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ON MODULAR PROJECTIVE REPRESENTATIONS OF FINITE NILPOTENT GROUPS

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Abstract. Our aim is to determine necessary and sufficient conditions for a finite nilpotent group to have a faithful irreducible projective representation over a field of characteristic $p \geq 0$.

1. Introduction. Frucht [4] proved that a finite abelian group G admits a faithful irreducible projective representation over an algebraically closed field K of characteristic not dividing the order of the group G if and only if G is of *symmetric type*, i.e. it decomposes into a direct product of two isomorphic groups. Yamazaki [11] showed that sufficiency of Frucht's theorem holds for an arbitrary field L containing the primitive $(\exp G)$th root of 1. Moreover, he established that the group G is of symmetric type if and only if for some factor system $\lambda \in Z^2(G, L^*)$ the twisted group algebra $L^\lambda G$ is a central simple algebra over the field L. Frucht's theorem is supplemented by Zhmud's result [14]: the minimal number of irreducible components of a faithful projective K-representation of the group G equals 1 if G is of symmetric type and equals 2 otherwise. A generalization of Frucht's and Zhmud's results to an arbitrary field with a restriction on the characteristic was given in [1]–[2]. A study of metabelian groups admitting a faithful irreducible projective representation over the field of complex numbers was performed by Ng [8]–[9]. Some general results on faithful projective representations of finite groups over a field with a restriction on the characteristic are obtained in [8]–[9] and [11]–[12]. Let us note that the above results are partially presented in Karpilovsky's monograph [6].

In this paper we look for necessary and sufficient conditions for finite nilpotent groups to have faithful irreducible projective representations over a field of any characteristic. In Section 2 we prove a number of propositions about semisimple twisted group algebras of finite groups. Since for an algebraically closed field there exists a close connection between the existence of

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simple twisted group algebras and the fact that any given group has a faithful irreducible projective representation, we show later how this connection is preserved for an arbitrary field. This fact leads to the description in Section 3 of simple twisted group algebras of finite abelian groups. In Section 4 we generalize the results of [9], [11] and [12] by giving conditions for a nilpotent group to have a class of faithful irreducible projective λ-representations (Propositions 8–10). However, the main Theorems 3 and 4 concern abelian groups and generalize the results of [2]. We assume in some propositions that cocycles are not taken from the whole second group of cocycles, but only from its subgroup. This approach to cocycles is more general than that in [2]. It is worth noting that the case of abelian groups can be investigated completely because, by [13], each finite abelian group is an extension of a group of symmetric type by a cyclic group.

2. Semisimple twisted group algebras. We use the following notations: G is a finite group; $o(g)$ is the order of $g \in G$; e is the unity of G; $Z(G)$ is the center of G; F is a field of characteristic $p \geq 0$; F^* is the multiplicative group of F; $F^m = \{\alpha^m : \alpha \in F\}$; $\lambda \in Z^2(G, F^*)$ is an F-factor system of the group G; $F^\lambda G$ is the twisted group algebra of the group G and the field F for the factor system λ; $J(F^\lambda G)$ is the Jacobson radical of the algebra $F^\lambda G$; $\operatorname{soc} G$ is the socle of the abelian group G; $\{u_g : g \in G\}$ is a natural F-basis of the algebra $F^\lambda G$, i.e. a basis satisfying $u_a u_e = u_e u_a = u_a,\ u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in G$. We often denote the restriction of $\lambda \in Z^2(G, F^*)$ to a subgroup H of G by λ as well. We identify u_e with the unity of the field F. Therefore, we write γ instead of γu_e ($\gamma \in F$).

Let $G = \langle a_1 \rangle \times \ldots \times \langle a_s \rangle$. The elements $u_{a_1}, \ldots, u_{a_s}$ of the natural F-basis of the algebra $F^\lambda G$ are generators of this algebra. Therefore, if

$$u_{a_i}^{o(a_i)} = \alpha_i \quad (\alpha_i \in F^*;\ i = 1, \ldots, s),$$

then we denote the algebra $F^\lambda G$ also by $[G, F, \alpha_1, \ldots, \alpha_s]$.

If the order of the group G is divisible by p, then we always assume that F is a field of positive characteristic p.

Let $G = \langle a_1 \rangle \times \ldots \times \langle a_s \rangle$ be an abelian p-group of type $(p^{n_1}, \ldots, p^{n_s})$, $F^\lambda G = [G, F, \alpha_1, \ldots, \alpha_s]$ and ϱ_i be a root of the polynomial $x^{p^{n_i}} - \alpha_i$ in the algebraic closure of the field F. We denote by $x^{p^{m_1}} - \beta_1$ an irreducible factor of the polynomial $x^{p^{n_1}} - \alpha_1$ over F and by

$$x^{p^{m_j}} - \beta_j(\varrho_1, \ldots, \varrho_{j-1})$$

an irreducible factor of $x^{p^{n_j}} - \alpha_j$ over the field $F(\varrho_1, \ldots, \varrho_{j-1})$ for $j \geq 2$. Moreover, $\beta_j(\varrho_1, \ldots, \varrho_{j-1})$ is the value of the polynomial $\beta_j(x_1, \ldots, x_{j-1}) \in F[x_1, \ldots, x_{j-1}]$ for $x_1 = \varrho_1, \ldots, x_{j-1} = \varrho_{j-1}$.

