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A RICCI FLAT PSEUDO-RIEMANNIAN METRIC ON THE TANGENT BUNDLE OF A RIEMANNIAN MANIFOLD

BY

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Abstract. We consider a certain pseudo-Riemannian metric G on the tangent bundle TM of a Riemannian manifold (M,g) and obtain necessary and sufficient conditions for the pseudo-Riemannian manifold (TM,G) to be Ricci flat (see Theorem 2).

1. Introduction. The tangent bundle TM of a Riemannian manifold (M,g) can be organized as a Riemannian or a pseudo-Riemannian manifold in many ways. The most known such structures are given by the Sasaki metrics (see [13], [1], [7]) and the complete lift type pseudo-Riemannian metrics (see [15], [14], [8]), both defined on TM with the help of g.

A slight generalization of the Sasaki metric is the special natural lift of g to TM considered by V. Oproiu [10] (for the definition of the natural lifts of g to TM see [4], [5], [3]). A Riemannian metric G on TM has been defined by using the Levi-Civita connection of g and two smooth real-valued functions $u(t), v(t)$ depending on the energy density only and such that $u(t) > 0$ and $u(t)+2tv(t) > 0$ for all $t \in [0,\infty)$. He has also considered an almost complex structure J on TM, related to the metric G and has studied the conditions under which (TM,G,J) is a Kähler Einstein manifold. Note that in [10], the author excludes some important cases which appeared, in a certain sense, as singular cases. These singular cases have been studied by V. Oproiu and the present author in [11], [9], [12]. Note also that one of the important cases studied in [11] is when the Riemannian metric G on TM is defined by using a certain Lagrangian L on the base manifold (M,g) depending on the energy density only (i.e. the case when $v(t) = u'(t)$). On the other hand, in [8], V. Oproiu has studied a pseudo-Riemannian structure on the tangent bundle of a Lagrange manifold M, considering the pseudo-Riemannian metric G on TM as being the complete lift of a quadratic form defined by the Lagrangian L considered.

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In the present note, inspired by [10] and [8], we consider a new pseudo-Riemannian metric G of natural lift type on TM (so that it is no longer obtained as the complete lift by using a Lagrangian on M). This new metric G is defined by using also the Levi-Civita connection of the Riemannian metric g and two smooth real-valued functions $u(t), v(t)$ such that $u(t) > 0$ and $u(t) + 2tv(t) > 0$ for all $t \in [0, \infty)$. Next, we study necessary and sufficient conditions for the pseudo-Riemannian manifold (TM, G) to be Ricci flat. The main result is: The pseudo-Riemannian manifold (TM, G) is Ricci flat if and only if the base manifold (M, g) is Ricci flat and the functions u and v which appear in the expression of G are related by $v = u'$. By using some known results from Lagrange geometry (see [8], [11]), it is shown that the condition $v = u'$ is equivalent to the fact that the pseudo-Riemannian metric G considered on TM is the complete lift of a quadratic form defined by a certain Lagrangian L on M (see Theorem 2).

The manifolds, tensor fields and geometric objects we consider in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). The well known summation convention is used throughout this paper, the range for the indices i, j, k, l, h, s, r being always $\{1, \ldots, n\}$. We denote by $\Gamma(TM)$ the module of smooth vector fields on TM.

2. The pseudo-Riemannian metric G on TM. Let (M, g) be a smooth n-dimensional Riemannian manifold, $n > 1$, and denote its tangent bundle by $\tau : TM \to M$. Recall that TM has the structure of a $2n$-dimensional smooth manifold induced from the smooth manifold structure of M. A local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ of M induces a local chart $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \ldots, x^n, y^1, \ldots, y^n)$ of TM in the usual way, in particular for $y \in \tau^{-1}(U)$ the coordinates y^i are given by

$$y = \sum y^i \frac{\partial}{\partial x^i}\bigg|_{\tau(y)}.$$

This special structure of TM allows us to introduce the notion of so-called M-tensor field on it (see [6]). An *M-tensor field* of type (p, q) on TM is defined by sets of functions

