COLLOQUIUM MATHEMATICUM

VOL. 137 2014 NO. 1

LINEAR EXTENSIONS OF ORDERS INVARIANT UNDER ABELIAN GROUP ACTIONS

BY

ALEXANDER R. PRUSS (Waco, TX)

Abstract. Let G be an abelian group acting on a set X, and suppose that no element of G has any finite orbit of size greater than one. We show that every partial order on X invariant under G extends to a linear order on X also invariant under G. We then discuss extensions to linear preorders when the orbit condition is not met, and show that for any abelian group acting on a set X, there is a linear preorder $\leq$ on the powerset $\mathcal{P}X$ invariant under G and such that if A is a proper subset of B, then $A < B$ (i.e., $A \leq B$ but not $B \leq A$).

1. Linear orders. Szpilrajn's theorem [S] says that given the Axiom of Choice, any partial order can be extended to a linear order, where $\leq^*$ *extends* $\leq$ provided that $x \leq y$ implies $x \leq^* y$. There has been much work on what properties of the partial order can be preserved in the linear order (see, e.g., [BP, DHLS, Y]) but the preservation of symmetry under a group acting on a partially ordered set appears to have been neglected.

Suppose a group G acts on a partially ordered set $(X, \leq)$ and the order is G-invariant, where a relation R is *G-invariant* provided that for all $g \in G$ and $x, y \in X$, we have xRy if and only if $(gx)R(gy)$. It is natural to ask about the condition under which $\leq$ extends to a G-invariant linear order. We shall answer this question in the case where G is abelian. Then we will discuss extensions where the condition is not met. In the latter case, the extension will be to a *linear preorder* (total, reflexive and transitive relation) but will nonetheless preserve strict comparisons. Finally, we will apply the results to show that for any abelian group G acting on a set X, there is a G-invariant linear preorder on the powerset $\mathcal{P}X$ preserving strict set inclusion.

Throughout the paper we will assume the Axiom of Choice and all our proofs will be elementary and self-contained.

An *orbit* of $g \in G$ is any set of the form $\{g^n x : n \in \mathbb{Z}\}$. An obvious necessary condition for X to have a G-invariant linear order is that no element of G has any finite orbit of size greater than 1. Surprisingly, this is sufficient not

2010 *Mathematics Subject Classification*: Primary 06A05; Secondary 06A06, 06F99.
Key words and phrases: linear orders, abelian groups, group actions, linear preorders.

DOI: 10.4064/cm137-1-8

just for the existence of an invariant linear order, but for invariant partial orders to have invariant linear extensions.

THEOREM 1.1. *Let G be an abelian group. The following are equivalent:*

(i) *No element of G has any finite orbit of size greater than one.*
(ii) *There is a G-invariant linear order on X.*
(iii) *Every G-invariant partial order on X extends to a G-invariant linear order.*

We will call (iii) the *invariant order extension property.*

Theorem 1.1 yields a positive answer to de la Vega's question [V] whether given an order automorphism f of a partially ordered set $(X, \leq)$, with f having no finite orbits, $\leq$ can be extended to a linear order $\leq^*$ in such a way that f is an order automorphism of $(X, \leq^*)$. Just let G be the group generated by f.

Both of the non-trivial implications in Theorem 1.1 are false for non-abelian groups. Any torsion-free group that is non-right-orderable [DPT, P] acting on itself would provide a counterexample to (i)⇒(ii), while the fundamental group of the Klein bottle acting on itself would be a counterexample to (ii)⇒(iii) [DDHPV].

For the proof of the theorem, define a relation $\sim_G$ (or $\sim_{G,X}$ if we need to make X clear) on X by $x \sim_G y$ if and only if there is a $g \in G$ such that $g^n y = y$ for some $n \in \mathbb{Z}^+$ and $gy = x$. Clearly $\sim_G$ is reflexive. To see that it is symmetric observe that if $g^n y = y$ and $gy = x$, then

$$g^n x = g^{n+1} g^{-1} x = g^{n+1} y = gy = x,$$

so $g^{-n} x = x$ and $x = g^{-1} y$. If G is abelian, $\sim_G$ is transitive. For if $g^m y = y$ and $gy = x$, and $h^n z = z$ and $hz = y$, then $(gh)z = x$ and

$$(gh)^{mn+1} z = gg^{mn} hh^{mn} z = gg^{mn} hz = gg^{mn} y = gy = x.$$

Also, given a G-invariant partial order $\leq$, we define the relation $\leq_G$ by $x \leq_G y$ if and only if there is a finite sequence $(g_i)_{i=1}^n$ in G such that $x \leq g_i y$ and $\prod_{i=1}^n g_i = e$.