THEOREM 1. *Let $G = \langle a_1 \rangle \times \ldots \times \langle a_s \rangle$ be an abelian p-group of type $(p^{n_1}, \ldots, p^{n_s})$, and*

$$\upsilon_j = \begin{cases} u_{a_j}^{p^{m_j}} - \beta_j(u_{a_1}, \ldots, u_{a_{j-1}}) & \text{for } m_j < n_j, \\ 0 & \text{for } m_j = n_j. \end{cases}$$

Then $J(F^\lambda G) = F^\lambda G\upsilon_1 + \ldots + F^\lambda G\upsilon_s$, $F^\lambda G/J(F^\lambda G) \cong F(\varrho_1, \ldots, \varrho_s)$ and $[F(\varrho_1, \ldots, \varrho_s) : F] = p^{m_1+\ldots+m_s}$.

Proof. By the hypothesis,

$$\varrho_j^{p^{n_j}} = \alpha_j, \quad [\beta_j(\varrho_1, \ldots, \varrho_{j-1})]^{p^{n_j-m_j}} = \alpha_j.$$

Therefore,

$$\alpha_j^{p^{n_1+n_2+\ldots+n_{j-1}}} = \{\widetilde{\beta}_j(\alpha_1^{p^{n_2+n_3+\ldots+n_{j-1}}}, \alpha_2^{p^{n_1+n_3+\ldots+n_{j-1}}}, \ldots, \alpha_{j-1}^{p^{n_1+n_2+\ldots+n_{j-2}}})\}^{p^{n_j-m_j}},$$

where $\widetilde{\beta}_j(x_1, \ldots, x_{j-1})$ is a polynomial of $x_1, \ldots, x_{j-1}$ over the field F^d, where $d = p^{n_1+\ldots+n_{j-1}}$. Consequently, $\upsilon_j^{p^t} = 0$, where $t = n_1 + \ldots + n_{j-1} + n_j - m_j$. Hence, the ideal

$$V = F^\lambda G\upsilon_1 + \ldots + F^\lambda G\upsilon_s$$

is nilpotent.

Let $w_i = u_{a_i} + V$ $(i = 1, \ldots, s)$. We identify $\alpha + V$ with α for each $\alpha \in F$. Since $\beta_1 \notin F^p$, it follows that $F[w_1]$ is a field. We can consider the F-algebra $F^\lambda G/V$ as an F_1-algebra, where $F_1 = F[w_1]$. Since

$$w_2^{p^{m_2}} = \beta_2(u_{a_1}) + V = \beta_2(w_1)$$

and $\beta_2(w_1) \notin F_1^p$, $F_2 = F_1[w_2]$ is a field and $F^\lambda G/V$ is an F_2-algebra. Continuing, we deduce that

$$F^\lambda G/V = F[w_1, \ldots, w_s]$$

is a field and its degree over F equals $p^{m_1+\ldots+m_s}$. Hence, V is the radical of the algebra $F^\lambda G$. ■

PROPOSITION 1. *Let $G = \langle a_1 \rangle \times \ldots \times \langle a_s \rangle$ be an abelian p-group, $F^\lambda G = [G, F, \alpha_1, \ldots, \alpha_s]$ and θ_i be a root of the polynomial $x^p - \alpha_i$. Then the following conditions are equivalent*:

(1) *the algebra $F^\lambda G$ is semisimple*;
(2) *$F^\lambda G$ is a field*;
(3) $[F(\theta_1, \ldots, \theta_s) : F] = p^s$.

Proposition 1 follows from Theorem 1 and from the criterion of irreducibility of a polynomial $x^{p^n} - \alpha$.

PROPOSITION 2 (see [10]). *Let G be a finite p-group. The algebra $F^\lambda G$ is semisimple if and only if G is abelian and $F^\lambda G$ is a field.*

Proof. Let $|G| = p^n$ and suppose $F^\lambda G$ is a semisimple algebra. By induction on n, we show that $F^\lambda G$ is a field. If $n = 1$, then G is a cyclic group and, by Proposition 1, $F^\lambda G$ is a field. Let H be the center of G. Then $F^\lambda H$ is a semisimple algebra of an abelian group and therefore, by Proposition 1, $F^\lambda H$ is a field. One can consider the algebra $F^\lambda G$ as a twisted group algebra of the group G/H and the field $F^\lambda H$. Since $|G/H| < |G|$, by the inductive assumption, $F^\lambda G$ is a field. ∎

THEOREM 2. *Let $G = G_p \times H$, where G_p is a Sylow p-subgroup. The algebra $F^\lambda G$ is semisimple if and only if $F^\lambda G_p$ is a field. If $F^\lambda G$ is semisimple, then each system of minimal pairwise orthogonal idempotents of the algebra $F^\lambda H$ (resp. of the center of $F^\lambda H$) is also a system of minimal pairwise orthogonal idempotents of the algebra $F^\lambda G$ (resp. of the center of $F^\lambda G$).*

