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# ON AN ESTIMATE FOR THE LINEARIZED COMPRESSIBLE NAVIER–STOKES EQUATIONS IN THE $L_p$-FRAMEWORK

BY

PIOTR BOGUSŁAW MUCHA (Warszawa)

**Abstract.** An $L_p$-estimate with a constant independent of time for solutions of the linearized compressible Navier–Stokes system in the whole space (under the assumption that solutions have compact supports in space) is obtained.

**1. Introduction.** In the paper we examine the following system in $\mathbb{R}^4$:

$$\begin{aligned} &u_t - \mu\Delta u - \nu\nabla \operatorname{div} u + a\nabla\eta = f,\\ &\eta_t + b \operatorname{div} u = g; \end{aligned} \tag{1.1}$$

we assume that

$$\operatorname{supp}(u,\eta) \subset B(0,1) \times (0,\infty), \tag{1.2}$$

where $B(0,1) = \{x \in \mathbb{R}^3 : |x| < 1\}$ and $\mu$, $\nu$, $a$, $b$ are constant positive coefficients. System (1.1)–(1.2) can be treated as a localization of the Cauchy problem for the linearized compressible Navier–Stokes equations in the whole space with vanishing initial data:

$$\begin{aligned} &v_t - \mu\Delta v - \nu\nabla \operatorname{div} v + a\nabla q = F,\\ &q_t + b \operatorname{div} v = G,\\ &v|_{t=0} = 0, \qquad q|_{t=0} = 0. \end{aligned} \tag{1.3}$$

To obtain (1.1)–(1.2) from (1.3) it is enough to multiply (1.3) by $\pi$, where $\pi$ is a smooth function with compact support, and consider the system for $u = \pi v$ and $\eta = \pi q$.

Our aim is to prove an $L_p$-estimate for solutions of (1.1)–(1.2) with a constant independent of time. This result can be a useful tool to prove the global existence of solutions to equations of motion of viscous compressible barotropic fluids. In this way we will be able to obtain global-in-time solu-

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tions for the compressible Navier–Stokes system with sharp regularity such that $u \in W_r^{2,1}$ with $r > 3$ (see [4]).

One can find similar results (more general, but with the constant depending on time) in [2, 5, 6].

The main result of the paper is the following:

THEOREM. *Let* $r \geq 2$, $f \in L_r(\mathbb{R}^4) \cap L_2(\mathbb{R}^4)$, $g \in W_r^{1,0}(\mathbb{R}^4) \cap W_2^{1,0}(\mathbb{R}^4)$. *Then for* $0 < T < \infty$ *the solution of* (1.1)–(1.2) *satisfies the following estimate*:

$$\begin{aligned}(1.4)\qquad &\|u\|_{W_r^{2,1}(\mathbb{R}^3\times[0,T])} + \|\eta\|_{W_r^{1,0}(\mathbb{R}^3\times[0,T])} + \|\eta_t\|_{W_r^{1,0}(\mathbb{R}^3\times[0,T])} \\ &\quad + \|u\|_{W_2^{2,1}(\mathbb{R}^3\times[0,T])} + \|\eta\|_{W_2^{1,0}(\mathbb{R}^3\times[0,T])} + \|\eta_t\|_{W_2^{1,0}(\mathbb{R}^3\times[0,T])} \\ &\leq A_0(\|f\|_{L_r(\mathbb{R}^3\times[0,T])} + \|g\|_{W_r^{1,0}(\mathbb{R}^3\times[0,T])} \\ &\quad + \|f\|_{L_2(\mathbb{R}^3\times[0,T])} + \|g\|_{W_2^{1,0}(\mathbb{R}^3\times[0,T])}),\end{aligned}$$

*where* $A_0$ *is independent of* $T$.

**2. Notation.** In our considerations we will need the anisotropic Sobolev spaces $W_r^{m,n}(Q_T)$, where $m, n \in \mathbb{R}_+ \cup \{0\}$, $r \geq 1$ and $Q_T = Q \times (0,T)$, with the norm