Since in Theorem 1.1, (iii)⇒(ii)⇒(i) is trivial, the theorem follows immediately from applying the following to a maximal G-invariant partial order on X extending $\leq$, which exists by Zorn, and obtaining a contradiction if that order is not linear.

PROPOSITION 1.2. *Let G be an abelian group acting freely on X. Let $\leq$ be a G-invariant partial order. If $\leq$ is not a linear order and G has no orbits of finite size greater than one, there exist x and y with $y \not\leq_G x$ and $x \not\leq y$. Moreover, whenever x and y in G are such that $y \not\leq_G x$, then there is a G-invariant partial order $\leq^*$ extending $\leq$ such that $x \leq^* y$.*

We now need to prove Proposition 1.2. Recall that R is *antisymmetric* provided that xRy and yRx implies $x = y$, so a partial order is an antisymmetric preorder. We then need:

LEMMA 1.3. *Suppose G is abelian and $\leq$ is a G-invariant partial order. Then:*

(i) *$\leq_G$ is a G-invariant preorder extending $\leq$.*

(ii) *For all $x, y \in X$, the following are equivalent:*

- (a) *$x \sim_G y$;*
- (b) *there is a finite sequence $(g_i)_{i=1}^n$ in G such that $x = g_i y$, $1 \leq i \leq n$, and $\prod_{i=1}^n g_i = e$;*
- (c) *$x \leq_G y$ and $y \leq_G x$.*

(iii) *The following are equivalent:*

- (a) *$\leq_G$ is antisymmetric;*
- (b) *for all $x, y \in X$, $x \sim_G y$ implies $x = y$;*
- (c) *no element of G has any finite orbit of size greater than one.*

Proof. (i) Invariance and reflexivity are clear. Suppose that $x \leq g_i y$, $1 \leq i \leq m$, and $y \leq h_j z$, $1 \leq j \leq n$, with the product of the g_i being e and that of the h_j being e as well. Then $g_i y \leq g_i h_j z$ by G-invariance of $\leq$, so $x \leq g_i h_j z$, and it is easy to see that the product of all the $g_i h_j$ is e, so $x \leq_G y$. Finally, if $x \leq y$, then $x \leq ey$ and so $x \leq_G y$.

(ii)(a)⇒(b). Assume (a). Then $g^n y = y$ and $x = gy$ for some $n \in \mathbb{Z}^+$ and $g \in G$, so $g^{-n} y = y$ and $x = g^{1-n} y$. Let $g_1 = g^{1-n}$, and let $g_i = g$ for $2 \leq i \leq n$. Then $x = g_i y$ for all i and the product of the g_i is e.

(ii)(b)⇒(a). Suppose $(g_i)_{i=1}^n$ in G are such that $x = g_i y$ and $\prod_{i=1}^n g_i = e$. Let G_y be the stabilizer of y, i.e., the subgroup $\{g \in G : gy = y\}$. We have $g_i^{-1} g_j y = g_i^{-1} x = y$ for all i, j, so the cosets $[g_i] = g_i G_y$ and $[g_j] = g_j G_y$ in G/G_y are equal for all i, j. Thus, $[g_1^n] = [\prod_{i=1}^n g_i] = e$, and so $g_1^n \in G_y$. Hence, $g_1^n y = y$ and $g_1 y = x$, so $x \sim_G y$. (I am grateful to Friedrich Wehrung for drawing my attention to the stabilizer subgroups in connection with condition (ii)(b).)

(ii)(b)⇒(c). Suppose $x = g_i y$ where the product of the g_i is e. Thus $x \leq g_i y$ for all i, and $x \leq_G y$. Let $h_i = g_i^{-1}$. Then $y = h_i x$, so $y \leq h_i x$, and the product of the h_i is e, so $y \leq_G x$.

(ii)(c)⇒(b). Suppose $x \leq_G y$ and $y \leq_G x$. Suppose thus that $x \leq g_i y$, $1 \leq i \leq m$, and $y \leq h_i x$, $1 \leq i \leq n$, with $\prod_{i=1}^m g_i = \prod_{i=1}^n h_i = e$. By invariance, $g_i y \leq g_i h_j x$ for $i \leq m$ and $j \leq n$, so

$$x \leq g_i y \leq g_i h_j x. \tag{1.1}$$

Fix $1 \leq i_1 \leq m$. Let $(i_k, j_k), 1 \leq k \leq mn$, enumerate $([1, m] \cap \mathbb{Z}) \times ([1, n] \cap \mathbb{Z})$.

