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ON THE RING OF CONSTANTS FOR DERIVATIONS OF POWER SERIES RINGS IN TWO VARIABLES

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Abstract. Let $k[[x, y]]$ be the formal power series ring in two variables over a field k of characteristic zero and let d be a nonzero derivation of $k[[x, y]]$. We prove that if $\mathrm{Ker}(d) \neq k$ then $\mathrm{Ker}(d) = \mathrm{Ker}(\delta)$, where δ is a jacobian derivation of $k[[x, y]]$. Moreover, $\mathrm{Ker}(d)$ is of the form $k[[h]]$ for some $h \in k[[x, y]]$.

1. Introduction. Let $k[x, y]$ be the ring of polynomials in two variables over a field k of characteristic zero. Let d be a nonzero derivation of $k[x, y]$ and let $\mathbb{A}$ be its ring of constants. It is well known (see for example [4]) that if $\mathbb{A} \neq k$ then $\mathbb{A} = \mathrm{Ker}(\delta)$, where δ is a jacobian derivation of $k[x, y]$. It is also well known ([5] or [4]) that $\mathbb{A}$ is of the form $k[h]$ for some $h \in k[x, y]$.

In 1975, A. Płoski [6] proved similar facts for derivations in the convergent power series ring in two variables. In this paper we show that the above facts are also true in the formal power series ring in two variables.

2. Preliminaries. If F is a nonzero power series from $k[[x, y]]$ then we denote by $\omega(F)$ the lowest homogeneous form of F, and by $o(F)$ the *order* of F, that is, $o(F) = \deg \omega(F)$. Moreover, we assume that $\omega(0) = 0$ and $o(0) = \infty$. It is clear that $o(FG) = o(F) + o(G)$ for all $F, G \in k[[x, y]]$.

By a *derivation* of $k[[x, y]]$ we mean every k-linear mapping $d : k[[x, y] \to k[[x, y]]$ such that $d(FG) = Fd(G)+Gd(F)$ for $F, G \in k[[x, y]]$. Recall (see for example [2], [4]) that each derivation d of $k[[x, y]]$ has a unique representation of the form $d = F\frac{\partial}{\partial x} + G\frac{\partial}{\partial y}$, where $F, G \in k[[x, y]]$. Recall also that if d is a derivation of $k[[x, y]]$ then its kernel

$$k[[x, y]]^d = \{F \in k[[x, y]] : d(F) = 0\}$$

is a subring of $k[[x, y]]$ containing k.

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If F, G are two power series from $k[[x,y]]$ then we denote by $J(F,G)$ the *jacobian* of F, G, that is,

$$J(F,G) = \frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x}.$$

If $F \in k[[x,y]]$ is fixed then the mapping $\delta : k[[x,y]] \to k[[x,y]$ defined by

$$\delta(G) = J(F,G) \quad \text{ for } G \in k[[x,y]]$$

is a derivation of $k[[x,y]]$; we call it a *jacobian derivation.*

Note the following useful lemma.

LEMMA 2.1. *Let f, g be nonzero homogeneous polynomials from $k[x,y]$. Assume that* $\deg(f) = sn$, $\deg(g) = sm$, *where* $\gcd(n,m) = 1$. *If $J(f,g) = 0$ then there exist a homogeneous polynomial $h \in k[x,y]$ of degree s and nonzero elements $a, b \in k$ such that $f = ah^n$ and $g = bh^m$.*

Proof. See, for example, [1]. ■

3. Jacobian derivations. In this section we prove the following

THEOREM 3.2. *Let d be a nonzero derivation of $k[[x,y]]$. If $F \in k[[x,y]]^d \setminus k$, then $k[[x,y]]^d = k[[x,y]]^\delta$, where $\delta = J(F,-)$.*

For the proof of this theorem we need two lemmas.