$$T^{i_1 \ldots i_p}_{j_1 \ldots j_q}(x, y), \quad i_1, \ldots, i_p, j_1, \ldots, j_q = 1, \ldots, n,$$

assigned to any induced local chart $(\tau^{-1}(U), \Phi)$ on TM, such that the transformation rule is that of the components of a tensor field of type (p, q) on the base manifold. Note that any ordinary tensor field on the base manifold may be thought of as an M-tensor field on TM, having the same type and with the components in the induced local chart on TM equal to the local coordinate components of the given tensor field in the chosen local chart on the base manifold. In the case of a covariant tensor field on the base manifold M the corresponding M-tensor field on TM may be thought of as the pullback of the initial tensor field by the smooth submersion $\tau : TM \to M$.

Recall that the Levi-Civita connection $\dot{\nabla}$ of g defines a direct sum decomposition $TTM = VTM \oplus HTM$ of the tangent bundle TTM into the vertical distribution $VTM = \mathrm{Ker}\,\tau_*$ and the horizontal distribution HTM. The vector fields $(\partial/\partial y^1), \ldots, \partial/\partial y^n)$ define a local frame field for VTM, and for HTM we have the local frame field $(\delta/\delta x^1, \ldots, \delta/\delta x^n)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{i0}\frac{\partial}{\partial y^h}, \qquad \Gamma^h_{i0} = \Gamma^h_{ik} y^k,$$

and $\Gamma^h_{ik}(x)$ are the Christoffel symbols of g.

The distributions VTM and HTM are isomorphic to each other and it is possible to derive an almost complex structure on TM which, together with the Sasaki metric, determines an almost Kählerian structure on TM (see [1], [14]).

Consider now the energy density (kinetic energy)

$$t = \tfrac{1}{2}\|y\|^2 = \tfrac{1}{2}g_{\tau(y)}(y,y) = \tfrac{1}{2}g_{ik}(x)y^i y^k, \qquad y \in \tau^{-1}(U),$$

of a tangent vector y, where g_{ik} are the components of g in the local chart (U, φ). Let $u, v : [0, \infty) \to \mathbb{R}$ be two smooth functions such that $u(t) > 0$ and $u(t) + 2tv(t) > 0$ for all $t \in [0, \infty)$. Then we may consider the symmetric M-tensor field of type (0,2) on TM with components (see [10], [9])

$$G_{ij} = u(t)g_{ij} + v(t)g_{0i}g_{0j},$$

where $g_{0i} = g_{hi}y^h$. The matrix (G_{ij}) is symmetric and positive definite and has an inverse with the entries

$$H^{kl} = \frac{1}{u}g^{kl} - \frac{v}{u(u+2tv)}y^k y^l,$$

where g^{kl} are the components of the inverse of the matrix (g_{ij}). The components $H^{kl}(x,y)$ define a symmetric M-tensor field of type $(2,0)$ on TM.

The following pseudo-Riemannian metric will be considered on TM:

$$G = 2G_{ij}\dot{\nabla}y^i dx^j = 2(ug_{ij} + vg_{0i}g_{0j})\dot{\nabla}y^i dx^j, \tag{1}$$

where $\dot{\nabla}y^i = dy^i + \Gamma^i_{j0}dx^j$ is the absolute differential of y^i with respect to the Levi-Civita connection $\dot{\nabla}$ of g. Equivalently, we have

$$G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 0, \quad G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0,$$

$$G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = G\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = G_{ij}.$$

Observe that the system of 1-forms $(dx^1, \dots, dx^n, \dot{\nabla} y^1, \dots, \dot{\nabla} y^n)$ defines a local frame of T^*TM, dual to the local frame $(\delta/\delta x^1, \dots, \delta/\delta x^n, \partial/\partial y^1, \dots \dots, \partial/\partial y^n)$.