Then by iterating (1.1) and using the invariance of $\leq$ we get

$$x \leq g_{i_1} y \leq g_{i_1} h_{j_1} x \leq g_{i_1} h_{j_1} g_{i_2} h_{j_2} x \leq \cdots \leq \prod_{k=1}^{mn} (g_{i_k} h_{j_k}) x = x.$$

Thus, $x = g_{i_1} y$. But i_1 was arbitrary. Hence, $x = g_i y$ for all i, and so $x \sim_G y$.

(iii) The equivalence of (a) and (b) follows from (ii). An element g has an orbit of finite size greater than 1 if and only if there is an x such that $gx \neq x$ but $g^n x = x$ for some n. The equivalence of (b) and (c) follows. ∎

Proof of Proposition 1.2. If $\leq$ is not a linear order, there are x and y such that $x \not\leq y$ and $y \not\leq x$. By the antisymmetry of $\leq_G$ (from Lemma 1.3), at least one of $y \not\leq_G x$ or $x \not\leq_G y$ must also hold.

Suppose now that $y \not\leq_G x$.

Let $a \leq^0 b$ provided that either $a \leq b$, or there is a $g \in G$ such that $a = gx$ and $b = gy$.

Let $\leq^*$ be the transitive closure of $\leq^0$. Then $\leq^*$ is G-invariant, reflexive, transitive and an extension of $\leq$. We need only show $\leq^*$ to be antisymmetric.

Since $\leq^*$ is the transitive closure of $\leq^0$ while $\leq$ is antisymmetric and transitive, if $\leq^*$ fails to be antisymmetric, by definition of $\leq^0$, there will have to be a loop of the form

$$g_1 x \leq^0 g_1 y \leq g_2 x \leq^0 g_2 y \leq \cdots \leq g_n x \leq^0 g_n y \leq g_1 x.$$

Let $g_{n+1} = g_1$. Thus, $g_i y \leq g_{i+1} x$ for $1 \leq i \leq n$. By G-invariance, $y \leq g_i^{-1} g_{i+1} x$. Let $h_i = g_i^{-1} g_{i+1}$, so $y \leq h_i x$, and observe that $\prod_{i=1}^n h_i = e$. Therefore, $y \leq_G x$ by Lemma 1.3, contrary to what we have assumed. ∎

Proposition 1.2 also yields:

COROLLARY 1.4. *If G is an abelian group acting on a set X with a G-invariant partial order $\leq$, and no element of G has a finite orbit of size greater than one, then $\leq_G$ is the intersection of all G-invariant linear orders extending $\leq$.*

Proof. Proposition 1.2 and Zorn's lemma show that if $y \not\leq_G x$, then there is a G-invariant linear order $\leq^*$ extending $\leq$ and such that $x \leq^* y$, and hence such that $y \not\leq^* x$. Thus the intersection of all G-invariant linear orders extending $\leq$ is contained in $\leq_G$.

For the other inclusion, we need to show that if $\leq^*$ is a G-invariant linear order extending $\leq$, then $x \leq_G y$ implies $x \leq^* y$.

Suppose $x \leq_G y$, so there are $(g_i)_{i=1}^n$ whose product is e and which satisfy $x \leq g_i y$. To obtain a contradiction, suppose $x \not\leq^* y$. Since $\leq^*$ is linear, $x \neq y$ and $y \leq^* x$. Thus, $x \leq g_i y \leq^* g_i x$ for all i. Hence, using the invariance of $\leq^*$ and iteratively applying $x \leq^* g_i x$, we get

$$x \leq g_1 y \leq^* g_1 x \leq^* g_1 g_2 x \leq^* \cdots \leq^* g_1 \cdots g_n x = x.$$

Thus $x = g_1 y$. Reordering the g_i as needed, we can prove that $x = g_i y$ for all i, and so $x \sim_G y$, and hence $x = y$ by Lemma 1.3, contrary to our assumptions. ∎

Note that if G is a partially ordered torsion-free abelian group considered as acting on itself, then it is easy to see that $x \leq_G y$ if and only if there is an $n \in \mathbb{Z}^+$ such that $x^n \leq_G y^n$. Thus, if $\leq$ is a *normal* order in the terminology of [F], i.e., one such that $0 \leq y^n$ implies $0 \leq y$ (and hence $x^n \leq y^n$ implies $x \leq y$), then $\leq_G$ coincides with $\leq$, and Corollary 1.4 yields classical results [E, F] on extensions of partial orders on abelian groups.