LEMMA 3.3. *Let d be a nonzero derivation of $k[[x,y]]$. If $F, G \in k[[x,y]]^d$ then the polynomials $\omega(F)$, $\omega(G)$ are algebraically dependent over k.*

Proof. This is obvious when $d(x) = 0$ or $d(y) = 0$ or $\omega(F) \in k$ or $\omega(G) \in k$. So we may assume that $d(x) \neq 0$, $d(y) \neq 0$, $\deg \omega(F) \geq 1$ and $\deg \omega(G) \geq 1$. Put $d(x) = U$, $d(y) = V$, and let $\overline{F} = F - \omega(F)$, $\overline{G} = G - \omega(G)$, $\overline{U} = U - \omega(U)$, $\overline{V} = V - \omega(V)$. Then

$$(1) \qquad \begin{cases} 0 = d(F) = \left(\dfrac{\partial \omega(F)}{\partial x} + \dfrac{\partial \overline{F}}{\partial x}\right)(\omega(U) + \overline{U}) \\ \qquad\qquad + \left(\dfrac{\partial \omega(F)}{\partial y} + \dfrac{\partial \overline{F}}{\partial y}\right)(\omega(V) + \overline{V}), \\ 0 = d(G) = \left(\dfrac{\partial \omega(G)}{\partial x} + \dfrac{\partial \overline{G}}{\partial x}\right)(\omega(U) + \overline{U}) \\ \qquad\qquad + \left(\dfrac{\partial \omega(G)}{\partial y} + \dfrac{\partial \overline{G}}{\partial y}\right)(\omega(V) + \overline{V}). \end{cases}$$

Assume that $\deg \omega(U) < \deg \omega(V)$. Then comparing the lowest forms in (1) we have

$$\frac{\partial \omega(F)}{\partial x}\omega(U) = 0 \quad \text{and} \quad \frac{\partial \omega(G)}{\partial x}\omega(U) = 0.$$

Hence $\partial\omega(F)/\partial x = \partial\omega(G)/\partial x = 0$ and the polynomials $\omega(F)$ and $\omega(G)$ are algebraically dependent because they belong to $k[y]$. If $\deg\omega(U) > \deg\omega(V)$ then analogously $\omega(F)$ and $\omega(G)$ are in $k[x]$ and are also dependent.

Assume now that $\deg\omega(U) = \deg\omega(V)$. Then comparing the lowest forms in (1) we get the following system of equations:

$$\begin{cases} \dfrac{\partial\omega(F)}{\partial x}\omega(U) + \dfrac{\partial\omega(F)}{\partial y}\omega(V) = 0, \\ \dfrac{\partial\omega(G)}{\partial x}\omega(U) + \dfrac{\partial\omega(G)}{\partial y}\omega(V) = 0. \end{cases}$$

Since $(\omega(U), \omega(V)) \neq (0,0)$, this system has a nonzero solution. This means that the jacobian $J(\omega(F), \omega(G))$ is equal to zero and so, the polynomials $\omega(F)$ and $\omega(G)$ are algebraically dependent. ■

LEMMA 3.4. *Let d_1 and d_2 be nonzero derivations of $k[[x,y]]$. Assume that the rings $k[[x,y]]^{d_1}$ and $k[[x,y]]^{d_2}$ both contain a series $F \in k[[x,y]] \setminus k$. Then $k[[x,y]]^{d_1} = k[[x,y]]^{d_2}$.*

Proof. We can assume that $\omega(F) \notin k$. By Lemmas 3.3 and 2.1 we know that $\omega(F)$ is a polynomial of h_1 and a polynomial of h_2 (for some $h_1, h_2 \in k[x,y]$). So we may assume that $h_1 = h_2 = h$.

Take any $G \in k[[x,y]]$ for which $\omega(G)$ is not a polynomial of h. Then $d_1(G) \neq 0$ and $d_2(G) \neq 0$. Consider the derivation

$$d_3 = d_2(G)d_1 - d_1(G)d_2.$$

It is clear that $d_3(F) = d_3(G) = 0$. But $\omega(F)$ and $\omega(G)$ are algebraically independent. By Lemma 3.3 it is possible only if $d_3 = 0$. So $d_2(G)d_1 = d_1(G)d_2$ and the kernels are the same. ■

Proof of Theorem 3.2. It is a simple consequence of Lemma 3.4 as $d \neq 0$, $\delta \neq 0$ and the rings $k[[x,y]]^d$ and $k[[x,y]]^\delta$ contain $F \in k[[x,y]] \setminus k$. ■

4. A generator of the ring of constants. Let $d : k[[x,y]] \to k[[x,y]]$ be a nonzero derivation and let $\mathbb{A} = k[[x,y]]^d$. We want to show that the ring $\mathbb{A}$ is of the form $k[[F]]$ for some series $F \in k[[x,y]]$. If $\mathbb{A} = k$, then $\mathbb{A} = k[[F]]$ for $F = 0$. Assume that $\mathbb{A} \neq k$.