$$\begin{aligned}(2.1)\quad \|u\|^r_{W_r^{m,n}(Q_T)} &= \int_0^T\int_Q |u(x,t)|^r\,dx\,dt \\ &+ \sum_{0\leq|m'|\leq[|m|]} \int_0^T\int_Q |D_x^{m'}u(x,t)|^r\,dx\,dt \\ &+ \sum_{|m'|=[|m|]} \int_0^T dt \int_Q\int_Q \frac{|D_x^{m'}u(x,t) - D_x^{m'}u(x',t)|^r}{|x-x'|^{s+r(|m|-[|m|])}}\,dx\,dx' \\ &+ \sum_{0\leq|n'|\leq[|n|]} \int_0^T\int_Q |D_t^{n'}u(x,t)|^r\,dx\,dt \\ &+ \int_Q dx \int_0^T\int_0^T \frac{|D_t^{[n]}u(x,t) - D_t^{[n]}(x,t')|^r}{|t-t'|^{1+r(n-[n])}}\,dt\,dt',\end{aligned}$$

where $s = \dim Q$, $[\alpha]$ is the integral part of $\alpha$, and $D_x^l = \partial_{x_1}^{l_1}\ldots\partial_{x_s}^{l_s}$, where $l = (l_1,\ldots,l_s)$ is a multiindex.

In the case when $Q_T = \mathbb{R}^s \times \mathbb{R}$ we can apply the Fourier transform and define the Bessel-potential spaces given by the norm

$$\begin{aligned}(2.2)\qquad \|u\|_{H_r^{m,n}(\mathbb{R}^{s+1})} &= \|u\|_{L_r(\mathbb{R}^{s+1})} + \|\mathcal{F}^{-1}_{t,x}[|\xi|^m\widehat{u}(\xi,\xi_0)]\|_{L_r(\mathbb{R}^{s+1})} \\ &\quad + \|\mathcal{F}^{-1}_{t,x}[|\xi_0|^n\widehat{u}(\xi,\xi_0)]\|_{L_r(\mathbb{R}^{s+1})},\end{aligned}$$

where $\widehat{u}(\xi,\xi_0)$ is the Fourier transform of $u(x,t)$:

$$\widehat{u}(\xi,\xi_0)=\int e^{-i\xi_0 t}\int e^{-i\xi\cdot x}u(x,t)\,dx\,dt\equiv\mathcal{F}_{t,x}[u](\xi,\xi_0),$$

and $\mathcal{F}^{-1}$ the inverse transformation

$$\mathcal{F}^{-1}_{t,x}[\widehat{u}](x,t)=(2\pi)^{-2(s+1)}\int e^{i\xi_0 t}\int e^{i\xi\cdot x}\widehat{u}(\xi,\xi_0)\,d\xi\,d\xi_0,$$

where $\xi=(\xi_1,\dots,\xi_s)$ and $\xi\cdot x=\xi_1x_1+\dots+\xi_sx_s$.

We also define the space $V_r(Q_T)$ with the norm

$$\|u\|_{V_r(Q_T)}=\|u\|_{W^{1,0}_r(Q_T)}+\|u_t\|_{W^{1,0}_r(Q_T)}. \tag{2.3}$$

In the proof we will use the following results.

THEOREM 2.1 (Marcinkiewicz theorem, see [3]). *Suppose that the function $\Phi:\mathbb{R}^m\to\mathbb{C}$ is smooth enough and there exists $M>0$ such that for every point $x\in\mathbb{R}^m$ we have*

$$|x_{j_1}\dots x_{j_k}|\left|\frac{\partial^k\Phi}{\partial x_{j_1}\dots\partial x_{j_k}}\right|\le M,\qquad 0\le k\le m,\ 1\le j_1<\dots<j_k\le m.$$

*Then the operator*

$$Pg(x)=(2\pi)^{-m}\int_{\mathbb{R}^m}dy\,e^{ixy}\Phi(y)\int_{\mathbb{R}^m}e^{-iyz}g(z)\,dz$$

*is bounded in $L_p(\mathbb{R}^m)$ and*

$$\|Pg\|_{L_p(\mathbb{R}^m)}\le A_{p,m}M\|g\|_{L_p(\mathbb{R}^m)}.$$

PROPOSITION 2.2 (see [7]). *If $r>2$ and $m,n>0$ then*

$$H^{m,n}_r(\mathbb{R}^s\times\mathbb{R})\subset W^{m,n}_r(\mathbb{R}^s\times\mathbb{R})$$

*and*

$$\|u\|_{W^{m,n}_r(\mathbb{R}^s\times\mathbb{R})}\le c\|u\|_{H^{m,n}_r(\mathbb{R}^s\times\mathbb{R})};$$