2. Preorders and orderings of subsets. Even if G's action on X lacks the invariant order extension property, we can extend a partial order to a linear preorder (i.e., a preorder where all elements are comparable). Of course this is trivially true: just take the preorder such that for all x, y we have $x \leq^* y$ and $y \leq^* x$. What is not trivially true is that if G is any abelian group, we can extend the partial order to a preorder while preserving all the strict inequalities in the partial order. In fact, this is even true if we start off with $\leq$ a preorder. Recall that $x < y$ is defined to hold if and only if $x \leq y$ and not $y \leq x$.

THEOREM 2.1. *If G is any abelian group acting on a space X, and $\leq$ is a G-invariant preorder on X, then there is a G-invariant linear preorder $\leq^*$ on X that extends $\leq$ and is such that if $x < y$, then $x <^* y$.*

The proof depends on two lemmas.

LEMMA 2.2. *Suppose G is an abelian group acting on a space X. Let $Y = X/{\sim_{G,X}}$ and extend the action of g to Y by $g[A] = [gA]$. This is a well-defined group action, and G acting on Y has the invariant order extension property.*

Proof. That the group action is well-defined follows from the fact that $x \sim_{G,X} y$ if and only if $gx \sim_{G,X} gy$, for any $x, y \in X$ and $g \in G$.

Suppose that $[x] \sim_{G,Y} [y]$ for $x, y \in X$. Choose $f \in G$ and $m \in \mathbb{Z}_+$ such that $f[y] = [x]$ and $f^m[y] = [y]$. Without loss of generality assume $m \geq 3$. Thus, $x \sim_{G,X} fy$ and $y \sim_{G,X} f^m y$. Hence there are $g, h \in G$ and $n, p \in \mathbb{Z}^+$ such that $gfy = x$, $g^n fy = fy$, $hf^m y = y$ and $h^p f^m y = f^m y$. Without loss of generality assume $n \geq 3$.

Thus, $y = g^{-n} y$, $y = f^{-m} h^{-1} y$ and $y = h^p y$. Since $x = fgy$, we have $x = h_i y$ for $1 \leq i \leq 4$, where

$$h_1 = fg, \quad h_2 = fg^{1-n}, \quad h_3 = f^{1-m} g h^{-1}, \quad h_4 = fgh^p.$$

Let $n_1 = m(n-1)p - n(p+1)$, $n_2 = mp$, $n_3 = np$ and $n_4 = n$ (the values were generated by computer). Given that $m \geq 3$ and $n \geq 3$, we have

$n_1 \geq 0$. Straightforwardly, $h_1^{n_1} h_2^{n_2} h_3^{n_3} h_4^{n_4} = e$. Then let the g_i be a sequence of $n_1+n_2+n_3+n_4$ entries from G, with the first n_1 being all equal to h_1, the next n_2 being h_2, the next n_3 being h_3 and the rest being h_4. Then $x = g_i y$, and the product of the g_i is e, so $x \sim_{G,X} y$. Thus $[x] = [y]$, and so we have the invariant order extension property. ∎

LEMMA 2.3. *Suppose G is an abelian group acting on a space X and $\leq$ is a G-invariant partial order on G.*

(i) *If $x < y$, then we do not have $x \sim_G y$.*
(ii) *If $x \sim_G x'$, $y \sim_G y'$ and $x \leq y$, then $x' \leq_G y'$.*
(iii) *If $x < y$ and $x \sim_G x'$ and $y \sim_G y'$, then we do not have $y' \leq x'$.*

Proof. (i) Suppose $x < y$. To obtain a contradiction, suppose $x \sim_G y$, so $gy = x$ and $g^n y = y$, for some n and g. By invariance, $g^k y > g^k x$ for all k. Thus,

$$y > x = gy > gx = g^2 y > \cdots > g^{n-1} x = g^n y = y,$$

a contradiction.