Proof. Suppose $K = F^\lambda G_p$ is a field. The algebra $F^\lambda H$ is separable, therefore the centers of its simple components are separable extensions of the field F [3, §71]. Let A be a simple component of $F^\lambda H$ and $Z(A)$ be its center. Since K is a purely inseparable extension of the field F, we conclude that $K \otimes_F Z(A)$ is a field [7], so that the algebra $K \otimes_F A$ is simple. Its index coincides with the index of the algebra A, since by [3, §68], [8], the index of A divides $|H|$ and therefore it is relatively prime to $[K : F]$. ∎

COROLLARY. *Let $G = G_p \times H$, where G_p is a Sylow p-subgroup. The algebra $F^\lambda G$ is simple if and only if $F^\lambda G_p$ is a field and $F^\lambda H$ is a simple algebra.*

3. Simple twisted group algebras

PROPOSITION 3. *Let G be an abelian q-group, $q \neq 2$, $q \neq p$; $F^\lambda G = [G, F, \beta_1, \ldots, \beta_m]$ be a commutative algebra; θ_i be a root of the polynomial $x^q - \beta_i$ $(i = 1, \ldots, m)$. Then the following conditions are equivalent:*

(1) *$F^\lambda G$ is a field;*

(2) $[F(\theta_1, \ldots, \theta_m) : F] = q^m$;

(3) *none of the elements $\beta_1^{t_1} \ldots \beta_m^{t_m}$ is the qth power of an element of the field F, where $0 \le t_1, \ldots, t_m < q$ and $t_1 + \ldots + t_m \neq 0$.*

Proof. Denote by ε a primitive qth root of 1. Without loss of generality, one can assume $\varepsilon \in F$. Suppose the condition (2) does not hold. Consider a sequence of fields

$$(*) \qquad F_0 = F \subset F_1 \subset \ldots \subset F_m,$$

where $F_i = F(\theta_1, \ldots, \theta_i)$ $(i = 1, \ldots, m)$. If $F(\theta_1) = F$, then $\beta_1 = \mu^q$, $\mu \in F$. Suppose $F_{r_1} \neq F_{r_1 - 1}$ and $F_{r_1} = F_{r_1-1}(\theta_d)$, where $1 \le r_1 < d \le m$.

Since for some $k_1,\ 1 \le k_1 < q$, the element $\theta_d/\theta_{r_1}^{k_1}$ cancels the action of automorphisms of the Galois group $G(F_{r_1}/F_{r_1-1})$, it follows that $\theta_d = \theta_{r_1}^{k_1}\varrho_{r_1}$, where $\varrho_{r_1} \in F_{r_1-1}$. If $\varrho_{r_1} \notin F_0$, then by a similar reasoning we obtain $\varrho_{r_1} = \theta_{r_2}^{k_2}\varrho_{r_2}$, where $\varrho_{r_2} \in F_{r_2-1}$ $(r_2 < r_1)$ and $1 \le k_2 < q$. Moving along the sequence $(*)$ from right to left, we obtain the equality

$$\theta_d = \theta_{r_1}^{k_1} \dots \theta_{r_v}^{k_v}\varrho_{r_v},$$

where $\varrho_{r_v} \in F$. Raising to the power q, we find

$$(**) \qquad \beta_d = \beta_{r_1}^{k_1} \dots \beta_{r_v}^{k_v}\varrho_{r_v}^{q}.$$

But this means that the condition (3) does not hold.

Conversely, if (3) does not hold, then $(**)$ holds for some $1 \le r_1 < \dots < r_v < d \le m$. Therefore $F_d = F_{d-1}$, hence $[F(\theta_1, \dots, \theta_m) : F] < q^m$, i.e. (2) does not hold. ∎

Remark 8 If $q = 2$, then, generally speaking, Proposition 3 is invalid. Indeed, let $F = \mathbb{Q}(\sqrt{2})$, where $\mathbb{Q}$ is the field of the rational numbers. If $G = \langle a \rangle$ is a group of order 4, then the algebra $F^\lambda G = [G, F, -1]$ is not a field. However, $-1 \ne \mu^2$ for each $\mu \in F$.

If F contains a primitive 4th root of 1, then Proposition 3 also holds for abelian 2-groups.

Let G be a finite group, $Z(G)$ be the center of G and $\lambda \in Z^2(G, F^*)$. The set $\{g \in Z(G) : \forall a \in G,\ \lambda_{a,g} = \lambda_{g,a}\}$ forms a subgroup of G. We call it the λ-*center* of G. If G is an abelian group and H is its λ-center, then the center of the algebra $F^\lambda G$ coincides with $F^\lambda H$. In this case, the algebra $F^\lambda G$ is simple if and only if its center $F^\lambda H$ is a field. Proposition 3 gives necessary and sufficient conditions for $F^\lambda H$ to be a field in the case where H is a q-group and $q \ne 2$, $q \ne p$.