*moreover if $m,n\in\mathbb{N}$ then $H^{m,n}_r=W^{m,n}_r$.*

PROPOSITION 2.3 (see [1]). *Let $u\in W^{m,n}_r(\Omega_T)$, $m,n\in\mathbb{R}_+$, and $q\ge r\ge 2$. If*

$$\kappa=\sum_{i=1}^{3}\left(\alpha_i+\frac{1}{r}-\frac{1}{q}\right)\frac{1}{m}+\left(\beta+\frac{1}{r}-\frac{1}{q}\right)\frac{1}{n}<1,$$

*then*

$$\|D^\beta_tD^\alpha_xu\|_{L_q(\Omega_T)}\le\varepsilon\|u\|_{W^{m,n}_r(\Omega_T)}+c(\varepsilon)\|u\|_{L_2(\Omega_T)}$$

*for each $\varepsilon\in(0,1)$, with $c(\varepsilon)\to\infty$ as $\varepsilon\to0$.*

During our considerations we will use well known results like the imbedding theorems for Sobolev spaces. All constants are denoted by $c$.

**3. Proof of Theorem.** In our considerations we assume that all functions are $C^\infty$ smooth. The result for such functions easily implies (1.4) in the general case. We examine the system

$$\begin{aligned} &u_t - \mu\Delta u - \nu\nabla \operatorname{div} u + a\nabla\eta = f,\\ &\eta_t + b \operatorname{div} u = g \end{aligned} \tag{3.1}$$

in $\mathbb{R}^4$; we assume that

$$\operatorname{supp}(u,\eta) \subset \Omega \times (0,\infty),$$

where $\Omega$ is a bounded domain with smooth boundary $S$ and $\operatorname{diam}\Omega \le 1$.

The first aim is to find an estimate on $\operatorname{div} u$. We set

$$d = \operatorname{div} u.$$

From (3.1) we get

$$\begin{aligned} &d_t - (\mu+\nu)\Delta d + a\Delta\eta = \operatorname{div} f,\\ &\eta_t + bd = g,\\ &d|_S = 0. \end{aligned} \tag{3.2}$$

To simplify (3.2) we solve the parabolic problem

$$\begin{aligned} &d_{1,t} - (\mu+\nu)\Delta d_1 = \operatorname{div} f,\\ &d_1|_S = 0. \end{aligned} \tag{3.3}$$

The solutions of (3.3) satisfy (see Appendix, Lemma 4A)

$$\begin{aligned} \|d_1\|_{W_r^{1,1/2}(\Omega\times(0,\infty))} &+ \|d_1\|_{W_2^{1,1/2}(\Omega\times(0,\infty))}\\ &\le c(\|f\|_{L_r(\Omega\times(0,\infty))} + \|f\|_{L_2(\Omega\times(0,\infty))}). \end{aligned} \tag{3.4}$$

We look for $d$ in the form

$$d = d_1 + d_2.$$

Hence (3.2) reduces to

$$\begin{aligned} &d_{2,t} - (\mu+\nu)\Delta d_2 + a\Delta\eta = 0,\\ &\eta_t + bd_2 = g - bd_1 = g',\\ &d_2|_S = 0. \end{aligned} \tag{3.5}$$

To examine (3.5) we apply the $L_2$-technique. Multiplying $(3.5)_1$ by $d_2$, integrating over $\Omega$, and using $(3.5)_2$ we get

$$\frac{1}{2}\frac{d}{dt}\int\left(d_2^2 + \frac{a}{b}|\nabla\eta|^2\right)dx + (\mu+\nu)\int|\nabla d_2|^2\,dx = \frac{a}{b}\int\nabla\eta\cdot\nabla g'\,dx, \tag{3.6}$$

which gives

$$\|d_2\|^2_{W_2^{1,0}(\Omega\times(0,\infty))} \le c\int_0^\infty\int|\nabla\eta\cdot\nabla g'|\,dx\,dt. \tag{3.7}$$

Inequality (3.7) and $(3.5)_2$ also give

$$\|\nabla\eta_t\|_{L_2(\Omega\times(0,\infty))} \le c\Big(\int_0^\infty\int|\nabla\eta\cdot\nabla g'|\,dx\,dt\Big)^{1/2} + \|\nabla g'\|_{L_2(\Omega\times(0,\infty))}. \tag{3.8}$$

From (3.5) and (3.8) we obtain the equation

$$\begin{aligned} &d_{2,tt} - (\mu+\nu)\Delta d_{2,t} = -a\operatorname{div}\nabla\eta_t,\\ &d_{2,t}|_S = 0. \end{aligned} \tag{3.9}$$