(ii) If $x \sim_G x'$ and $y \sim_G y'$, then by Lemma 1.3 there are $(g_i)_{i=1}^m$ with product e, and $(h_i)_{i=1}^n$ with product e, such that $x = g_i x'$ and $y = h_j y'$. Thus, $g_i x' \leq h_j y'$, and by G-invariance of $\leq$, we have $x' \leq g_i^{-1} h_j y'$. The product of the $g_i^{-1} h_j$, as (i,j) ranges over $([1,m] \cap \mathbb{Z}) \times ([1,n] \cap \mathbb{Z})$, is e, so $x' \leq_G y'$.

(iii) Now suppose that $x < y$, $x \sim_G x'$ and $y \sim_G y'$. Then $x' \leq_G y'$ by (ii). To obtain a contradiction, suppose $y' \leq x'$. So $y' \leq_G x'$. Thus, $x' \sim_G y'$ by Lemma 1.3. Since $\sim_G$ is an equivalence relation, $x \sim_G y$, which contradicts $x < y$ by (i). ∎

Proof of Theorem 2.1. First note that we only need to prove the result for $\leq$ a partial order. For if $\leq$ is a preorder, then we can replace X by $X/\simeq$ where $x \simeq y$ if and only if $x \leq y$ and $y \leq x$. Define the natural group action of G by $g[x]_\simeq = [gx]_\simeq$, and note that stipulating that $[x]_\simeq \preceq [y]_\simeq$ if and only if $x \leq y$ gives a well-defined G-invariant partial order. The partial order version of the theorem then yields a linear preorder extending $\preceq$, which lifts to a linear preorder on X satisfying the required conditions.

Suppose thus that $\leq$ is a G-invariant partial order on X. For $a, b \in Y = X/{\sim_{G,X}}$, let $a \leq^0 b$ if and only if there are representatives $x \in a$ and $y \in b$ such that $x \leq y$.

Clearly, $\leq^0$ is reflexive and G-invariant. Suppose that $a \leq^0 b$ and $b \leq^0 c$. Choose $x \in a$, $y_1, y_2 \in b$ and $z \in c$ such that $x \leq y_1$ and $y_1 \leq z$. Since $y_1 \sim_G y_2$, by Lemma 1.3 we have $y_1 \leq y_2$, so $x \leq z$ and $a \leq^0 c$.

We now check that $\leq^0$ is antisymmetric. Suppose $a \leq^0 b$ and $b \leq^0 a$. Thus there are representatives $x, x' \in a$ and $y, y' \in b$ such that $x \leq y$ and

$y' \leq x'$. If $x = y$, we have $a = b$ as desired. Otherwise, $x < y$. Moreover, $x \sim_{G,X} x'$ and $y \sim_{G,X} y'$. But that would contradict Lemma 2.3(iii).

Thus $\leq^0$ is a partial order. By Lemma 2.2 and Theorem 1.1, extend it to a G-invariant linear order $\leq^1$ on Y. Now let $x \leq^* y$ if and only if $[x] \leq^1 [y]$. This is a G-invariant linear preorder.

Suppose $x < y$. We then have $[x] \leq^1 [y]$. Thus $x \leq^* y$. To complete our proof, we must show $y \not\leq^* x$. By Lemma 2.3(i), we do not have $x \sim_{G,X} y$, and so $[x] \neq [y]$. Since $\leq^1$ is a partial order, $[y] \not\leq^1 [x]$, and so $y \not\leq^* x$. Thus $x <^* y$. ∎

Corollary 2.4. *Suppose G is an abelian group acting on a space X. Then there is a G-invariant linear preorder $\leq$ on the powerset $\mathcal{P}X$ such that if A is a proper subset of B, then $A < B$.*

In particular, there is a translation-invariant "size comparison" for subsets of $\mathbb{R}^n$ for all n as well as a rotationally-invariant "size comparison" for subsets of the circle $\mathbb{T}$ that preserves the intuition that proper subsets are "smaller".

Corollary 2.4 is not true in general for non-abelian G, even in the case of isometry groups that are "very close" to abelian. For instance, suppose G is all isometries on the line $\mathbb{R}$. This has the translations as a subgroup of index two and is supramenable, i.e., for every non-empty subset A of any set X that G acts on, there is a finitely-additive G-invariant measure μ of X with $\mu(A) = 1$ [W, Chapter 12]. But we shall shortly see that there is no G-invariant preorder $\leq$ on $\mathcal{P}\mathbb{R}$ such that $A < B$ whenever A is a proper subset of B.

