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WIENER'S INVERSION THEOREM FOR A CERTAIN CLASS OF *-ALGEBRAS

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Abstract. We generalize Wiener's inversion theorem for Fourier transforms on closed subsets of the dual group of a locally compact abelian group to cosets of ideals in a class of non-commutative *-algebras having specified properties, which are all fulfilled in the case of the group algebra of any locally compact abelian group.

Introduction. Let G be a locally compact abelian group with dual group Γ, and let E be a closed subset of Γ. Furthermore, let $A(\Gamma)$ denote the set of all Fourier transforms $\hat{f}$ of integrable complex-valued functions $f \in L^1(G)$. If E is compact and $Z(\hat{f}) \cap E = \emptyset$ for some $\hat{f} \in A(\Gamma)$, where $Z(\hat{f})$ denotes the zero set of $\hat{f}$, then *Wiener's inversion theorem* (see e.g. [2, Proposition 1.1.5(b)]) says that there exists a $\hat{g} \in A(\Gamma)$ such that $\hat{g}(\gamma) = 1/\hat{f}(\gamma)$ for all $\gamma \in E$.

A first step towards a generalization of Wiener's inversion theorem is to note that $C_0(\Gamma)$, which consists of all continuous complex-valued functions on Γ vanishing at infinity, is the enveloping C^*-algebra of $A(\Gamma)$. Now, in the non-commutative situation, we replace $A(\Gamma)$ by an arbitrary *-algebra A equipped with the Gelfand–Naĭmark seminorm γ_A. Then A is called a G^*-*algebra*, and we may construct the enveloping C^*-algebra $C^*(A)$ of A. For simplicity, we will always assume that A is *reduced*, which means that the *-*radical* of A, i.e., the intersection of the kernels of all *-representations of A on a Hilbert space, is trivial.

As a second step, we notice that spectral synthesis holds in $C_0(\Gamma)$, giving a one-to-one correspondence between the closed subsets E of Γ and the closed ideals in $C_0(\Gamma)$, whose common zero set is equal to E. Hence, we may replace, in the case of an arbitrary G^*-algebra A, a given closed subset E of Γ by a closed two-sided ideal in $C^*(A)$, which we will also denote by E. In particular, we replace the zero set of a function in $A(\Gamma)$ by the closed two-sided ideal in $C^*(A)$ generated by a given element a in A, which will be

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denoted by $Z(a)$. Now, we consider the subset

$$k(E) := \{\hat{f} \in A(\Gamma) : E \subseteq Z(\hat{f})\}$$

of $A(\Gamma)$, which is the largest closed ideal in $A(\Gamma)$ such that its common zero set is equal to E, i.e., $Z(k(E)) = E$. Its importance stems from the fact that elements of the quotient Banach algebra $A(\Gamma)/k(E)$ may be identified with the restrictions of functions in $A(\Gamma)$ to E. In our abstract framework, the closed ideal $k(E)$ in $A(\Gamma)$ then becomes the *-ideal $\{a \in A : Z(a) \subseteq E\}$ in A, which we will also denote by $k(E)$. We show in Lemma 2.7 that $k(E) = E \cap A$.

In fact, in order to generalize Wiener's inversion theorem, we have to impose additional conditions on the reduced G^*-algebra A. We need the concept of *-*regularity* of A, saying that the *structure space* of $C^*(A)$, which consists of all primitive ideals in $C^*(A)$, is homeomorphic to the *-structure space of A consisting of all kernels of topologically irreducible *-representations of A on a Hilbert space, where both spaces are equipped with the hull-kernel topology. This notion is due to H. Leptin et al. [6]. Since the *-representation theory of G^*-algebras may be poorly behaved, we are led to the subclass of the so-called BG^*-algebras, having essentially all of the features of the *-representation theory of Banach *-algebras. Actually, every Banach *-algebra is a BG^*-algebra. It is shown by B. A. Barnes [1] that a reduced BG^*-algebra A is *-regular if and only if $E \cap A$ is dense in E for each closed two-sided ideal E in $C^*(A)$. Applying this characterization, we are able to prove in Theorem 2.11 that $A/k(E)$ is unital if and only if $C^*(A)/E$ is unital.

In addition, for the investigation of invertible elements in $A/k(E)$, we need to assume that the Gelfand–Naĭmark seminorm γ_A satisfies a certain spectral condition. The class of G^*-algebras having this property forms the so-called γS^*-algebras. They generalize the notion of hermitian Banach *-algebras, constituting a class which has already played an important part in C. E. Rickart's [19, 20] in connection with invertibility questions.