By Lemma 4A (see Appendix) we have

$$\|d_{2,t}\|_{W_2^{1,1/2}(\Omega\times(0,\infty))} \le c\|\nabla\eta_t\|_{L_2(\Omega\times(0,\infty))}. \tag{3.10}$$

By the imbedding theorem (Proposition 2.3), (3.10) and (3.8) we get

$$\|d_{2,t}\|_{L_r(\Omega\times(0,\infty))} \le c\Big(\int_0^\infty\int|\nabla\eta\cdot\nabla g'|\,dx\,dt\Big)^{1/2} + c\|\nabla g'\|_{L_2(\Omega\times(0,\infty))}, \tag{3.11}$$

where $2 \le r \le 10/3$.

Now we return to (3.2) in the form

$$\begin{aligned} &-(\mu+\nu)\Delta d + a\Delta\eta = \operatorname{div} f - d_{1,t} - d_{2,t},\\ &\eta_t + bd = g,\\ &d|_{t=0} = 0, \qquad \eta|_{t=0} = 0. \end{aligned} \tag{3.12}$$

We recall that $d$ and $\eta$ have compact supports in space. From (3.12) we get the equation in the whole space

$$\begin{aligned} &-\Delta\Big(\frac{\mu+\nu}{b}\eta_t + a\eta\Big) = \frac{1}{b}\operatorname{div}\nabla g + \operatorname{div} f - d_{1,t} - d_{2,t},\\ &\eta|_{t=0} = 0. \end{aligned} \tag{3.13}$$

To solve (3.13) we consider two systems

$$\begin{aligned} &-\Delta\Big(\frac{\mu+\nu}{b}\eta_{1,t} + a\eta_1\Big) = \frac{1}{b}\operatorname{div}\nabla g + \operatorname{div} f,\\ &\eta_1|_{t=0} = 0 \end{aligned}$$

and

$$\begin{aligned} &-\Delta\Big(\frac{\mu+\nu}{b}\eta_{2,t} + a\eta_2\Big) = -d_{1,t} - d_{2,t},\\ &\eta_2|_{t=0} = 0. \end{aligned}$$

We see that

$$\eta = \eta_1 + \eta_2. \tag{3.14}$$

Solving for $\eta_1$ and $\eta_2$, and applying the Fourier transform we get

$$\eta_1 = \mathcal{F}^{-1}\left[\frac{i\xi}{|\xi|^2(a+\frac{\mu+\nu}{b}i\xi_0)}\mathcal{F}\left[f+\frac{1}{b}\nabla g\right]\right],$$
$$\eta_2 = \mathcal{F}^{-1}\left[\frac{1}{|\xi|^2(a+\frac{\mu+\nu}{b}i\xi_0)}\mathcal{F}\left[-d_{1,t}-d_{2,t}\right]\right].$$

Since

$$\left||\xi_0|^\alpha \partial^\alpha_{\xi_0}\frac{1}{a+\frac{\mu+\nu}{b}i\xi_0}\right| < c,$$

by Theorem 2.1 we have

$$\begin{aligned}\|\nabla\eta_1\|_{L_r(\mathbb{R}^4)} &\le c\|f\|_{L_r(\mathbb{R}^4)} + c\|\nabla g\|_{L_r(\mathbb{R}^4)},\\ \|\nabla^2\eta_2\|_{L_r(\mathbb{R}^4)} &\le c\|d_1\|_{L_r(\mathbb{R}^4)} + c\|d_{2,t}\|_{L_r(\mathbb{R}^4)}.\end{aligned} \tag{3.15}$$

We see that from (3.15) we cannot obtain an estimate for $\eta$. Take the equation for $\eta_1$:

$$\begin{aligned}&-\Delta\left(\frac{\mu+\nu}{b}\eta_{1,t}+a\eta_1\right) = \operatorname{div}\left(\frac{1}{b}\nabla g + f\right),\\ &\eta_1|_{t=0} = 0.\end{aligned} \tag{3.16}$$

Multiplying $(3.16)_1$ by $\eta_1$ and integrating over $\mathbb{R}^3$ we obtain

$$\frac{\mu+\nu}{2b}\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla\eta_1|^2\,dx + a\int_{\mathbb{R}^3}|\nabla\eta_1|^2\,dx = -\int_{\mathbb{R}^3}\left(\frac{1}{b}\nabla g+f\right)\cdot\nabla\eta_1\,dx. \tag{3.17}$$