Let G be an abelian group of exponent $o(G)$, $a \in G$ be an element of order $o(G)$ and m be the exponent of the group $G/\langle a \rangle$. If $\lambda \in Z^2(G, F^*)$, then

$$(\lambda_{a,b} \cdot \lambda_{b,a}^{-1})^m = 1$$

for all $a, b \in G$. This condition can also hold for some divisors of m. If d is such a divisor, then we write $\lambda \in Z^2(G, F^*, d)$.

The number $t_q = \sup\{0, m\}$ is important in describing simple twisted group algebras of abelian q-groups, where m is a natural number such that for some $\gamma_1, \dots, \gamma_m \in F^*$ the algebra

$$F[x]/(x^q - \gamma_1) \otimes_F \dots \otimes_F F[x]/(x^q - \gamma_m)$$

is a field. The dimension of $F^*/(F^*)^q$ as a vector space over a field of q elements is said to be the *rank* of the group $F^*/(F^*)^q$. By Proposition 3, t_q for $q \ne p$ equals the rank of the group $F^*/(F^*)^q$.

Proposition 4. *Let F^* contain a primitive q^nth root of 1, where $n \geq 2$ for $q = 2$; G_q be an abelian q-group; s_q be the number of invariants of the group G_q, exceeding q^n. The group G_q has a simple algebra $F^\lambda G_q$ for some $\lambda \in Z^2(G, F^*, q^n)$ if and only if the following conditions hold*:

(1) *if $t_q > 0$, then $s_q \leq t_q$*;
(2) *if $t_q = 0$, then $s_q = 0$ and G_q is a group of symmetric type.*

Proof. The center of the algebra $F^\lambda G_q$ coincides with $F^\lambda H_q$, where H_q is the λ-center of the group G_q. If $a \in G_q$ and $o(a) > q^n$, then $a^{q^n} \neq e$ and $a^{q^n} \in H_q$. It follows that for each cocycle $\lambda \in Z^2(G_q, F^*, q^n)$ the group H_q decomposes into a direct product of no less than s_q cyclic subgroups. The algebra $F^\lambda G_q$ is simple if and only if $F^\lambda H_q$ is a field. If $t_q = 0$, then $F^\lambda H_q$ is a field if and only if $H_q = \{e\}$. In the case $H_q = \{e\}$, we have $s_q = 0$ and, by [11], G_q is a group of symmetric type. Conversely, if the last condition holds, then $F^\lambda G_q$ is a central simple algebra over F.

Let $t_q > 0$. If $F^\lambda G_q$ is a simple algebra, then $s_q \leq t_q$ by Proposition 3. Conversely, let $s_q \leq t_q$. If $s_q = 0$, then there exists [13] a cyclic subgroup H_q of G_q such that G_q/H_q is a group of symmetric type and $\exp(G_q/H_q) \leq q^n$. If $s_q > 0$, then by Lemma 5 of [2], the group G_q has a subgroup $H_q = \langle a_1 \rangle \times \ldots \times \langle a_{s_q} \rangle$ such that G_q/H_q is a group of symmetric type and the exponent of G_q/H_q does not exceed q^n. By Proposition 3, there exists a cocycle $\mu \in Z^2(H_q, F^*)$ such that $F^\mu H_q$ is a field. By Lemma 6 of [2], there exists a cocycle $\lambda \in Z^2(G, F^*, q^n)$ such that the center of $F^\lambda G_q$ coincides with $F^\mu H_q$. ■

Proposition 5. *Let G be a finite abelian group and G_q be a Sylow q-subgroup of G. The algebra $F^\lambda G$ is simple if and only if $F^\lambda G_q$ is simple for each prime $q \mid |G|$. If $F^\lambda G$ is simple and m_q is the index of $F^\lambda G_q$, then the index of $F^\lambda G$ equals $\prod_{q||G|} m_q$.*

Proof. By Theorem 2, one can assume that p does not divide $|G|$. Let $G = G_{q_1} \times \ldots \times G_{q_s}$ be a decomposition into a direct product of Sylow subgroups. Denote by λ_i the restriction of $\lambda \in Z^2(G, F^*)$ to G_{q_i}. If H_i is the λ_i-center of G_{q_i}, then $F^{\lambda_i} H_i$ is the center of $F^{\lambda_i} G_{q_i}$. Moreover, the center K of $F^\lambda G$ is isomorphic to

$$F^{\lambda_1} H_1 \otimes_F \ldots \otimes_F F^{\lambda_s} H_s.$$

It follows that K is a field if and only if $F^{\lambda_i} H_i$ is a field for each $i = 1, \ldots, s$. For this reason, $F^\lambda G$ is simple if and only if each $F^\lambda G_{q_i}$ is.