Now, we arrive at our announced generalization of Wiener's inversion theorem. For that purpose, let A be simultaneously a *-regular reduced BG^*-algebra and a γS^*-algebra. If we further assume the existence of an identity element in either $A/k(E)$ or $C^*(A)/E$ for any fixed closed two-sided ideal E in $C^*(A)$, we prove in Theorem 2.14 that a coset $a + k(E)$ for any given $a \in A$ is invertible in $A/k(E)$ if and only if the coset $a + E$ is invertible in $C^*(A)/E$. In fact, in the commutative situation of a locally compact abelian group G with dual group Γ, the Fourier algebra $A(\Gamma)$ is a *-regular, hermitian, reduced Banach *-algebra. An application of the above-mentioned possibility of spectral synthesis in $C_0(\Gamma)$ shows that an arbitrary closed ideal in $C_0(\Gamma)$ may be identified with a closed subset E of Γ. Furthermore, E is compact if and only if $A(\Gamma)/k(E)$ is unital. Hence, we get Wiener's classical inversion theorem by regarding elements of $A(\Gamma)/k(E)$ as the restrictions of functions in $A(\Gamma)$ to E.

1. Preliminaries. In this section, we briefly recall basic definitions and facts that we need. For a comprehensive exposition of the general theory of *-algebras, we refer to [17, 18].

Let A be an algebra, which is always assumed to be associative and complex. A two-sided ideal P in A is called *primitive* if P is equal to the kernel of an algebraically irreducible representation of A on a vector space. Then Π_A denotes the set of all primitive ideals in A. Let S be a subset of A. Then the set $h^{\Pi}(S) := \{P \in \Pi_A : S \subset P\}$ is called the *hull* of S with respect to Π_A. A subset of Π_A having the form $h^{\Pi}(S)$ with a subset S of A is called a *hull in* Π_A. Let B be a subset of Π_A. Then the set $k^{\Pi}(B) := \bigcap\{P \in \Pi_A : P \in B\}$ is called the *kernel* of B. The family $\{\Pi_A \setminus H : H \text{ is a hull in } \Pi_A\}$ of complements of all hulls in Π_A is called the *hull-kernel topology* or the *Jacobson topology* on Π_A. Equipped with this topology, Π_A is said to be the *structure space* of A.

A two-sided ideal I in A is called *regular* in A if the quotient algebra A/I is unital. The *spectrum* $\mathrm{spec}_A(a)$ of $a \in A$ is defined by

$$\mathrm{spec}_A(a) := \mathrm{spec}_{A^1}(a) := \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not invertible in } A^1\},$$

where $A^1 := A \oplus \mathbb{C}$ denotes the unitization of A with identity 1 and $\mathbb{C}$ the complex numbers. Let $a \in A$. Then $\rho(a) := \sup\{|\lambda| : \lambda \in \mathrm{spec}_A(a)\}$ is called the *spectral radius* of a. An algebra seminorm q on A is called *spectral* if $q(a) \geq \rho(a)$ for all $a \in A$, and A is called *spectral* if it has a spectral algebra seminorm.

Now, let A be an algebra with an involution, i.e., let A be a *-algebra. The **-radical* A_R of A is defined by

$$A_R := \bigcap_{\pi} \ker \pi,$$

where π runs through all *-representations of A on a Hilbert space. If $A_R = \{0\}$, then A is called *reduced* or **-semisimple*. In particular, if G is an arbitrary locally compact group, the Banach *-algebra $L^1(G)$ consisting of all integrable complex-valued functions on G is reduced. Let Π_A^* denote the set of all kernels of topologically irreducible *-representations of A on a Hilbert space. If, as above, one defines the hull-kernel topology on Π_A^*, then Π_A^* equipped with this topology is called the **-structure space* of A.

For each $a \in A$, the mapping $\gamma_A : A \to \mathbb{R}^+ \cup \{\infty\}$ is defined by

$$\gamma_A(a) := \sup_{\pi} \|\pi(a)\|,$$

where π runs through all *-representations of A on a Hilbert space and $\mathbb{R}^+ :=$

$\{\lambda \in \mathbb{R} : \lambda \geq 0\}$. If $\gamma_A(a)$ is finite for all $a \in A$, then A is called a *G^*-algebra* and γ_A the *Gelfand–Naĭmark seminorm* on A. It is a C^*-seminorm on A and the largest one that can be defined on A. An arbitrary Banach *-algebra is a G^*-algebra. If A is a G^*-algebra, the completion of A/A_R with respect to the quotient Gelfand–Naĭmark C^*-norm γ_{A/A_R} is called the *enveloping C^*-algebra* of A and is denoted by $C^*(A)$. The closure of $\pi_u(C^*(A))$ with respect to the σ-weak operator topology, where π_u denotes the universal representation of $C^*(A)$, is called the *universal enveloping von Neumann algebra* of $C^*(A)$, and it will be denoted by $W^*(A)$.

A *-algebra A is called a *γS^*-algebra* if A is a G^*-algebra and the Gelfand–Naĭmark seminorm γ_A on A is spectral in the above sense. For a *-algebra A, one defines $A_h := \{a \in A : a^* = a\}$. Then A is called *hermitian* if $\mathrm{spec}_A(a) \subseteq \mathbb{R}$ for all $a \in A_h$. The γS^*-algebras generalize the hermitian Banach *-algebras.