We already know (by Lemmas 3.3 and 2.1) that all lowest homogeneous forms of nonzero elements in $\mathbb{A}$ are scalar multiples of powers of a homogeneous form φ. For each $F \in \mathbb{A} \setminus \{0\}$ denote by $\gamma(F)$ the degree of φ in $\omega(F)$, that is, if $\omega(F) = a\varphi^n$ where $0 \neq a \in k$, then $\gamma(F) = n$. Assume moreover that $\gamma(0) = \infty$.

Consider the semigroup $\pi = \{\gamma(F) : 0 \neq F \in \mathbb{A}\}$. Since $\mathbb{A} \neq k$ this semigroup contains positive numbers. Let γ be the greatest common divisor of the elements of π.

Lemma 4.5. *There exist* $F, G \in \mathbb{A} \setminus \{0\}$ *such that* $\gamma = \gamma(F) - \gamma(G)$.

Proof. Since γ is the greatest common divisor, there exist nonnegative integers $i_1, \dots, i_n, j_1, \dots, j_m$ and nonzero series $F_1, \dots, F_n, G_1, \dots, G_m$ from $\mathbb{A}$ such that

$$\gamma = i_1\gamma(F_1) + \ldots + i_n\gamma(F_n) - j_1\gamma(G_1) - \ldots - j_m\gamma(G_m).$$

Put $F = F_1^{i_1} \ldots F_n^{i_n}$ and $G = G_1^{j_1} \ldots G_m^{j_m}$. Then $F, G \in \mathbb{A} \setminus \{0\}$ and $\gamma = \gamma(F) - \gamma(G)$. ■

Lemma 4.6. *Let* $F, G \in \mathbb{A} \setminus \{0\}$ *be as in Lemma* 4.5. *Let* $\gamma(G) = s\gamma$. *Then* $n\gamma \in \pi$ *for any* $n > s^2 - s - 1$.

Proof. Since $\gamma = \gamma(F) - \gamma(G)$, we have $\gamma(F) = (s+1)\gamma$. Assume that $n > s^2 - s - 1$. Let $n = us + r$, where u, r are integers such that $0 \le r < s$. Put $i = r$, $j = u - r$. Then $i \ge 0$ and $j \ge 0$ (since $u \ge s - 1 \ge r$), and moreover

$$i(s+1) + js = r(s+1) + (u-r)s = us + r = n.$$

Let $H = F^iG^j$. Then $0 \neq H \in \mathbb{A}$ and

$$\gamma(H) = i\gamma(F) + j\gamma(G) = i(s+1)\gamma + js\gamma = n\gamma,$$

that is, $n\gamma \in \pi$. ■

We may extend the mapping $\gamma(-)$ to $\mathbb{A}_0 \setminus \{0\}$ (where $\mathbb{A}_0$ is the field of fractions of $\mathbb{A}$) by defining

$$\gamma(A/B) = \gamma(A) - \gamma(B)$$

for all nonzero $A, B \in \mathbb{A}$.

Lemma 4.7. *Let* $f, g \in \mathbb{A}_0 \setminus \{0\}$. *If* $\gamma(f) = \gamma(g)$, *then there exists* $c \in k \setminus \{0\}$ *such that* $\gamma(f - cg) > \gamma(f)$.

Proof. This is clear if $f, g \in \mathbb{A}$. Let $f = A/B$, $g = C/D$, where $A, B, C, D \in \mathbb{A} \setminus \{0\}$. Since $\gamma(f) = \gamma(g)$, we have

$$\gamma(AD) = \gamma(A) + \gamma(D) = \gamma(C) + \gamma(B) = \gamma(CB),$$

and so, there exists a nonzero $c \in k$ such that $\gamma(AD - cCB) > \gamma(AD)$. Then we have

$$\begin{aligned} \gamma(f - cg) &= \gamma((AD - cCB)/BD) = \gamma(AD - cCB) - \gamma(BD) \\ &> \gamma(AD) - \gamma(BD) = \gamma(A/B) = \gamma(f), \end{aligned}$$

that is, $\gamma(f - cg) > \gamma(f)$. ■

Consider now the fraction

$$h = F/G,$$

where F and G are nonzero series from $\mathbb{A}$ as in Lemma 4.5. We know that $\gamma(h) = \gamma$. We want to show that $h \in \mathbb{A}$.

