\mathbb{Z}_2})$ *is a smash product Hom-algebra.*

Lemma 5.1.2. *With the notations above, define a comodule action* $\rho : H_\alpha \to K\mathbb{Z}_2\otimes H_\alpha$ *by*

$$1_{H_\alpha}\mapsto 1_{K\mathbb{Z}_2}\otimes 1_{H_\alpha},\quad g\mapsto 1_{K\mathbb{Z}_2}\otimes g,\quad x\mapsto ka\otimes x,\quad y\mapsto ka\otimes y.$$

Then (H_α, ρ, α) *is a left* $(K\mathbb{Z}_2, \mathrm{id}_{K\mathbb{Z}_2})$*-comodule Hom-coalgebra. Therefore,* $(H_\alpha\natural K\mathbb{Z}_2, \alpha\otimes\mathrm{id}_{K\mathbb{Z}_2})$ *is a smash coproduct Hom-coalgebra.*

From the above two lemmas and a direct computation, we have

Theorem 5.1.3. *With the notations above,* $(H_\alpha\natural_\diamond K\mathbb{Z}_2,\ \mu_{H_\alpha\natural K\mathbb{Z}_2},\ 1_{H_\alpha}\otimes 1_{K\mathbb{Z}_2},\ \Delta_{H_\alpha\diamond K\mathbb{Z}_2},\ \varepsilon_{H_\alpha}\otimes\varepsilon_{K\mathbb{Z}_2},\ \alpha\otimes\mathrm{id}_{K\mathbb{Z}_2})$ *is a Radford biproduct Hom-bialgebra. Furthermore,* $(H_\alpha\natural_\diamond K\mathbb{Z}_2, \alpha\otimes\mathrm{id}_{K\mathbb{Z}_2}, S_{H_\alpha\natural_\diamond K\mathbb{Z}_2})$ *is a Hom-Hopf algebra, where* $S_{H_\alpha\natural_\diamond K\mathbb{Z}_2}$ *is defined by*

$$\begin{aligned}
S_{H_\alpha\natural_\diamond K\mathbb{Z}_2}(1_{H_\alpha}\otimes 1_{K\mathbb{Z}_2}) &= 1_{H_\alpha}\otimes 1_{K\mathbb{Z}_2}, & S_{H_\alpha\natural_\diamond K\mathbb{Z}_2}(1_{H_\alpha}\otimes a) &= 1_{H_\alpha}\otimes a,\\
S_{H_\alpha\natural_\diamond K\mathbb{Z}_2}(g\otimes 1_{K\mathbb{Z}_2}) &= g\otimes 1_{K\mathbb{Z}_2}, & S_{H_\alpha\natural_\diamond K\mathbb{Z}_2}(g\otimes a) &= g\otimes a,\\
S_{H_\alpha\natural_\diamond K\mathbb{Z}_2}(x\otimes 1_{K\mathbb{Z}_2}) &= y\otimes a, & S_{H_\alpha\natural_\diamond K\mathbb{Z}_2}(x\otimes a) &= y\otimes 1_{K\mathbb{Z}_2},\\
S_{H_\alpha\natural_\diamond K\mathbb{Z}_2}(y\otimes 1_{K\mathbb{Z}_2}) &= -x\otimes a, & S_{H_\alpha\natural_\diamond K\mathbb{Z}_2}(y\otimes a) &= -x\otimes 1_{K\mathbb{Z}_2}.
\end{aligned}$$

Example 5.2. Let $K\mathbb{Z}_2 = K\{1, a\}$ be a Hopf group algebra as in Example 5.1.

Let $A = K\{1, z\}$ be a vector space. Define the multiplication μ_A by

$$1z = z1 = lz,\quad z^2 = 0,$$

and the automorphism $\beta : A \to A$ by

$$\beta(1) = 1,\quad \beta(z) = lz,$$

for some $0 \neq l \in K$. Then (A, β) is a Hom-algebra.