Let A be a G^*-algebra with the canonical mapping $\Phi : A \to C^*(A)$. Then A is said to have a *unique C^*-norm* if γ_{A/A_R} is the only C^*-norm on A/A_R which can be defined on A/A_R. Furthermore, A is called **-regular* if the continuous surjection

$$\breve{\Phi} : \Pi_{C^*(A)} \to \Pi_A^*, \qquad P \mapsto \Phi^{-1}(P),$$

is a homeomorphism. If, in addition, A is reduced, we have $\Phi^{-1}(P) = P \cap A$ for all $P \in \Pi_{C^*(A)}$. According to [18, Theorem 10.5.12(c)], any *-regular G^*-algebra always has a unique C^*-norm.

The disadvantage of the class of G^*-algebras is that comparatively little of the representation theory of Banach *-algebras can be extended to G^*-algebras. Consequently, we have to add another key property to the class of G^*-algebras. For that purpose, let A be a *-algebra, $\mathcal{L}(X)$ the algebra of all linear mappings of a pre-Hilbert space X to X and $\mathcal{L}_*(X)$ the *-algebra of all elements T from $\mathcal{L}(X)$ having an adjoint element T^* in $\mathcal{L}(X)$, i.e., such that $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$ for all $\xi, \eta \in X$. A *pre-*-representation* π of A on a pre-Hilbert space X is a *-algebra homomorphism of A to $\mathcal{L}_*(X)$. Then, a *-algebra A is called a *BG^*-algebra* if every pre-*-representation π of A on a pre-Hilbert space X is *normed*, i.e., $\pi(a)$ is a bounded linear operator on X for all $a \in A$. In view of [18, Theorem 10.2.8(a)], an arbitrary Banach *-algebra is a BG^*-algebra, and any BG^*-algebra is a G^*-algebra by [18, Proposition 10.1.19(a)].

In contrast to the class of G^*-algebras, the smaller class of BG^*-algebras now enables the following construction: Define a pre-*-representation on a pre-Hilbert space first and then extend it to the Hilbert space completion. Hence, essentially all of the features of the *-representation theory of Banach *-algebras are reproduced in the *-representation theory of BG^*-algebras. In particular, this yields the fundamental result that any *-representation of a *-ideal in a BG^*-algebra can be extended to a *-representation of the whole *-algebra on the same Hilbert space (see [18, Theorem 10.1.21]).

Now, for the class of reduced BG^*-algebras A, we obtain the following characterizations. By [18, Proposition 10.5.19(a)], A has a unique C^*-norm if and only if, for every closed two-sided ideal $E \neq \{0\}$ in $C^*(A)$, we have $E \cap A \neq \{0\}$. According to [18, Proposition 10.5.19(b)], A is *-regular if and only if $E \cap A$ is dense in E for every closed two-sided ideal E in $C^*(A)$. This is the key result that we will use throughout the paper. Both characterizations are due to B. A. Barnes [1, Proposition 2.4].

In the whole paper, we use the following notations:

$$\mathcal{C} := C^*(A)$$

always denotes the enveloping C^*-algebra of a given G^*-algebra A, and

$$\mathcal{N} := W^*(A)$$

the universal enveloping von Neumann algebra of $C^*(A)$.

2. Wiener's inversion theorem

PROPOSITION 2.1. *Let A be a G^*-algebra. If A is unital, then so is $\mathcal{C}$.*

Proof. Let A be unital. Then the quotient G^*-algebra A/A_R is unital, too, where A_R denotes the *-radical of A. Since A/A_R is dense in $\mathcal{C}$ with respect to the quotient Gelfand–Naĭmark C^*-norm γ_{A/A_R} and since left and right multiplication are continuous with respect to γ_{A/A_R}, the identity of A/A_R is the identity of $\mathcal{C}$, i.e., $\mathcal{C}$ is unital. ■

PROPOSITION 2.2. *Let A be a *-regular BG^*-algebra. Then so also is the unitization $A^1 := A \oplus \mathbb{C}$ of A.*

Proof. Let A be a *-regular BG^*-algebra. In the case of C^*-algebras, the *-structure space coincides with the structure space according to [18, Corollary 10.5.4]. Hence, the C^*-algebra $\mathbb{C}$ of all complex numbers is *-regular. Since $A^1/A = \mathbb{C}$, A^1/A is *-regular, too. Since A is a BG^*-algebra, A^1 is also a BG^*-algebra in view of [18, Theorem 10.1.20(f)]. By [18, Theorem 10.5.15(d)], we conclude from the *-regularity of both A^1/A and A that A^1 is *-regular. ■

The converse of Proposition 2.1 holds for special classes of G^*-algebras.

PROPOSITION 2.3.