Suppose $F^\lambda G$ is simple. One can consider $F^\lambda G$ as a twisted group algebra $K^\mu(G/H)$ of the field K and the group G/H, where $H = H_1 \times \ldots \times H_s$. Hence,

$$F^\lambda G \cong K^{\mu_1}(G_{q_1}/H_1) \otimes_K \ldots \otimes_K K^{\mu_s}(G_{q_s}/H_s).$$

Let m_i be the index of $F^{\lambda_i}G_{q_i}$. It is known ([3, §68], [8]) that m_i is a divisor of $|G_{q_i}|$. Since $[K : F^{\lambda_i}H_i]$ is not divisible by q_i, it follows that the index of $K^{\mu_i}(G_{q_i}/H_i)$ equals the index of $F^{\lambda_i}G_{q_i}$. Since the numbers $m_1, \ldots, m_s$ are pairwise relatively prime, we conclude [5] that the index of $F^\lambda G$ equals $m_1 \ldots m_s$. ■

4. Faithful projective representations of nilpotent groups. Let $G = G_{q_1} \times \ldots \times G_{q_r}$ be a decomposition of a nilpotent group G into a direct product of Sylow subgroups. Then

$$F^\lambda G \cong F^\lambda G_{q_1} \otimes_F \ldots \otimes_F F^\lambda G_{q_r}$$

and

$$F^\lambda G/J(F^\lambda G) \cong F^\lambda G_p/J(F^\lambda G_p) \otimes_F \prod_{q \neq p} F^\lambda G_q.$$

Since each simple $F^\lambda G$-module is isomorphic to a component of the semisimple module $F^\lambda G/J(F^\lambda G)$, it follows that each simple $F^\lambda G$-module is isomorphic to a component of the module

$$M_1 \# \ldots \# M_r,$$

where M_j is a simple $F^\lambda G_{q_j}$-module $(j = 1, \ldots, r)$.

PROPOSITION 6. *Let $G = G_p \times H$, where G_p is a Sylow p-subgroup. Each irreducible λ-representation of G over a field F is equivalent to a representation $\Gamma \# \Delta$, where Γ is an irreducible λ_1-representation of G_p, Δ is an irreducible λ_2-representation of H and the cocycle λ is cohomologous to $\lambda_1 \times \lambda_2$. Conversely, each representation of the form $\Gamma \# \Delta$ is an irreducible $\lambda_1 \times \lambda_2$-representation of G. Representations $\Gamma \# \Delta$ and $\Gamma' \# \Delta'$ of this type are linearly equivalent if and only if Γ is linearly equivalent to Γ' and Δ is linearly equivalent to Δ'.*

The proof of Proposition 6 is similar to that of Theorem 2.

PROPOSITION 7. *Let $G = G_{q_1} \times \ldots \times G_{q_s}$ be a decomposition of an abelian group G into a direct product of Sylow subgroups. Suppose F contains a primitive qth root of 1 for each prime $q \mid |G|$ different from p. Each irreducible (faithful irreducible) λ-representation of the group G over F is then equivalent to a representation*

$$\Gamma_1 \# \ldots \# \Gamma_s, \tag{$***$}$$

where Γ_i is an irreducible (faithful irreducible) λ_i-representation of G_{q_i} over F $(i = 1, \ldots, s)$. Moreover, the cocycle λ is cohomologous to $\lambda_1 \times \ldots \times \lambda_s$ and vice versa, each representation of the form $(***)$ is an irreducible (faithful irreducible) $\lambda_1 \times \ldots \times \lambda_s$-representation of G. Representations $\Gamma_1 \# \ldots \# \Gamma_s$

and $\Gamma_1' \# \ldots \# \Gamma_s'$ of the form $(***)$ are linearly equivalent if and only if Γ_i is linearly equivalent to Γ_i' for each $i = 1, \ldots, s$.

Proof. By Proposition 6 one can assume that $|G|$ is not divisible by p. Let λ_i be the restriction of λ to G_{q_i}. If H_i is the λ_i-center of G_{q_i}, then $F^{\lambda_i}H_i$ is the center of $F^{\lambda_i}G_{q_i}$. Since $F^{\lambda_i}H_i$ decomposes into a tensor product of algebras of the form $F[x]/(x^{q_i^m} - \alpha)$ over F, and F contains a primitive q_ith root of 1, the degree of each simple direct summand of the algebra $F^{\lambda_i}H_i$ with respect to F is a divisor of $|H_i|$. Let A_i be a simple component of $F^{\lambda_i}G_{q_i}$, F_i be the center of A_i, $n_i = [F_i : F]$ and m_i be the index of A_i. The numbers n_i and m_i are powers of q_i. Since $n_1, \ldots, n_s$ are pairwise relatively prime, it follows that $K = F_1 \otimes_F \ldots \otimes_F F_s$ is a field. Hence, $A = A_1 \otimes_F \ldots \otimes_F A_s$ is a simple algebra with center K. We will assume that F_i is a subfield of K. Let $B_i = K \otimes_{F_i} A_i$. Since $[K : F_i]$ is not divisible by q_i, the index of B_i equals the index of A_i. As $m_1, \ldots, m_s$ are pairwise relatively prime and $A \cong B_1 \otimes_K \ldots \otimes_K B_s$, we conclude [5, Lemma 4.4.8] that the index of A equals $m_1 \ldots m_s$. This proves that each irreducible λ-representation of G is equivalent to a representation of the form $(***)$ and, moreover, each representation of the form $(***)$ is an irreducible $\lambda_1 \times \ldots \times \lambda_s$-representation of G.