Define the comultiplication Δ_A by

$$\Delta_A(1) = 1\otimes 1,\quad \Delta_A(z) = lz\otimes 1 + l1\otimes z,\quad \text{and}\quad \varepsilon_A(1) = 1,\quad \varepsilon_A(z) = 0.$$

Then (A, β) is a Hom-coalgebra. By a direct computation we get:

Lemma 5.2.1. *With the notations above, define a module action* $\trianglerighteq : K\mathbb{Z}_2\otimes A \to A$ *by*

$$\begin{aligned}
1_{K\mathbb{Z}_2}\trianglerighteq 1_A &= 1_A, & 1_{K\mathbb{Z}_2}\trianglerighteq z &= lz,\\
a\trianglerighteq 1_A &= 1_A, & a\trianglerighteq z &= -lz.
\end{aligned}$$

Then $(A, \trianglerighteq, \beta)$ *is a* $(K\mathbb{Z}_2, \mathrm{id}_{K\mathbb{Z}_2})$*-module Hom-algebra. Therefore,* $(A\natural K\mathbb{Z}_2, \beta\otimes\mathrm{id}_{K\mathbb{Z}_2})$ *is a smash product Hom-algebra.*

LEMMA 5.2.2. *With the notations above, define a comodule action* $\psi : A \to K\mathbb{Z}_2 \otimes A$ *by*

$$1_A \mapsto 1_{K\mathbb{Z}_2} \otimes 1_A, \quad z \mapsto la \otimes z.$$

Then (A, ψ, β) *is a left* $(K\mathbb{Z}_2, \mathrm{id}_{K\mathbb{Z}_2})$*-comodule Hom-coalgebra. Therefore,* $(A \natural K\mathbb{Z}_2, \beta \otimes \mathrm{id}_{K\mathbb{Z}_2})$ *is a smash coproduct Hom-coalgebra.*

By the above two lemmas and a direct computation, we have

THEOREM 5.2.3. *With the notations above,* $(A^{\natural}_{\diamond}K\mathbb{Z}_2, \mu_{A\natural K\mathbb{Z}_2}, 1_A \otimes 1_{K\mathbb{Z}_2}, \Delta_{A\diamond K\mathbb{Z}_2}, \varepsilon_A \otimes \varepsilon_{K\mathbb{Z}_2}, \beta \otimes \mathrm{id}_{K\mathbb{Z}_2})$ *is a Radford biproduct Hom-bialgebra. Furthermore,* $(A^{\natural}_{\diamond}K\mathbb{Z}_2, \beta \otimes \mathrm{id}_{K\mathbb{Z}_2}, S_{A^{\natural}_{\diamond}K\mathbb{Z}_2})$ *is a Hom-Hopf algebra, where* $S_{A^{\natural}_{\diamond}K\mathbb{Z}_2}$ *is defined by*

$$S_{A^{\natural}_{\diamond}K\mathbb{Z}_2}(1_A \otimes 1_{K\mathbb{Z}_2}) = 1_A \otimes 1_{K\mathbb{Z}_2}, \quad S_{A^{\natural}_{\diamond}K\mathbb{Z}_2}(1_A \otimes a) = 1_A \otimes a,$$
$$S_{A^{\natural}_{\diamond}K\mathbb{Z}_2}(z \otimes 1_{K\mathbb{Z}_2}) = z \otimes a, \quad S_{A^{\natural}_{\diamond}K\mathbb{Z}_2}(z \otimes a) = -z \otimes 1_{K\mathbb{Z}_2}.$$

REMARK. If $\beta = \mathrm{id}_A$, i.e., $l = 1$, then Example 5.2 coincides with the biproduct $B\star H$ (which is isomorphic to Sweedler's Hopf algebra $T_{2,\omega}$) of [12, Example 4.3].

In the following, let us recall the definition of a quasitriangular Hom-Hopf algebra from [26] or [10].