Integrating (3.17) with respect to $t$ over $[0,\infty)$ and applying the Young inequality we get

$$\sup_{t\in(0,\infty)}\int_{\mathbb{R}^3}|\nabla\eta_1|^2\,dx + \int_0^\infty\int_{\mathbb{R}^3}|\nabla\eta_1|^2\,dx\,dt \le c\int_0^\infty\int_{\mathbb{R}^3}(|\nabla g|^2+|f|^2)\,dx\,dt. \tag{3.18}$$

Since $\eta$ has compact support in space, from (3.13) we get (in the same way as for (3.16))

$$\sup_{t\in(0,\infty)}\int_{\mathbb{R}^3}|\nabla\eta|^2\,dx + \int_0^\infty\int_{\mathbb{R}^3}(|\nabla\eta|^2+|\eta|^2)\,dx\,dt \le c\int_0^\infty\int_{\mathbb{R}^3}(|\nabla g|^2+|f|^2)\,dx\,dt. \tag{3.19}$$

Since $\eta_2 = \eta - \eta_1$, from (3.18) and (3.19) we get

$$\|\nabla\eta_2\|_{L_r(0,\infty;L_2(\mathbb{R}^3))} \le c(\|\nabla g\|_{L_2(\mathbb{R}^4)} + \|f\|_{L_2(\mathbb{R}^4)}). \tag{3.20}$$

From the imbedding theorem we have

$$\|\nabla\eta_2\|_{L_r(\Omega)} \le c(\|\nabla^2\eta_2\|_{L_r(\Omega)} + \|\nabla\eta_2\|_{L_2(\Omega)}). \tag{3.21}$$

From (3.20) and (3.21) we obtain

$$\|\nabla\eta_2\|_{L_r(\Omega\times(0,\infty))} \le c(\|\nabla^2\eta_2\|_{L_r(\mathbb{R}^4)} + \|\nabla g\|_{L_2(\mathbb{R}^4)} + \|f\|_{L_2(\mathbb{R}^4)}). \tag{3.22}$$

By (3.4), (3.11), (3.15) and (3.22) we get

$$\begin{aligned}\|\nabla\eta\|_{L_r(\Omega\times(0,\infty))} &\le c\|f\|_{L_r(\Omega\times(0,\infty))} + c\|\nabla g\|_{L_r(\Omega\times(0,\infty))} \\ &\quad + c\|\nabla g\|_{L_2(\Omega\times(0,\infty))} + c\|f\|_{L_2(\Omega\times(0,\infty))} \\ &\quad + c\Big(\int_0^\infty\int|\nabla\eta\cdot\nabla g'|\,dt\,dx\Big)^{1/2}.\end{aligned}$$

In particular when $r = 2$, we can estimate the last term of the r.h.s. from the Young inequality:

$$\begin{aligned}&c\Big(\int_0^\infty\int\nabla\eta\cdot\nabla g'\,dt\,dx\Big)^{1/2} \\ &\qquad \le \frac{1}{2}\|\nabla\eta\|_{L_2(\Omega\times(0,\infty))} + c\|\nabla g\|_{L_2(\Omega\times(0,\infty))} + c\|f\|_{L_2(\Omega\times(0,\infty))},\end{aligned}$$

and this gives

$$\begin{aligned}&\|\nabla\eta\|_{L_2(\Omega\times(0,\infty))} + \|\nabla\eta\|_{L_r(\Omega\times(0,\infty))} \\ &\qquad \le c(\|f\|_{L_r(\Omega\times(0,\infty))} + \|\nabla g\|_{L_r(\Omega\times(0,\infty))} \\ &\qquad\quad + \|f\|_{L_2(\Omega\times(0,\infty))} + \|\nabla g\|_{L_2(\Omega\times(0,\infty))}).\end{aligned} \tag{3.23}$$

We have already got an estimate of $\nabla\eta$ in $L_r$, so we can treat $(3.1)_1$ as a parabolic system with a solution with compact support in space. Hence by (3.23) we easily get

$$\begin{aligned}\|u\|_{W_r^{2,1}(\Omega\times(0,\infty))} &\le c(\|f\|_{L_r(\Omega\times(0,\infty))} + \|\nabla g\|_{L_r(\Omega\times(0,\infty))} \\ &\quad + \|f\|_{L_2(\Omega\times(0,\infty))} + \|\nabla g\|_{L_2(\Omega\times(0,\infty))}).\end{aligned} \tag{3.24}$$