(i) *Let A be a *-regular BG^*-algebra. If $\mathcal{C}$ is unital, then so is A.*
(ii) *Let A be a γS^*-algebra. If $\mathcal{C}$ is unital, then so is A.*

Proof. (i) Let $\mathcal{C}$ be unital. Suppose that A is non-unital. From Proposition 2.2, A^1 is also a *-regular BG^*-algebra. Furthermore, the *-regularity of A and A^1 implies that A and A^1 have unique C^*-norms. Since A is assumed to be non-unital, it follows from [18, Theorem 10.5.26] that $\mathcal{C}$ is non-unital, too. But this contradicts our assumption.

(ii) The contraposition of [18, Proposition 10.4.27] gives the assertion. ∎

Remark 2.4. We may, for example, apply Propositions 2.1 and 2.3 to the case of algebraic tensor products of *-algebras (see e.g. [18] and [4]), and to complete m*-convex algebras with Arens–Michael decompositions and their enveloping pro-C^*-algebras (see e.g. [7, 10, 11, 16] and [4]), and obtain similar results.

Generalizing the situation of commutative harmonic analysis (see e.g. [2]), we make the following

Definition 2.5. Let A be a reduced G^*-algebra. Then we set, for all $a \in A$,

$$Z(a) := \bigcap_{E \ni a} \{E : E \text{ a closed two-sided ideal in } \mathcal{C}\}$$

and, for all $X \subseteq A$,

$$Z(X) := \bigcap_{E \supseteq X} \{E : E \text{ a closed two-sided ideal in } \mathcal{C}\}.$$

Let E be a closed two-sided ideal in $\mathcal{C}$. Then we put

$$k(E) := \{a \in A : Z(a) \subseteq E\}.$$

Remark 2.6. (i) Let G be a locally compact abelian group with dual group Γ and let E be a closed subset of Γ. Furthermore, let $A(\Gamma)$ denote the set of Fourier transforms $\hat{f}$ of all integrable complex-valued functions $f \in L^1(G)$. It is clear that $A(\Gamma)$ is a reduced G^*-algebra with the enveloping C^*-algebra $C_0(\Gamma)$. Following [2, p. 22] (see also [22, 7.1.3] and [13, Example 39.10(b)]), let $Z(\hat{f})$ denote the zero set of some $\hat{f} \in A(\Gamma)$, $Z(X) := \bigcap_{\hat{f} \in X} Z(\hat{f})$ the common zero set of some $X \subseteq A(\Gamma)$, and $k(E) := \{\hat{f} \in A(\Gamma) : E \subseteq Z(\hat{f})\}$, which is equal to the kernel of E with respect to the hull-kernel topology of Γ according to [14, Definition VIII.5.3]. Based on classical spectral synthesis in $C_0(\Gamma)$ (see e.g. [13, Example 39.10(a)]), giving a one-to-one correspondence between the closed subsets E of Γ and the closed ideals in $C_0(\Gamma)$ whose hull is equal to E, or equivalently, by [14, Definition VIII.5.3], whose common zero set is equal to E, a closed subset E of Γ can be identified with the closed ideal $\{\varphi \in C_0(\Gamma) : \varphi(\gamma) = 0\ \forall \gamma \in E\}$ in $C_0(\Gamma)$. Hence, our Definition 2.5 generalizes all the above notations from the case of $A(\Gamma)$ to the non-commutative situation, i.e., to any reduced G^*-algebra with enveloping C^*-algebra.

(ii) As in (i) for $A(\Gamma)$, our Definition 2.5 is also motivated by corresponding notations from the case of any commutative semisimple Banach algebra A regarded, via Gelfand transform, as a subalgebra of $C_0(\Pi_A)$, where the structure space Π_A of A is a locally compact Hausdorff space with respect to the hull-kernel topology if A is completely regular (see [13, Definition 39.7 and Example 39.10(a)] and [21, Theorem 3.7.1]).

The following lemma is essential for proving all our further results.

LEMMA 2.7. *Let A be a reduced G^*-algebra, and let E be a closed two-sided ideal in $\mathcal{C}$. Then*

$$Z(k(E)) \subseteq E \quad \text{and} \quad k(E) = E \cap A.$$

Thus $k(E)$ is a *-ideal in A.

Proof. The inclusion $Z(k(E)) \subseteq E$ follows from the definition of $k(E)$. Now, we show that $k(E) = E \cap A$.

"$\subseteq$": Let $a \in k(E)$, i.e., let $a \in A$ and $Z(a) \subseteq E$. Since $a \in Z(a)$, it follows that $a \in E$. Hence, $a \in E \cap A$.

"$\supseteq$": Let $a \in E \cap A$, i.e., let $a \in A$ and $a \in E$. Since E is a closed two-sided ideal in $\mathcal{C}$ and since $Z(a)$ is the smallest closed two-sided ideal in $\mathcal{C}$ containing a, we get $Z(a) \subseteq E$. Consequently, $k(E) = E \cap A$.