To see this, say that a preorder $\leq$ is *strongly G-invariant* provided that $x \leq y$ if and only if $gx \leq y$ if and only if $x \leq gy$, for all $g \in G$ and $x, y \in X$. Then there is no *strongly* G-invariant preorder $\leq$ on $\mathcal{P}\mathbb{R}$ such that $A \subset B$ implies $A < B$, since if $\leq$ were such a preorder, then we would have $\mathbb{Z}^+ < \mathbb{Z}_0^+$ even though $1 + \mathbb{Z}_0^+ = \mathbb{Z}^+$.

But it turns out that if G is all isometries on $\mathbb{R}$, then invariance implies strong invariance, and so there is no invariant G-invariant preorder on $\mathcal{P}\mathbb{R}$ which preserves strict inclusion. For the isometry group G is generated by elements of finite order, namely reflections, and elements of finite order have finite orbits, while:

Proposition 2.5. *If $\leq$ is a G-invariant linear preorder on X, and G is any group generated by elements all of whose orbits are finite, then $\leq$ is strongly G-invariant.*

Proof. We only need to prove that if $g \in G$ has only finite orbits, then $x \leq y$ implies $gx \leq y$. Suppose $x \leq y$ and $g^n x = x$. By linearity, we have $x \leq gx$ or $gx \leq x$ (or both). If $x \leq gx$, then $g^k x \leq g^{k+1} x$ for all k by invariance, and so

$$x \le gx \le g^2x \le \cdots \le g^n x = x,$$

hence $gx \le x$. So in either case, $gx \le x$. By transitivity, $x \le y$ implies $gx \le y$. ∎

The following generalizes the remarks about the isometries on $\mathbb{R}$:

Corollary 2.6. *If G is any group acting on a set X and there are $g, h \in G$ with only finite orbits, while gh has at least one infinite orbit, then there is no G-invariant preorder $\le$ on $\mathcal{P}X$ such that if A is a proper subset of B, then $A < B$.*

Proof. Without loss of generality, G is generated by g and h. Let A be an infinite orbit of gh, fix $x \in A$, and let $A^+ = \{(gh)^n x : n \in \mathbb{Z}_0^+\}$. Then ghA^+ is a proper subset of A^+, and there is no strongly G-invariant preorder $\le$ on $\mathcal{P}X$ such that $ghA^+ < A^+$. By Proposition 2.5, there is no G-invariant preorder like that, either. ∎

Acknowledgements. I am grateful to David Arnold, Dietrich Burde, Ramiro de la Vega, Trent Dougherty, A. Paul Pedersen and Friedrich Wehrung for discussions.

REFERENCES

[BP] R. Bonnet and M. Pouzet, *Linear extensions of ordered sets*, in: I. Rival (ed.), Ordered Sets, Reidel, Boston, 1982, 125–170.

[DDHPV] M. A. Dąbkowska, M. K. Dąbkowski, V. S. Harizanov, J. H. Przytycki and M. A. Veve, *Compactness of the space of left orders*, J. Knot Theory Ramif. 16 (2007), 257–266.

[DPT] M. K. Dąbkowski, J. H. Przytycki and A. A. Togha, *Non-left-orderable 3-manifold groups*, Canad. Math. Bull. 48 (2005), 32–40.

[DHLS] R. G. Downey, D. R. Hirschfeldt, S. Lempp and R. Solomon, *Computability-theoretic and proof-theoretic aspects of partial and linear orderings*, Israel J. Math. 138 (2003), 271–289.

[E] C. J. Everett, *Note on a result of L. Fuchs on ordered groups*, Amer. J. Math. 72 (1950), 216.

[F] L. Fuchs, *On the extension of the partial order of groups*, Amer. J. Math. 72 (1950), 191–194.

[P] S. D. Promislow, *A simple example of a torsion-free, non unique product group*, Bull. London Math. Soc. 20 (1988), 302–304.

[S] E. Szpilrajn, *Sur l'extension de l'ordre partiel*, Fund. Math. 16 (1930), 386–389.

[V] R. de la Vega, *Extending a partial order while preserving an automorphism*, MathOverflow, 2013, http://mathoverflow.net/questions/141979.

[W] S. Wagon, *The Banach–Tarski Paradox*, Cambridge Univ. Press, Cambridge, 1993.

[Y] Z.-Q. Yang, *Every self-dual ordering has a self-dual linear extension*, Order 4 (1987), 97–100.

Alexander R. Pruss
Department of Philosophy
Baylor University
One Bear Place #97273
Waco, TX 76798-7273, U.S.A.
E-mail: alexander_pruss@baylor.edu

Received 26 October 2013;
revised 14 August 2014 (6056)