From $(3.1)_2$ we have

$$\begin{aligned}\|\nabla\eta_t\|_{L_r(\Omega\times(0,\infty))} &\le c(\|f\|_{L_r(\Omega\times(0,\infty))} + \|\nabla g\|_{L_r(\Omega\times(0,\infty))} \\ &\quad + \|f\|_{L_2(\Omega\times(0,\infty))} + \|\nabla g\|_{L_2(\Omega\times(0,\infty))}).\end{aligned} \tag{3.25}$$

If $r \le 10/3$, from (3.23)–(3.25) we obtain

$$\begin{aligned}&\|u\|_{W_r^{2,1}(\Omega\times(0,\infty))} + \|\eta\|_{L_r(\Omega\times(0,\infty))} + \|\eta_t\|_{L_r(\Omega\times(0,\infty))} \\ &\le c(\|f\|_{L_r(\Omega\times(0,\infty))} + \|f\|_{L_2(\Omega\times(0,\infty))} + \|\nabla g\|_{L_r(\Omega\times(0,\infty))} + \|\nabla g\|_{L_2(\Omega\times(0,\infty))}),\end{aligned} \tag{3.26}$$

but if $r > 10/3$ we have to return to (3.9) with $\nabla\eta_t \in L_{10/3}$. Hence in the same way as in (3.10) we get $d_{2,t} \in L_r$ if $r \le 10$, because $W_{10/3}^{1,1/2} \subset L_{10}$. And if $r > 10$ then we repeat the above procedure to obtain (3.26) for $r < \infty$ ($W^{1,1/2}_{10} \subset L_\infty$).

Inequality (1.4) comes easily from (3.26) and the uniqueness in time of solutions of system (1.1).

**4. Appendix.** We consider the following problem in a bounded domain $\Omega$ with smooth boundary $S$:

$$\begin{aligned} &d_t - \alpha \Delta d = \operatorname{div} f, \\ &d|_S = 0, \\ &d|_{t=0} = 0. \end{aligned} \tag{4.1}$$

LEMMA 4A. *For solutions of problem* (4.1) *we have the following estimate*:

$$\|d\|_{W^{1,1/2}_r(\Omega\times(0,\infty))} + \|d\|_{W^{1,1/2}_2(\Omega\times(0,\infty))} \le c(\|f\|_{L_r(\Omega\times(0,\infty))} + \|f\|_{L_2(\Omega\times(0,\infty))}), \tag{4.2}$$

*where* $r \ge 2$.

COROLLARY 4A. *We also have*

$$\|d\|_{W^{1,1/2}_r(\Omega\times[0,T])} + \|d\|_{W^{1,1/2}_2(\Omega\times[0,T])} \le A(\|f\|_{L_r(\Omega\times[0,T])} + \|f\|_{L_2(\Omega\times[0,T])}), \tag{4.3}$$

*where* $r \ge 2$ *and* $A$ *is a constant independent of* $T$.

*Proof of Lemma 4A.* To obtain a suitable estimate we introduce a smooth function $\zeta$ such that

$$\zeta(x) = \begin{cases} 1 & \text{for } B(y_0, \lambda), \\ 0 & \text{for } B(y_0, 2\lambda), \end{cases}$$

and $0 \le \zeta \le 1$, $|\nabla \zeta| \le c/\lambda$, $\lambda$ is a parameter which will be defined later.

Using the function $\zeta$ we define a new variable

$$D = \zeta d.$$

From (4.1) we obtain an equation for $D$:

$$\begin{aligned} &D_t - \alpha \Delta D = \operatorname{div} \zeta f - \nabla\zeta \cdot f + 2\nabla\zeta \cdot \nabla d + \Delta\zeta d, \\ &D|_S = 0. \end{aligned} \tag{4.4}$$

If $B(y_0, 2\lambda) \cap S = \emptyset$ equation (4.4) can be treated as a problem in $\mathbb{R}^4$; to solve it we can use the Fourier transform to get

$$\begin{aligned} D &= \mathcal{F}^{-1}\left[\frac{1}{i\xi_0 + \alpha|\xi|^2}\mathcal{F}[\operatorname{div}\zeta f + 2\operatorname{div}(\nabla\zeta d) - \nabla\zeta f - \Delta\zeta d]\right] \\ &= D_1 + D_2, \end{aligned} \tag{4.5}$$

where

$$D_1 = \mathcal{F}^{-1}\left[\frac{i\xi}{i\xi_0 + \alpha|\xi|^2}\mathcal{F}\left[\zeta f + 2d\nabla\zeta\right]\right],$$
$$D_2 = \mathcal{F}^{-1}\left[\frac{1}{i\xi_0 + \alpha|\xi|^2}\mathcal{F}\left[-\nabla\zeta\cdot f - \Delta\zeta d\right]\right].$$