By [24, Theorem I.8.1], E is a *-ideal in $\mathcal{C}$. Hence, $E \cap A = k(E)$ is a *-ideal in A. ∎

NOTATION 2.8. Let A be a reduced G^*-algebra, and let E be a closed two-sided ideal in $\mathcal{C}$. We set

$$A(E) := A/k(E) \quad \text{and} \quad \mathcal{C}(E) := \mathcal{C}/E.$$

REMARK 2.9. According to [18, Theorem 10.1.7(k)], $A(E)$ is a G^*-algebra, and by [24, Theorem I.8.1], $\mathcal{C}(E)$ is a C^*-algebra.

PROPOSITION 2.10. *Let A be a *-regular reduced BG^*-algebra, and let E be a closed two-sided ideal in $\mathcal{C}$. Then we have the isometric *-isomorphism*

$$C^*(A(E)) \cong \mathcal{C}(E).$$

Proof. According to [18, Theorem 10.1.22], for each *-ideal I in a BG^*-algebra A, there is a short exact sequence $C^*(I) \to C^*(A) \to C^*(A/I)$. By Lemma 2.7, $k(E)$ is a *-ideal in A such that $k(E) = E \cap A$. Thus, an application to $k(E)$ yields an isometric *-isomorphism $C^*(A(E)) = C^*(A/k(E)) \cong C^*(A)/\overline{k(E)}^{\gamma_A}$. Since A is also *-regular, we know that $\overline{k(E)}^{\gamma_A} = \overline{E \cap A}^{\gamma_A} = E$. ∎

THEOREM 2.11. *Let A be a *-regular reduced BG^*-algebra, and let E be a closed two-sided ideal in $\mathcal{C}$. Then E is regular in $\mathcal{C}$ if and only if $k(E)$ is regular in A.*

Proof. "$\Rightarrow$": Let E be regular in $\mathcal{C}$, i.e., let $\mathcal{C}(E)$ be unital. By Proposition 2.10, $C^*(A(E))$ is unital, too. Since $A(E)$ is a quotient algebra and

A is a BG^*-algebra, $A(E)$ is also a BG^*-algebra according to [18, Theorem 10.1.20(g)]. From [18, Theorem 10.5.15(a)], the *-regularity of A implies the *-regularity of $A(E)$. Altogether, we conclude from Proposition 2.3(i) that $A(E)$ is unital, i.e., $k(E)$ is regular in A.

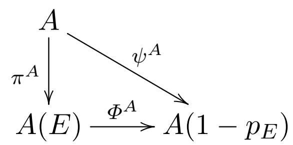
"$\Leftarrow$": Let $k(E)$ be regular in A, i.e., let $A(E)$ be unital. By Remark 2.9, the quotient algebra $A(E)$ is a G^*-algebra. Thus, according to Proposition 2.1, we know that $C^*(A(E))$ is unital, too. Now, Proposition 2.10 shows that $\mathcal{C}(E)$ is unital, i.e., E is regular in $\mathcal{C}$. ■

Proposition 2.12. *Let A be a reduced G^*-algebra, and let E be a closed two-sided ideal in $\mathcal{C}$ with the weak* closure $\overline{E}^{w^*}$ of E in the universal enveloping von Neumann algebra $\mathcal{N} := W^*(A)$ of A. Let p_E denote the central projection in $\mathcal{N}$ such that $\overline{E}^{w^*} = \mathcal{N}p_E$. Then we have the *-algebra isomorphisms*

$$A(E) \cong A(1-p_E) \quad \text{and} \quad \mathcal{C}(E) \cong \mathcal{C}(1-p_E).$$

Proof. According to [23, Proposition 1.10.5], for every weak* closed two-sided ideal $\overline{E}^{w^*}$ in $\mathcal{N}$, there is a uniquely determined central projection $p_E \in \mathcal{N}$ such that $\overline{E}^{w^*} = \mathcal{N}p_E = p_E\mathcal{N}$.

Now, we consider the following diagram:



Since $A(E) := A/k(E)$ and since $1 - p_E$ is a central projection in $\mathcal{N}$, the mappings π^A and ψ^A are canonical surjective *-algebra homomorphisms. Furthermore, we have

$$\ker\psi^A = k(E),$$

since

$$\ker\psi^A = \{a \in A : 0 = \psi^A(a) = a(1-p_E)\} = \{a \in A : a = ap_E\}.$$

Hence, $\ker\psi^A = \overline{E}^{w^*} \cap A$. Since $A \subseteq \mathcal{C}$ and $\overline{E}^{w^*} \cap \mathcal{C} = E$, we have $\overline{E}^{w^*} \cap A = \overline{E}^{w^*} \cap \mathcal{C} \cap A = E \cap A$. Together with Lemma 2.7, we get $\ker\psi^A = k(E)$. Thus we conclude that Φ^A is a *-algebra isomorphism from $A(E)$ onto $A(1-p_E)$.