In the following we determine the Levi-Civita connection ∇ of the pseudo-Riemannian metric G defined by (1). To do this we need the following well known formulas for the brackets of the vector fields $\partial/\partial y^i, \delta/\delta x^i$:

$$\left[\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right] = 0, \quad \left[\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right] = -\Gamma^h_{ij}\frac{\partial}{\partial y^h}, \quad \left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = -R^h_{0ij}\frac{\partial}{\partial y^h},$$

where $R^h_{0ij} = R^h_{kij} y^k$ and R^h_{kij} are the local coordinate components of the curvature tensor field of $\dot{\nabla}$ on M.

Recall that the Levi-Civita connection ∇ on the pseudo-Riemannian manifold (TM, G) is obtained from the formula

$$\begin{aligned}2G(\nabla_X Y, Z) = X(G(Y,Z)) + Y(G(X,Z)) - Z(G(X,Y)) + G([X,Y],Z) \\ - G([X,Z],Y) - G([Y,Z],X), \quad X,Y,Z \in \Gamma(TM).\end{aligned}$$

PROPOSITION 1. *Let (M, g) be a Riemannian manifold. Then the Levi-Civita connection ∇ of the pseudo-Riemannian metric G defined by* (1) *on TM has the following expression in the local adapted frame $(\partial/\partial y^1, \dots \dots, \partial/\partial y^n, \delta/\delta x^1, \dots, \delta/\delta x^n)$:*

$$\begin{aligned}\nabla_{\partial/\partial y^i}\frac{\partial}{\partial y^j} &= Q^h_{ij}\frac{\partial}{\partial y^h}, \quad \nabla_{\delta/\delta x^i}\frac{\partial}{\partial y^j} = \Gamma^h_{ij}\frac{\partial}{\partial y^h} + P^h_{ji}\frac{\delta}{\delta x^h},\\ \nabla_{\partial/\partial y^i}\frac{\delta}{\delta x^j} &= P^h_{ij}\frac{\delta}{\delta x^h}, \quad \nabla_{\delta/\delta x^i}\frac{\delta}{\delta x^j} = \Gamma^h_{ij}\frac{\delta}{\delta x^h} + S^h_{ij}\frac{\partial}{\partial y^h},\end{aligned}$$

where the components P^h_{ij}, Q^h_{ij}, S^h_{ij} define M-tensor fields of type $(1,2)$ *on TM and are given by*

$$\begin{aligned}P^h_{ij} &= \frac{u'-v}{2u}\left(g_{0i}\delta^h_j - \frac{u}{u+2tv} g_{ij} y^h - \frac{v}{u+2tv} g_{0i} g_{0j} y^h\right),\\ Q^h_{ij} &= \frac{u'+v}{2u}(g_{0i}\delta^h_j + g_{0j}\delta^h_i) + \frac{v}{u+2tv} g_{ij} y^h + \frac{v'u - u'v - v^2}{u(u+2tv)} g_{0i} g_{0j} y^h,\\ S^h_{ij} &= g^{hk} R_{0ikj} + \frac{v}{u+2tv} R_{0ij0} y^h,\end{aligned}$$

R_{likj} denoting the local coordinate components of the Riemann–Christoffel tensor of $\dot{\nabla}$ on M and $R_{0ikj} = R_{likj} y^l$, $R_{0ij0} = R_{lijk} y^l y^k$.