Since

$$\left||\xi_0|^\alpha \partial_{\xi_0}^\alpha |\xi|^\beta \partial_\xi^\beta \frac{\xi|\xi_0|^{1/2}}{i\xi_0 + \alpha|\xi|^2}\right| < c,$$
$$\left||\xi_0|^\alpha \partial_{\xi_0}^\alpha |\xi|^\beta \partial_\xi^\beta \frac{|\xi_0| + |\xi|^2}{i\xi_0 + \alpha|\xi|^2}\right| < c,$$

by the Marcinkiewicz theorem (Theorem 2.1) we have

$$\begin{aligned} &\|\mathcal{F}^{-1}[|\xi_0|^{1/2}\mathcal{F}D_1]\|_{L_r(\mathbb{R}^4)} + \|\nabla D_1\|_{L_r(\mathbb{R}^4)} \\ &\qquad\qquad \le c\|\zeta f\|_{L_r(\mathbb{R}^4)} + c\|d\nabla\zeta\|_{L_r(\mathbb{R}^4)}, \\ &\|D_{2,t}\|_{L_r(\mathbb{R}^4)} + \|\nabla^2 D_2\|_{L_r(\mathbb{R}^4)} \le c\|\nabla\zeta\cdot f\|_{L_r(\mathbb{R}^4)} + c\|\Delta\zeta d\|_{L_r(\mathbb{R}^4)}. \end{aligned} \tag{4.6}$$

Hence $D_1 \in H_r^{1,1/2}$ and $D_2 \in H_r^{2,1}$, but locally, we only have estimates for the highest derivatives. We need a Poincaré inequality. We assume that $B(y_0, 2\lambda) \subset \mathbb{R}^3_+$. Then we extend the problem to the whole space using the transformation

$$\widetilde{h}(x) = \begin{cases} h(x', x_3), & x_3 \ge 0, \\ -h(x', -x_3), & x_3 < 0. \end{cases} \tag{4.7}$$

Note that this transformation preserves the equation. It is easily seen that $\Delta\widetilde{h}$ is a regular distribution, hence on $x_3 = 0$ there are no singularities ($h(x', 0) = 0$, $\partial_{x_3}\widetilde{h}(x', 0)$ is continuous, $\partial_{x_3}^2\widetilde{h}(x', x_3)$ in $L_r$ is well defined as a function).

The transformation (4.7) changes (4.4) into the following problem in the whole space:

$$\widetilde{D}_t - \alpha\Delta\widetilde{D} = \operatorname{div}\bar{f}_{11} + \widetilde{f}_{12}, \tag{4.8}$$

where $f_{11}$ and $f_{12}$ comes from the r.h.s. of (4.4) and $\bar{f}_{11}^1 = \widetilde{f}_{11}^1$, $\bar{f}_{11}^2 = \widetilde{f}_{11}^2$,

$$\bar{f}_{11}^3 = \begin{cases} f_{11}^3(x', x_3), & x_3 \ge 0, \\ f_{11}^3(x', -x_3), & x_3 < 0. \end{cases}$$

Since (4.8) has the same structure as (4.4), for $\widetilde{D}_1$ and $\widetilde{D}_2$ we have estimates (4.6). As $\widetilde{D}_1(x', 0) = 0$, we get a Poincaré inequality which gives estimates for $\|\widetilde{D}_1\|_{L_r(\operatorname{supp}\widetilde{D}\times\mathbb{R})}$ (in particular for $r = 2$). Since $\widetilde{D}$ has compact support (in space), from (4.8) we have the following energy estimate:

$$\|\widetilde{D}\|_{L_2(\mathbb{R}^4)} \le c(\|\bar{f}_{11}\|_{L_2(\mathbb{R}^4)} + \|\widetilde{f}_{12}\|_{L_2(\mathbb{R}^4)}). \tag{4.9}$$

Together with (4.6) and Proposition 2.3 we obtain $D \in H_r^{1,1/2}$ and by Proposition 2.2 if $r \geq 2$ we have $D \in W_r^{1,1/2}$. Thus

$$\|D\|_{W_r^{1,1/2}(\mathbb{R}^4)} \leq c(\|f\|_{L_r(\operatorname{supp} D\times(0,\infty))} + \|f\|_{L_2(\operatorname{supp} D\times(0,\infty))} + \|d\|_{L_r(\operatorname{supp} D\times(0,\infty))} + \|d\|_{L_2(\operatorname{supp} D\times(0,\infty))}). \tag{4.10}$$