Similarly, we obtain $\mathcal{C}(E) \cong \mathcal{C}(1-p_E)$. ■

Proposition 2.13. *Let A be a *-regular reduced BG^*-algebra, and let E be a closed two-sided ideal in $\mathcal{C}$. Then $A(E)$ is reduced.*

Proof. Since A is reduced, we get $A \subseteq \mathcal{C}$. Thus $A(1-p_E) \subseteq \mathcal{C}(1-p_E)$ with the central projection $p_E \in \mathcal{N} := W^*(A)$ from Proposition 2.12, showing that $A(E)$ may be identified with a subset of $\mathcal{C}(E)$ and, by Proposition 2.10, with a subset of $C^*(A(E))$. Consequently, $A(E)$ is reduced. ■

THEOREM 2.14. *Let A be simultaneously a *-regular reduced BG^*-algebra and a γS^*-algebra, and let E be a closed two-sided ideal in $\mathcal{C}$. If E is regular in $\mathcal{C}$ (i.e., $k(E)$ is regular in A), the following assertions are equivalent for all $a \in A$:*

(i) *$a + k(E)$ is invertible in $A(E)$;*
(ii) *$a + E$ is invertible in $\mathcal{C}(E)$.*

Hence, letting $A(E)_G$ and $\mathcal{C}(E)_G$ denote the groups of invertible elements in $A(E)$ and $\mathcal{C}(E)$, respectively, we obtain

$$A(E)_G = \mathcal{C}(E)_G \cap A(E).$$

Proof. Without loss of generality, by Theorem 2.11, let E be regular in $\mathcal{C}$, i.e., let $\mathcal{C}(E)$ be unital. From Proposition 2.13 we know that $A(E)$ is reduced. Since A is a γS^*-algebra, the quotient algebra $A(E)$ is also a γS^*-algebra according to [18, Theorem 10.4.12]. Therefore, the desired equivalence follows from Proposition 2.10 and [18, Corollary 10.4.20(a)]. ■

REMARK 2.15. (i) Let G be a locally compact abelian group with dual group Γ, and let E be a closed subset of Γ. In the classical notations from Remark 2.6(i), by [2, p. 22], elements of the quotient Banach *-algebra $A(\Gamma)/k(E)$ can be identified with the restrictions of functions from $A(\Gamma)$ to E. Furthermore, E is compact if and only if $A(\Gamma)/k(E)$ is unital. Since $A(\Gamma)$ is isometrically isomorphic to $L^1(G)$, it is clear that $A(\Gamma)$ is simultaneously a *-regular reduced BG^*-algebra and a γS^*-algebra. Consequently, we obtain Wiener's classical inversion theorem (see e.g. [2, Proposition 1.1.5(b)]) from Theorem 2.14.

(ii) The investigation of invertible elements in inclusions of algebras has already been carried out by C. E. Rickart in [19, 20] (see also [21, Theorem 4.1.9]), in the case of closed *-subalgebras of hermitian Banach *-algebras.

(iii) If G is any locally compact group, then $A := L^1(G)$ is a reduced Banach *-algebra, and the enveloping C^*-algebra $\mathcal{C} = C^*(A)$ is called the *full group C^*-algebra* of G and denoted by $C^*(G)$. The class of locally compact groups for which $L^1(G)$ is both *-regular and hermitian includes all connected groups of polynomial growth and all nilpotent groups (see [6, 15]). It also includes all groups in $[FC]^-$ consisting of those groups such that each conjugacy class has compact closure (see [12]). In particular, the class includes all locally compact abelian groups and all compact groups, since locally compact abelian groups are nilpotent and compact groups are in $[FC]^-$. In the case of locally compact abelian groups as well as compact groups G, it is also known that the full group C^*-algebra $C^*(G)$ is isometrically *-isomorphic to the *reduced group C^*-algebra* $C_r^*(G)$, which is generated by the left regular representation of $L^1(G)$.

(iv) Let $\mathbb{K} = (M, \Delta, \kappa, \varphi)$ be a Kac algebra, generalizing the situation of a locally compact group. For a comprehensive exposition of its theory, we refer to [8] and also to [4, 5]. Since the predual M_* of M is a Banach *-algebra, M_* is a BG^*-algebra and thus a G^*-algebra. By [8, Theorem 2.5.3], the Fourier representation λ is a faithful *-representation of M_*. Hence, M_* is also reduced. Let $\hat{\mathbb{K}} = (\hat{M}, \hat{\Delta}, \hat{\kappa}, \hat{\varphi})$ denote the dual Kac algebra of $\mathbb{K}$. If, in addition, $\mathbb{K}$ is compact such that $\varphi(1) = 1$, we conclude from [8, Introduction 1.6.1 and Theorem 6.2.5(i)] that the enveloping C^*-algebra $C^*(M_*)$ is isometrically *-isomorphic to $\hat{M}_c = C_0(\hat{\mathbb{K}})$, where $\hat{M}_c$ denotes the C^*-algebra $\overline{\lambda(M_*)}^{\text{norm}}$ associated with $\hat{M}$. Consequently, for a compact Kac algebra, in Theorem 2.14 we may replace A by M_*, $\mathcal{C} := C^*(A)$ by $\hat{M}_c = C_0(\hat{\mathbb{K}})$, and furthermore the universal enveloping von Neumann algebra $W^*(A)$ of A by $\hat{M} = L^\infty(\hat{\mathbb{K}})$. Since, if G is a compact group, the predual $L^1(G)$ of $L^\infty(G)$ is both *-regular and hermitian (see (iii)), it may be interesting to ask if M_* automatically has these properties in the case of an arbitrary compact Kac algebra $\mathbb{K} = (M, \Delta, \kappa, \varphi)$.