Suppose the irreducible λ-representation $\Gamma = \Gamma_1 \# \ldots \# \Gamma_s$ is not faithful. Then $\Gamma(a) = \alpha E$ for some non-identity $a \in G$. Let $a = bc$, where $b \in G_{q_j}$, $c \in \prod_{i \neq j} G_{q_i}$ and $b \neq e$. There exists a natural number m, relatively prime to q_j, such that $a^m = b^m$. It follows that Γ is not a faithful representation of G_{q_j}. However, $\Gamma|_{G_{q_j}} = \Gamma_j \dotplus \ldots \dotplus \Gamma_j$. Hence, Γ_j is not a faithful representation. ∎

Proposition 7, generally speaking, is not valid in the case when F does not contain a primitive qth root of 1 for some prime $q \mid |G|$ different from p. Indeed, let $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, where $a^3 = e$, $b^3 = e$, $c^{19} = e$ and $F = \mathbb{Z}_7(y)$ be the field of rational functions of the variable y over the residue class field $\mathbb{Z}_7$ modulo 7. The field $\mathbb{Z}_7$ contains a primitive 3rd root ω of 1, but no primitive 19th root of 1. Let

$$F^\lambda G \cong F[x]/(x^3 - \omega) \otimes_F F[x]/(x^3 - y) \otimes_F F[x]/(x^{19} - 1).$$

It is obvious that $G = G_3 \times G_{19}$, where $G_3 = \langle a \rangle \times \langle b \rangle$, $G_{19} = \langle c \rangle$. By Proposition 3, the algebra $F^\lambda G_3$ is a field and the algebra $F^\lambda G_{19}$decomposes into a direct sum of fields and only one of them coincides with F. Hence, there exists a faithful irreducible λ_1-representation Γ_1 of G_3 and a faithful irreducible λ_2-representation Γ_2 of G_{19}. Moreover, $\lambda_1 \times \lambda_2$ is cohomologous to λ. However, the field $\mathbb{Z}_7(\xi)$, $\xi^3 = \omega$, contains a primitive 19th root of 1. Therefore $F^\lambda G \cong K \dotplus \ldots \dotplus K$, where $K = F^\lambda G_3$. It follows that the degree of each irreducible λ-representation of G equals 9. In consequence, $\Gamma_1 \# \Gamma_2$ is a reducible representation.

PROPOSITION 8. *Let G be a finite q-group, $q \neq p$, $\lambda \in Z^2(G, F^*)$ and N be the λ-center of G. The group G has a faithful irreducible λ-representation if and only if N has a faithful irreducible λ-representation.*

Proof. If Γ is an irreducible λ-representation of G, then by Clifford's theorem, $\Gamma|_N = \Delta \dot{+} \ldots \dot{+} \Delta$, where Δ is an irreducible λ-representation of N. It follows that if Γ is faithful, then so is Δ.

Conversely, let Δ be a faithful irreducible λ-representation of N and Γ be an irreducible component of the induced representation Δ^G. If Γ is not faithful, then $\operatorname{Ker}\Gamma \cap Z(G) \neq \{e\}$. Let $b \in \operatorname{Ker}\Gamma \cap Z(G)$ and $b \neq e$. Then $\Gamma(b) = \mu E$ ($\mu \in F^*$) and for each $g \in G$ we have $\Gamma(b)\Gamma(g) = \Gamma(g)\Gamma(b)$ and $\lambda_{b,g}\Gamma(bg) = \lambda_{g,b}\Gamma(gb)$. Since $bg = gb$, we conclude $\lambda_{b,g} = \lambda_{g,b}$. Therefore, b is a non-identity element of the λ-center N. Since $\Delta^G|_N = \Delta \dot{+} \ldots \dot{+} \Delta$, it follows that $\Gamma|_N = \Delta \dot{+} \ldots \dot{+} \Delta$. Ultimately, we get $\Delta(b) = \mu E$. This is a contradiction. Hence, Γ is a faithful representation. ∎

PROPOSITION 9. *Let G be a finite p-group and $\lambda \in Z^2(G, F^*)$. If G has a faithful irreducible λ-representation, then G is abelian. Let H be the socle of an abelian p-group G. Then the following conditions are equivalent*:

(1) *G has a faithful irreducible λ-representation*;
(2) *H has a faithful irreducible λ-representation*;
(3) *if $F^\lambda H = [H, F, \delta_1, \ldots, \delta_m]$, then none of the products*

$$\delta_1^{t_1} \ldots \delta_m^{t_m} \qquad (0 \leq t_i < p,\ t_1 + \ldots + t_m \neq 0)$$

is the pth power of an element of F.