If $B(y_0, 2\lambda) \cap S \neq \emptyset$ then we have to transform the problem to the half-space. Since $S$ is smooth, the transformation $F$ is also smooth. And we have

$$F : B(y_0, 2\lambda) \cap \Omega \to \mathbb{R}^2_{z'} \times [0,\infty)_{z_3}, \qquad \partial_x = \partial_z - \nabla F \partial_z.$$

Then (4.4) reads

$$\begin{aligned} &D_t - \alpha \Delta_z D = \operatorname{div}_z f_{21} + f_{22} + \alpha(\Delta_x - \Delta_z)D, \\ &D|_{z_3=0} = 0, \end{aligned} \tag{4.11}$$

where $f_{21}$ and $f_{22}$ comes from the r.h.s. of (4.4).

We have

$$(\Delta_x - \Delta_z)D = \nabla(\nabla F \nabla D) - \nabla(\nabla^2 F D) + \nabla(\nabla F \nabla F \nabla D) + \nabla^3 F D - \nabla(\nabla F \nabla F)\nabla D + \nabla F \nabla^2 F \nabla D. \tag{4.12}$$

We extend equation (4.11) in the same way as in (4.7) to get a problem in the whole space. This is possible since $(4.11)_2$ holds. From the considerations from the first part of the proof we get, by (4.12),

$$\|D\|_{W_r^{1,1/2}(\mathbb{R}^4_+)} \leq c\|f\|_{L_r(\operatorname{supp} D\times(0,\infty))} + c\|D\|_{L_r(\operatorname{supp} D\times(0,\infty))} + c\|d\|_{L_r(\operatorname{supp} D\times(0,\infty))} + c|\nabla F| \cdot \|\nabla D\|_{L_r(\operatorname{supp} D\times(0,\infty))}, \tag{4.13}$$

where $\mathbb{R}^4_+ = \mathbb{R}^2_{z'} \times [0,\infty)_{z_3} \times \mathbb{R}$. But the function $F$ satisfies the following relations (by smoothness of the boundary $S$):

$$F(0) = 0, \qquad \nabla F(0) = 0, \qquad F \in C^3, \qquad |\nabla F| \leq c\lambda. \tag{4.14}$$

Thus taking $\lambda$ small enough, using the interpolation theorem, by (4.12) and (4.14) we obtain

$$\|D\|_{W_r^{1,1/2}(\mathbb{R}^4_+)} \leq c\|f\|_{L_r(\operatorname{supp} D\times(0,\infty))} + c\|d\|_{L_2(\operatorname{supp} D\times(0,\infty))} + c\|d\|_{L_r(\operatorname{supp} D\times(0,\infty))}. \tag{4.15}$$

Taking a cover of $\Omega$ consisting of such balls, from (4.10) and (4.15), remembering that the functions $D$ have compact supports in space, we obtain

$$\|d\|_{W_r^{1,1/2}(\Omega\times(0,\infty))} \leq c\|f\|_{L_r(\Omega\times(0,\infty))} + c\|d\|_{L_2(\Omega\times(0,\infty))}. \tag{4.16}$$

To estimate the last term of the r.h.s. of (4.16) we write the energy estimate for (4.1):

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega} d^2\,dx + \alpha\int\limits_{\Omega} |\nabla d|^2\,dx = -\int\limits_{\Omega} f\cdot\nabla d\,dx.$$

Since $d|_{t=0} = 0$, $d|_S = 0$ and $\Omega$ is bounded, using the Poincaré inequality, we easily get

$$\|d\|_{L_2(\Omega\times(0,\infty))} \le c\|f\|_{L_2(\Omega\times(0,\infty))}. \tag{4.17}$$

This gives by (4.16) the estimate

$$\|d\|_{W_r^{1,1/2}(\Omega\times(0,\infty))} \le c(\|f\|_{L_r(\Omega\times(0,\infty))} + \|f\|_{L_2(\Omega\times(0,\infty))}), \tag{4.18}$$

where $r \ge 2$. From (4.18) we immediately obtain (4.2).

The proof of Corollary 4A follows easily from the uniqueness in time for system (4.1).

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Institute of Applied Mathematics and Mechanics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: mucha@hydra.mimuw.edu.pl

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