Corollary 2.16. *Under the assumptions of Theorem* 2.14*:*

(i) *For all $a \in A$, we have*

$$\operatorname{spec}_{A(E)}(a + k(E)) = \operatorname{spec}_{\mathcal{C}(E)}(a + E).$$

(ii) *Let $a \in A$. Then $a + k(E)$ is not contained in any maximal left or right ideal of $A(E)$ if and only if $a + E$ is not contained in any maximal left or right ideal of $\mathcal{C}(E)$.*

(iii) *If one of the equivalent assertions of Theorem* 2.14 *holds for some $a \in A$, then, for each $b \in A$, there is a $c \in A$ such that*

$$b + k(E) = (c + k(E))(a + k(E)).$$

Proof. (i) Since the spectra depend only on invertibility, the assertion follows from Theorem 2.14.

(ii) Let $a \in A$. By [21, Corollary 2.1.2], $a + k(E)$ (resp. $a + E$) is not contained in any maximal left or right ideal of $A(E)$ (resp. $\mathcal{C}(E)$) if and only if $a + k(E)$ (resp. $a + E$) is invertible in $A(E)$ (resp. $\mathcal{C}(E)$). Hence, the equivalence follows from Theorem 2.14.

(iii) Without loss of generality, by Theorem 2.11, let $k(E)$ be regular in A and $a + k(E)$ invertible in $A(E)$ for some $a \in A$ according to Theorem 2.14. Now, let $b \in A$. Then we set

$$c + k(E) := (b + k(E))(a + k(E))^{-1} \in A(E).$$

Hence,

$$b + k(E) = (b + k(E))(a + k(E))^{-1}(a + k(E)) = (c + k(E))(a + k(E)). \blacksquare$$

Next, we give a necessary condition for the equivalent assertions in Theorem 2.14.

PROPOSITION 2.17. *Under the assumptions of Theorem* 2.14*, suppose that one of the equivalent assertions of that theorem holds for some* $a \in A$*. Then*

$$Z(a, k(E)) = \mathcal{C}.$$

Proof. Take $b \in A$. By Corollary 2.16(iii), there exists $c \in A$ with $b+k(E) = (c+k(E))(a+k(E))$. Hence, $b = ca+d$ for some $d \in k(E)$. Since A is reduced, we have $b \in E'$ for any two-sided ideal E' in $\mathcal{C}$ containing $\{a\} \cup k(E)$. Thus $A \subseteq Z(a, k(E)) \subseteq \mathcal{C}$. The density of A in $\mathcal{C}$ now implies that $Z(a, k(E)) = \mathcal{C}$, since $Z(a, k(E))$ is closed. ∎

REMARK 2.18. (i) Let A be a reduced γS^*-algebra. If $\mathcal{C}$ is unital (resp. A is unital) and if some $a \in A$ is invertible in $\mathcal{C}$ (resp. invertible in A), we can show that $Z(a) = \mathcal{C}$ in like manner as the above Proposition 2.17 using Proposition 2.3(ii) and [18, Corollary 10.4.20(a)] directly.

(ii) We can also prove Proposition 2.17 by using Proposition 2.12 and the theory of projections in von Neumann algebras.

PROPOSITION 2.19. *Let A be a *-regular reduced BG^*-algebra, and let E be a closed two-sided ideal in $\mathcal{C}$. Then:*

(i) *If $a \in A$ with $Z(a) \subseteq Z(k(E))$, then $a \in k(E)$.*
(ii) *If $Z(k(E)) = \mathcal{C}$, then $k(E) = A$.*
(iii) *If $k(E) \neq A$ and E is regular in $\mathcal{C}$, then there is a maximal regular ideal J in A containing $k(E)$.*

Proof. (i) Let $a \in A$ with $Z(a) \subseteq Z(k(E))$. Since A is *-regular, we conclude from Lemma 2.7 that $Z(k(E)) = \overline{k(E)}^{\gamma_A} = \overline{E \cap A}^{\gamma_A} = E$. Since A is reduced, we have $a \in Z(a)$. Hence, $a \in E$. Thus $a \in E \cap A = k(E)$.

(ii) Let $Z(k(E)) = \mathcal{C}$. Similarly to (i), we get $Z(k(E)) = E$. Thus $E = \mathcal{C}$. Since A is reduced, Lemma 2.7 shows that $k(E) = E \cap A = \mathcal{C} \cap A = A$.