A *quasitriangular Hom-Hopf algebra* is an octuple $(H, \mu, 1_H, \Delta, \varepsilon, S, \beta, R)$ (abbr. (H, β, R)) in which $(H, \mu, 1_H, \Delta, \varepsilon, S, \beta)$ is a Hom-Hopf algebra and $R = R^1 \otimes R^2 \in H \otimes H$, satisfying the following axioms (for all $h \in H$ and $R = r$):

(QHA1) $\varepsilon(R^1)R^2 = R^1\varepsilon(R^2) = 1,$
(QHA2) ${R^1}_1 \otimes {R^1}_2 \otimes \beta(R^2) = \beta(R^1) \otimes \beta(r^1) \otimes R^2r^2,$
(QHA3) $\beta(R^1) \otimes {R^2}_1 \otimes {R^2}_2 = R^1r^1 \otimes \beta(r^2) \otimes \beta(R^2),$
(QHA4) $h_2R^1 \otimes h_1R^2 = R^1h_1 \otimes R^2h_2,$
(QHA5) $\beta(R^1) \otimes \beta(R^2) = R^1 \otimes R^2.$

Let (H, β, S) be a Hom-Hopf algebra and $R = R^1 \otimes R^2 \in H \otimes H$. Define

$$\rho^H : H \to H \otimes H, \quad h \mapsto h_{-1} \otimes h_0 = \beta^{-3}(R^2) \otimes R^1h.$$

PROPOSITION 5.3. *Let* (H, β, R) *be a quasitriangular Hom-Hopf algebra. Then* (H, β, ρ^H) *is a left* (H, β)*-comodule Hom-coalgebra and* $(H, \mu_H, \rho^H, \beta)$ *is a Hom-Yetter–Drinfeld module.*

Proof. We compute as follows:

$$\begin{aligned} \beta(h_{-1}) \otimes \beta(h_0) &= \beta(\beta^{-3}(R^2)) \otimes \beta(R^1h) \\ &\overset{(\mathrm{HA1})}{=} \beta(\beta^{-3}(R^2)) \otimes \beta(R^1)\beta(h) \\ &\overset{(\mathrm{QHA5})}{=} \beta^{-3}(R^2) \otimes R^1\beta(h) = \beta(h)_{-1} \otimes \beta(h)_0, \end{aligned}$$

so (HCM1) holds. Now,

$$\begin{aligned}
h_{-11}\otimes h_{-12}\beta(h_0) &= \beta^{-3}(R^2)_1\otimes\beta^{-3}(R^2)_2\otimes\beta(R^1h)\\
&\overset{\text{(HC1),(HA1)}}{=} \beta^{-3}(R^2{}_1)\otimes\beta^{-3}(R^2{}_2)\otimes\beta(R^1)\beta(h)\\
&\overset{\text{(QHA3)}}{=} \beta^{-2}(R^2)\otimes\beta^{-2}(r^2)\otimes(r^1R^1)\beta(h)\\
&\overset{\text{(HA2)}}{=} \beta^{-2}(R^2)\otimes\beta^{-2}(r^2)\otimes\beta(r^1)(R^1h)\\
&\overset{\text{(QHA5)}}{=} \beta^{-2}(R^2)\otimes\beta^{-3}(r^2)\otimes r^1(R^1h)\\
&= \beta(h_{-1})\otimes h_{0-1}\otimes h_{00},
\end{aligned}$$

thus we get (HCM2). Next,

$$\begin{aligned}
\beta^2(h_{-1})\otimes h_{01}\otimes h_{02} &= \beta^{-1}(R^2)\otimes(R^1h)_1\otimes(R^1h)_2\\
&= \beta^{-1}(R^2)\otimes R^1{}_1h_1\otimes R^1{}_2h_2\\
&\overset{\text{(QHA2)}}{=} \beta^{-2}(R^2r^2)\otimes\beta(R^1)h_1\otimes\beta(r^1)h_2\\
&\overset{\text{(QHA5),(HA1)}}{=} \beta^{-3}(R^2)\beta^{-3}(r^2)\otimes R^1h_1\otimes r^1h_2\\
&= h_{1-1}h_{1-1}\otimes h_{10}\otimes h_{20},
\end{aligned}$$

therefore we obtain (HCMC1).

(HCMC2) can be checked by using (QHA1).