Proof. It is known [6] that an irreducible λ-representation Γ of G is realized in the field $F^\lambda G/J(F^\lambda G)$. Hence, $\Gamma(a)\Gamma(b) = \Gamma(b)\Gamma(a)$ for all $a, b \in G$. If Γ is faithful, then from the equality

$$\Gamma(a^{-1}b^{-1}ab) = \gamma\Gamma(a)^{-1}\Gamma(b)^{-1}\Gamma(a)\Gamma(b) = \gamma E \qquad (\gamma \in F^*)$$

it follows that $a^{-1}b^{-1}ab = e$, i.e. $ab = ba$ for all $a, b \in G$. Therefore, G is abelian.

Let Γ be an irreducible λ-representation of an abelian group G. If Γ is not faithful, then $\Gamma(a) = \gamma E$ ($\gamma \in F$) for some non-identity element $a \in H$. Since $\Gamma(a)^p = \lambda_{a,a}\lambda_{a,a^2} \ldots \lambda_{a,a^{p-1}}E$, we have $\lambda_{a,a}\lambda_{a,a^2} \ldots \lambda_{a,a^{p-1}} = \gamma^p$. Conversely, if the last equality holds, then an irreducible λ-representation Δ of the subgroup $\langle a \rangle$ is one-dimensional: $\Delta(a^i) = \gamma^i$, $i = 0, 1, \ldots, p-1$. Hence, by Clifford's theorem, $\Gamma(a) = \Delta(a) \dot{+} \ldots \dot{+} \Delta(a) = \gamma E$.

Let $H = \langle b_1 \rangle \times \ldots \times \langle b_m \rangle$ and $a = b_1^{t_1} \ldots b_m^{t_m}$. Then $\lambda_{a,a} \ldots \lambda_{a,a^{p-1}} \in F^p$ if and only if $\delta_1^{t_1} \ldots \delta_m^{t_m} \in F^p$. ∎

COROLLARY. *An abelian p-group G has a faithful irreducible projective representation over a field F if and only if the number of invariants of the group G does not exceed the rank of the group $F^*/(F^*)^p$.*

PROPOSITION 10. *Let G_q be a Sylow q-subgroup of a nilpotent group G, $\lambda \in Z^2(G, F^*)$ and λ_q be the restriction of λ to G_q. The group G has a faithful irreducible λ-representation over the field F if and only if for each prime $q \mid |G|$ the group G_q has a faithful irreducible λ_q-representation over F.*

Proof. Let Γ be a faithful irreducible λ-representation of G. By Clifford's theorem, the restriction of Γ to G_q is a completely reducible representation and all its irreducible components are pairwise conjugate. Let Δ be one of them. If Δ is not faithful, then as in the proof of Proposition 8, the λ_q-center of G_q contains a non-identity element a. Since a belongs to the λ-center of G, it follows that $\Gamma(a) = \mu E \ \ (\mu \in F^*)$. The contradiction obtained proves the necessity.

Let $G = G_{q_1} \times \ldots \times G_{q_s}$ be a decomposition into a direct product of Sylow subgroups, Γ_i be a faithful irreducible λ_{q_i}-representation of G_{q_i} and Δ be an irreducible component of $\Gamma = \Gamma_1 \# \ldots \# \Gamma_s$. Since $\Gamma|_{G_{q_i}} = \Gamma_i \dotplus \ldots \dotplus \Gamma_i$, we have $\Delta|_{G_{q_i}} = \Gamma_i \dotplus \ldots \dotplus \Gamma_i$. Therefore, as in the proof of Proposition 7, Δ is a faithful irreducible λ-representation of G. This proves the sufficiency. ■

THEOREM 3. *Let G be a finite abelian q-group, $q \neq p$, $\lambda \in Z^2(G, F^*)$, N be the λ-center of G and ε be a primitive qth root of 1. If $N = \{e\}$, then G is a group of symmetric type and G has a faithful irreducible λ-representation. Let $N \neq \{e\}$ and H be the socle of N. Then the following conditions are equivalent*:

(1) *G has a faithful irreducible λ-representation*;

(2) *H has a faithful irreducible λ-representation*;

(3) *if $\varepsilon \in F$, then $F^\lambda H$ is a field, and if $\varepsilon \notin F$, then $F^\lambda H$ is a twisted group algebra of a group of order q and of a field K, $K \supset F$*;

(4) *let $H = \langle b_1 \rangle \times \ldots \times \langle b_m \rangle$ and $F^\lambda H = [H, F, \delta_1, \ldots, \delta_m]$; if $\varepsilon \notin F$, then no more than one of the products*

$$\delta_1^{t_1} \ldots \delta_m^{t_m} \qquad (0 \leq t_i < q,\ t_1 + \ldots + t_m \neq 0)$$

is the qth power of an element of F, and if $\varepsilon \in F$, then none of them is.

Proof. The case $N = \{e\}$ was studied by Yamazaki [11]. Consider the case $N \neq \{e\}$. Let Δ be a faithful irreducible λ-representation of the subgroup H and Δ^G be the projective λ-representation of G induced by Δ. Arguing as in the proof of Proposition 8, we deduce that each irreducible component of Δ^G is a faithful λ-representation of G. Together with Clifford's theorem, this proves that conditions (1) and (2) are equivalent.