LEMMA 4.8. *There exists a natural number n such that $h^n \in \mathbb{A}$.*

Proof. Let $\gamma(G) = s\gamma$, $\gamma(F) = (s+1)\gamma$ and let n be a natural number such that $n > s^2 - s$. We shall show that $h^n \in k[[x, y]]$.

We know, by Lemma 4.6 and its proof, that there exist integers $i_1 \geq 0$, $j_1 \geq 0$ such that $\gamma(H_1) = n\gamma$, where $H_1 = F^{i_1}G^{j_1}$. Then $\gamma(h^n) = n\gamma = \gamma(H_1)$, so (by Lemma 4.7) $\gamma(h^n - c_1H_1) > \gamma(h^n)$ for some $c_1 \in k \setminus \{0\}$.

Put $h_1 = h^n - c_1H_1$. If $h_1 = 0$ then $h^n = c_1H_1 \in \mathbb{A}$ and we are done. Assume that $h_1 \neq 0$. Then $\gamma(h_1) = n_1\gamma$, where $n_1 > n \geq n_0$. Using again Lemmas 4.6 and 4.7 we see that $\gamma(h_1 - c_2H_2) > \gamma(h_1)$ for some $c_2 \in k \setminus \{0\}$, $H_2 = F^{i_2}G^{j_2}$, $i_2 \geq 0$, $j_2 \geq 0$. Put $h_2 = h_1 - c_2H_2 = h^n - c_1H_1 - c_2H_2$. If $h_2 = 0$, then $h^n \in \mathbb{A}$ and we are done, and so on.

If the above procedure has no end, then we obtain an infinite sequence (c_mH_m) of nonzero elements from $\mathbb{A}$ such that $o(H_m) < o(H_{m+1})$ for any natural m and $o(h^n - U_{m+1}) > o(h^n - U_m)$, where $U_m = c_1H_1 + \ldots + c_mH_m$. This means that h^n is the limit of the convergent sequence (U_m). Since each U_m belongs to the ring $k[[x, y]]$, which is complete, the limit h^n also belongs to $k[[x, y]]$. Therefore $h^n \in k[[x, y]] \cap \mathbb{A}_0 = \mathbb{A}$. ∎

LEMMA 4.9. *$h \in k[[x, y]]$.*

Proof. The ring $k[[x, y]]$ is a unique factorization domain (see, for example, [3], p. 163), hence it is integrally closed. Lemma 4.8 implies that h is integral over $k[[x, y]]$, so $h \in k[[x, y]]$. ∎

LEMMA 4.10. *$\mathbb{A} = k[[h]]$.*

Proof. We already know (by the previous lemma) that $h \in k[[x, y]] \cap \mathbb{A}_0 = A$. We also know that $\gamma(h) = \gamma \geq 1$, so $o(h) \geq 1$ (that is, h has no constant term). It is clear that $k[[h]] \subseteq A$. Let $U \in \mathbb{A} \setminus k$. We shall show that $U \in k[[h]]$.

Since $\gamma(U) = n_1\gamma$ for some natural n_1, there exists a nonzero $c_1 \in k$ such that $\gamma(U_1) > \gamma(U)$ for $U_1 = U - c_1h^{n_1}$. If $U_1 = 0$ then $U \in k[[h]]$ and we are done. Assume that $U_1 \neq 0$. Since $U_1 \in \mathbb{A}$ there exist $n_2 > n_1$ and $0 \neq c_2 \in k$ such that $\gamma(U_2) > \gamma(U_1)$ for $U_2 = U_1 - c_2h^{n_2} = U - c_1h^{n_1} - c_2h^{n_2}$. If $U_2 = 0$ then $U \in k[[h]]$ and we are done, and so on.

If the above procedure terminates, we see that $U \in k[[h]]$. In the opposite case U is the limit of the infinite convergent sequence (h_m), where each h_m is of the form

$$h_m = c_1h^{n_1} + \ldots + c_mh^{n_m},$$

where $c_1, \ldots, c_m$ are nonzero elements of k and $n_1 < \ldots < n_m$. Therefore $U \in k[[h]]$. ∎

From the above lemmas we get the following main result of our paper.

THEOREM 4.11. *If d is a nonzero derivation of $k[[x, y]]$, then $k[[x, y]]^d = k[[h]]$ for some $h \in k[[x, y]]$.* ∎

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