Finally, we verify that (HYD) is satisfied:

$$\begin{aligned}
(\beta^2(h_1)\triangleright g)_{-1}h_2\otimes(\beta^2(h_1)\triangleright g)_0 &= \beta^{-3}(R^2)h_2\otimes R^1(\beta^2(h_1)g)\\
&\overset{\text{(HA2)}}{=} \beta^{-3}(R^2)h_2\otimes(\beta^{-1}(R^1)\beta^2(h_1))\beta(g)\\
&\overset{\text{(HA1)(HC1)}}{=} \beta^{-3}(R^2\beta^3(h)_2)\otimes\beta^{-1}(R^1\beta^3(h)_1)\beta(g)\\
&\overset{\text{(QHA4)}}{=} \beta^{-3}(\beta^3(h)_1R^2)\otimes\beta^{-1}(\beta^3(h)_2R^1)\beta(g)\\
&\overset{\text{(HA1)(HC1)}}{=} h_1\beta^{-3}(R^2)\otimes(\beta^2(h_2)\beta^{-1}(R^1))\beta(g)\\
&\overset{\text{(HA2)}}{=} h_1\beta^{-3}(R^2)\otimes\beta^3(h_2)(\beta^{-1}(R^1)g)\\
&\overset{\text{(QHA5)}}{=} h_1\beta^{-2}(R^2)\otimes\beta^3(h_2)(R^1g) = h_1\beta(g_{-1})\otimes(\beta^3(h_2)\triangleright g_0),
\end{aligned}$$

finishing the proof. ∎

PROPOSITION 5.4. *Let (H,β,S) be a Hom-Hopf algebra, with the notations as above. If (H,β,ρ^H) is a left (H,β)-comodule Hom-coalgebra and (H,μ_H,ρ^H,β) is a Hom-Yetter–Drinfeld module, then (H,β,R) is a quasitriangular Hom-Hopf algebra.*

Proof. This is straightforward. ∎

By Propositions 5.3 and 5.4, we have:

THEOREM 5.5. *With the notations above, (H,β,R) is a quasitriangular Hom-Hopf algebra if and only if (H,β,ρ^H) is a left (H,β)-comodule Hom-coalgebra and (H,μ_H,ρ^H,β) is a Hom-Yetter–Drinfeld module.*

Dually, we have

THEOREM 5.6. *Let (H, β, S) be a Hom-Hopf algebra and $\sigma : H \otimes H \to K$ a bilinear map. Define $\triangleright_H : H \otimes H \to H$ by*

$$h \otimes g \mapsto h \triangleright_H g = \sigma(g_1, \beta^{-3}(h))g_2$$

for $h, g \in H$. Then (H, β, σ) is a cobraided Hom-Hopf algebra (see [11,27]) if and only if $(H, \beta, \triangleright_H)$ is a left (H, β)-module Hom-algebra and $(H, \triangleright_H, \Delta_H, \beta)$ is a Hom-Yetter–Drinfeld module.

Acknowledgements. This work was partially supported by the NNSF of China (No. 11101128) and the Foundation of Henan Province (No. 14IRT-STHN023).

REFERENCES

[1] N. Andruskiewitsch and H.-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. of Math. 171 (2010), 375–417.

[2] S. Caenepeel and I. Goyvaerts, *Monoidal Hom-Hopf algebras*, Comm. Algebra 39 (2011), 2216–2240.

[3] Y. Y. Chen, Z. W. Wang and L. Y. Zhang, *Integrals for monoidal Hom-Hopf algebras and their applications*, J. Math. Phys. 54 (2013), 073515.

[4] Y. Y. Chen, Z. W. Wang and L. Y. Zhang, *Quasitriangular Hom-Hopf algebras*, Colloq. Math. 137 (2014), 67–88.

[5] Y. Y. Chen and L. Y. Zhang, *The category of Yetter–Drinfel'd Hom-modules and the quantum Hom-Yang–Baxter equation*, J. Math. Phys. 55 (2014), 031702.

[6] S. Dăscălescu, C. Năstăsescu and S. Raianu, *Hopf Algebras. An Introduction*, Lecture Notes in Pure Appl. Math. 235, Dekker, New York, 2001.

[7] C. Kassel, *Quantum Groups*, Grad. Texts in Math. 155, Springer, Berlin, 1995.