The equivalence of (2) and (3) is established in Lemma 4 of [2]. Let us prove the equivalence of (2) and (4).

Let $\varrho_a = \lambda_{a,a}\lambda_{a,a^2}\ldots\lambda_{a,a^{q-1}}$. Consider first the case $\varepsilon \notin F$. Assume that H contains a subgroup $Q = \langle a\rangle \times \langle b\rangle$ of the type (q,q) such that $\varrho_a = \gamma^q$ and $\varrho_b = \delta^q$ $(\gamma, \delta \in F)$. Since $F^\lambda Q \cong FQ$, no irreducible λ-representation of Q is faithful. Hence, by Clifford's theorem, H does not have faithful irreducible λ-representations. If $H = H' \times \langle a\rangle$, $\varrho_a = \gamma^q$ and $\varrho_b \notin F^q$ for each non-identity element $b \in H'$, then by Proposition 3, $F^\lambda H'$ is a field and therefore $F^\lambda H$ satisfies (3). It follows that H has a faithful irreducible λ-representation. In the case $\varepsilon \in F$, the condition $\varrho_a = \gamma^q$ is equivalent to the fact that each irreducible λ-representation of the subgroup $\langle a\rangle$ is one-dimensional. Together with Clifford's theorem, this proves that H does not have faithful irreducible λ-representations. ■

PROPOSITION.*. *Let G be a finite abelian group, $p \nmid |G|$, $\lambda \in Z^2(G, F^*)$ and for each prime $q \mid |G|$ let the field F contain a primitive qth root of 1. Assume also that if the exponent of a Sylow 2-subgroup of G is greater than 2, then F also contains a primitive 4th root of 1. The group G has a faithful irreducible λ-representation if and only if $F^\lambda G$ is a simple algebra.*

Proposition 11 is an immediate consequence of Propositions 3, 5, 10 and Theorem 3.

Suppose the field F contains a primitive dth root of 1 and let d be a divisor of the exponent of a finite abelian group G. For each prime $q \mid |G|$, we denote by G_q a Sylow q-subgroup of G and by d_q the q-part of d. The value of t_q is the same as in Proposition 4 and s_q is the number of invariants of G_q, exceeding d_q.

THEOREM 4. *A finite abelian group G has a faithful irreducible λ-representation over F for some cocycle $\lambda \in Z^2(G, F^*, d)$ if and only if, for $p \mid |G|$, the number of invariants of the subgroup G_p does not exceed the rank of the group $F^*/(F^*)^p$ and for each prime $q \mid |G|$ different from p one of the following conditions holds*:

(1) *if $t_q = 0$ and $d_q \neq 1$, then $s_q = 0$ and G_q is a group of symmetric type*;

(2) *if $t_q \neq 0$ and $d_q \neq 1$, then $s_q \le t_q$*;

(3) *if $d_q = 1$, then $s_q \le t_q + 1$.*

Proof. By Proposition 10 one can assume $G = G_q$. If $q = p$, then we apply the Corollary to Proposition 9. Let $q \neq p$. If $s_q > 0$, then by Lemma 5 of [2], the group G_q has a subgroup $N_q = \langle a_1\rangle \times \ldots \times \langle a_{s_q}\rangle$ such that G_q/N_q is a group of symmetric type and the exponent of G_q/N_q does not exceed d_q.

By Lemma 6 of [2], for each symmetric cocycle $\mu \in Z^2(N_q, F^*)$ there exists a cocycle $\lambda \in Z^2(G_q, F^*, d_q)$ such that $\lambda|_{N_q} = \mu$ and the λ-center of G_q coincides with N_q. If $s_q = 0$ then there exists a cocycle $\lambda \in Z^2(G_q, F^*, d_q)$ such that the λ-center of G_q is a cyclic group [13]. By Theorem 3, the above reasoning shows that if one of conditions (1)–(3) holds, then G_q has a faithful irreducible λ-representation for some $\lambda \in Z^2(G_q, F^*, d_q)$.

Assume that none of (1)–(3) holds for G_q. If $d_q = q^m$, $c \in G_q$ and $o(c) = q^t$, where $t > m$, then for each cocycle $\lambda \in Z^2(G_q, F^*, d_q)$ a basis element $u_c^{q^m}$ belongs to the center of the algebra $F^\lambda G_q$. It follows that if $s_q > 0$, then the number of invariants of the λ-center of G_q is not less than s_q. If $s_q = 0$ and G_q is not a group of symmetric type, then the λ-center of G_q differs from $\langle e \rangle$. Let H_q be the socle of the λ-center of G_q. Since, for $d_q \neq 1$, the number of invariants of H_q is greater than t_q, we conclude that $F^\lambda H_q$ is not a field. For $d_q = 1$, the number of invariants of H_q is greater than $t_q + 1$ and therefore $F^\lambda H_q$ is not a twisted group algebra of a group of order q and of a field K, $K \supset F$. Hence, by Theorem 3, we conclude that G_q has a faithful irreducible λ-representation for no cocycle $\lambda \in Z^2(G_q, F^*, d_q)$. ∎

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