(iii) Let $k(E) \neq A$, and let E be regular in $\mathcal{C}$. According to Theorem 2.11, $k(E)$ is regular in A. Since $k(E) \neq A$, there is a maximal regular ideal J in A containing $k(E)$, by [17, Theorem 2.4.6(d)]. ∎

If A is simultaneously a *-regular reduced BG^*-algebra and a γS^*-algebra, then Proposition 2.19 holds for the primitive ideals in A.

COROLLARY 2.20. *Let A be simultaneously a *-regular reduced BG^*-algebra and a γS^*-algebra. Furthermore, let $P \in \Pi_A$ be a primitive ideal in A. Then:*

(i) *If $a \in A$ with $Z(a) \subseteq Z(P)$, then $a \in P$.*
(ii) *If $Z(P) = \mathcal{C}$, then $P = A$.*

(iii) *If $P \neq A$ and $P = k(E)$ with a closed regular ideal E in $\mathcal{C}$, then there is a maximal regular ideal J in A containing P.*

Proof. Since A is *-regular, the *-structure space Π_A^* of A is homeomorphic to the structure space $\Pi_{\mathcal{C}}$ of $\mathcal{C}$. Since A is reduced, we conclude from [18, Corollary 10.5.7] that for every $I \in \Pi_A^*$ there is a primitive ideal $E \in \Pi_{\mathcal{C}}$ such that $I = E \cap A$. Since A is also a γS^*-algebra, we get, by [18, Theorem 10.5.1],

$$\Pi_A \subseteq \Pi_A^*.$$

So, altogether, each primitive ideal $P \in \Pi_A$ has the form $E \cap A$ with a primitive ideal $E \in \Pi_{\mathcal{C}}$. Since, by [17, Corollary 2.2.8], every Banach algebra is a spectral normed algebra, it follows from [17, Proposition 4.2.6] that each primitive ideal in a Banach algebra is closed. Hence, E is a closed two-sided ideal in $\mathcal{C}$. Consequently, according to Lemma 2.7, each primitive ideal $P \in \Pi_A$ has the form $k(E)$ with a closed two-sided ideal E in $\mathcal{C}$.

Therefore, the three assertions follow from the corresponding assertions in Proposition 2.19. ∎

In conclusion, we give an application of our results to the problem of *spectral synthesis*.

REMARK 2.21. Let G be a locally compact abelian group with dual group Γ of G, and let E be a closed subset of Γ. Then E is called a *set of spectral synthesis*, or an *S-set*, if E is the hull of a unique closed ideal in $A(\Gamma)$ (see e.g. [22, 7.1.4] or [2, p. 54]). We say that *spectral synthesis holds in* $A(\Gamma)$ if each closed subset of Γ is an S-set. In fact, by Malliavin's theorem (see e.g. [22, Theorem 7.6.1]), this is true if and only if G is compact so that Γ is discrete.

More generally, spectral synthesis may be defined for any completely regular, commutative, semisimple Banach algebra (see e.g. [13, Definition 39.9]).

Now, Remark 2.6 leads us to the following non-commutative generalization of spectral synthesis for a certain class of *-algebras:

Let A be a *-regular, hermitian, reduced Banach *-algebra. We call a closed two-sided ideal E in $\mathcal{C}$ an *ideal of spectral synthesis* for A, or an *S-ideal* for A, if there is a unique closed two-sided ideal in A which is dense in E. Furthermore, we say that *spectral synthesis holds in* A if each closed two-sided ideal E in $\mathcal{C}$ is an S-ideal for A. In this case, $k(E) = E \cap A$ is the only closed two-sided ideal in A which is dense in E, since the *-regularity of A implies that $k(E)$ is always dense in E. We also note that if A is in addition commutative, then, by [18, Proposition 10.5.9], A is completely regular, too.

Our definition of spectral synthesis for *-regular, hermitian, reduced Banach *-algebras A with enveloping C^*-algebra $\mathcal{C} := C^*(A)$ turns out to be equivalent to the "usual" one by E. Kaniuth et al. [9] saying the following: *Spectral synthesis holds in A* if each closed subset of the *-structure space Π^*_A of A is the hull of a unique closed two-sided ideal in A. The equivalence follows from the *-regularity of A, which means that the structure space $\Pi_{\mathcal{C}}$ of $\mathcal{C}$ is homeomorphic to Π^*_A, and since each closed two-sided ideal in $\mathcal{C}$ is an intersection of primitive ideals in $\mathcal{C}$, implying that there is a natural one-to-one correspondence between the closed subsets of $\Pi_{\mathcal{C}}$ and the closed two-sided ideals in $\mathcal{C}$, i.e., spectral synthesis holds in every C^*-algebra (see e.g. [3, II.6.5.3]).

In [9], it is further suggested that, since spectral synthesis is a very strong property, it seems unlikely that in a Banach *-algebra A spectral synthesis could hold when A fails to be *-regular and hermitian. This also justifies our common assumptions on A.

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