[8] H. Y. Li and T. S. Ma, *On Hom-Yetter–Drinfeld category*, submitted, 2014.

[9] L. Liu and B. L. Shen, *Radford's biproducts and Yetter–Drinfeld modules for monoidal Hom-Hopf algebras*, J. Math. Phys. 55 (2014), 031701.

[10] T. S. Ma, H. Y. Li and L. L. Liu, *Double biproduct Hom-bialgebra and related quasitriangular structures*, submitted, 2014.

[11] T. S. Ma, H. Y. Li and T. Yang, *Cobraided smash product Hom-Hopf algebras*, Colloq. Math. 134 (2014), 75–92.

[12] T. S. Ma, H. Y. Li and W. Z. Zhao, *On the braided structures of Radford's biproduct*, Acta Math. Sci. Ser. B English Ed. 31 (2011), 701–715.

[13] T. S. Ma and S. H. Wang, *Bitwistor and quasitriangular structures of bialgebras*, Comm. Algebra 38 (2010), 3206–3242.

[14] S. Majid, *Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group*, Comm. Math. Phys. 156 (1993), 607–638.

[15] S. Majid, *Algebras and Hopf algebras in braided categories*, in: Advances in Hopf Algebras (Chicago, IL, 1992), Lecture Notes in Pure Appl. Math. 158, Dekker, New York, 1994, 55–105.

[16] A. Makhlouf and F. Panaite, *Yetter–Drinfeld modules for Hom-bialgebras*, J. Math. Phys. 55 (2014), 013501.

[17] A. Makhlouf and F. Panaite, *Twisting operators, twisted tensor products and smash products for Hom-associative algebras*, arXiv:1402.1893 (2014).

[18] A. Makhlouf and S. D. Silvestrov, *Hom-algebra stuctures*, J. Gen. Lie Theory Appl. 2 (2008), 51–64.
[19] A. Makhlouf and S. D. Silvestrov, *Hom-algebras and hom-coalgebras*, J. Algebra Appl. 9 (2010), 553–589.
[20] D. E. Radford, *The structure of Hopf algebra with a projection*, J. Algebra 92 (1985), 322–347.
[21] D. E. Radford, *Hopf Algebras*, Ser. Knots and Everything 49, World Sci., 2012.
[22] D. Simson, *Coalgebras of tame comodule type, comodule categories, and a tame-wild dichotomy problem*, in: Representation of Algebras and Related Topics (Tokyo, 2010), A. Skowroński and K. Yamagata (eds.), EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2010, 561–660.
[23] M. E. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.
[24] D. Yau, *Module Hom-algebras*, arXiv:0812.4695v1 (2008).
[25] D. Yau, *Hom-bialgebras and comodule Hom-algebras*, Int. Electron. J. Algebra 8 (2010), 45–64.
[26] D. Yau, *Hom-quantum groups I: Quasitriangular Hom-bialgebras*, J. Phys. A 45 (2012), 065203, 23 pp.
[27] D. Yau, *Hom-quantum groups II: Cobraided Hom-bialgebras and Hom-quantum geometry*, arXiv:0907.1880 (2009).
[28] D. Yau, *Hom-quantum groups III: Representations and module Hom-algebras*, arXiv:0911.5402 (2009).
[29] D. Yau, *The Hom-Yang–Baxter equation and Hom-Lie algebras*, J. Math. Phys. 52 (2011), 053502.
[30] D. Yau, *The Hom-Yang–Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras*, J. Phys. A 42 (2009), 165202, 12 pp.
[31] T. Zhang, *Comodule Hom-coalgebras*, arXiv:1301.4152 (2013).
[32] N. F. Zheng, *The braided structures over Hom-ω-smash coproduct Hopf algebras*, Acta Math. Sci. Ser. A 33 (2013), 1068–1088 (in Chinese).

Haiying Li, Tianshui Ma (corresponding author)
College of Mathematics and Information Science
Henan Normal University
453007 Xinxiang, China
E-mail: haiyingli2012@yahoo.com
matianshui@yahoo.com

Received 12 June 2014 (6288)