The curvature tensor field K of the Levi-Civita connection ∇ is defined by

$$K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X,Y,Z \in \Gamma(TM).$$

By straightforward computations we obtain

$$\begin{cases} K\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} = XXX^h_{kij}\frac{\delta}{\delta x^h} + y^l\left(\dot{\nabla}_l R^h_{kij} + \frac{v}{u+2tv}\dot{\nabla}_l R_{k0ij}y^h\right)\frac{\partial}{\partial y^h}, \\ K\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} = XXY^h_{kij}\frac{\partial}{\partial y^h}, \quad K\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^k} = YYX^h_{kij}\frac{\delta}{\delta x^h}, \\ K\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k} = YYY^h_{kij}\frac{\partial}{\partial y^h}, \quad K\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} = YXX^h_{kij}\frac{\partial}{\partial y^h}, \\ K\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} = YXY^h_{kij}\frac{\delta}{\delta x^h}, \end{cases}$$

where the components XXX^h_{kij}, XXY^h_{kij}, YYX^h_{kij}, YYY^h_{kij}, YXX^h_{kij}, YXY^h_{kij} define M-tensor fields of type $(1,3)$ on TM and are given by

$$\begin{cases} XXX^h_{kij} = R^h_{kij} + \frac{u'-v}{2(u+2tv)}(R_{0j0k}\delta^h_i - R_{0i0k}\delta^h_j), \\ XXY^h_{kij} = R^h_{kij} + \frac{v}{u}g_{0k}R^h_{0ij} - \frac{v}{u+2tv}R_{0kij}y^h - \frac{u'-v}{2(u+2tv)}(g_{kj}R^h_{00i} - g_{ki}R^h_{00j}) \\ \qquad - \frac{v(u'-v)}{2u(u+2tv)}(g_{0j}g_{0k}R^h_{00i} - g_{0i}g_{0k}R^h_{00j}), \\ YYX^h_{kij} = \frac{\alpha-2uv(u'-v)}{4u(u+2tv)^2}(g_{0j}g_{ik} - g_{0i}g_{jk})y^h + \frac{u'-v}{2(u+2tv)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i) \\ \qquad + \frac{v(u'-v)}{2u(u+2tv)}(g_{0i}g_{0k}\delta^h_j - g_{0j}g_{0k}\delta^h_i), \\ YYY^h_{kij} = \frac{\alpha}{4u^2(u+2tv)}(g_{0i}g_{0k}\delta^h_j - g_{0j}g_{0k}\delta^h_i) + \frac{u'-v}{2(u+2tv)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i), \\ YXX^h_{kij} = R^h_{kij} + \frac{v}{u}g_{0i}R^h_{k0j} + \frac{v^2}{u(u+2tv)}g_{0i}R_{0jk0}y^h - \frac{v}{u+2tv}R_{0kij}y^h \\ \qquad - \frac{u'-v}{2(u+2tv)}R_{0jk0}\delta^h_i + \frac{u'-v}{2(u+2tv)}g_{ik}R^h_{00j} + \frac{v(u'-v)}{2u(u+2tv)}g_{0i}g_{0k}R^h_{00j}, \\ YXY^h_{kij} = \frac{u'-v}{2(u+2tv)}(g_{ik}\delta^h_j - g_{jk}\delta^h_i) + \frac{v(u'-v)}{2u(u+2tv)}(g_{0i}g_{0k}\delta^h_j - g_{0j}g_{0k}\delta^h_i) \\ \qquad + \frac{\alpha-2uv(u'-v)}{4u(u+2tv)} \\ \qquad \times\left[\frac{1}{u}g_{0i}g_{0k}\delta^h_j - \frac{1}{u+2tv}g_{0i}g_{jk}y^h - \frac{v}{u(u+2tv)}g_{0i}g_{0j}g_{0k}y^h\right], \end{cases}$$

$$\alpha = 2u(u+2tv)u'' - 3u(u')^2 - 2u^2v' + 3uv^2 - 4tuu'v' - 2t(u')^2v + 2tv^3. \tag{2}$$

From the above formulas, we get the Ricci tensor $S(Y,Z) = \operatorname{trace}(X \to K(X,Y)Z)$,

$$\begin{cases} S\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = YYY^i_{kij} - YXY^i_{kji} = \frac{(1-n)(u'-v)}{u+2tv}g_{jk} \\ \qquad + \frac{(1-n)\alpha}{2u^2(u+2tv)}g_{0j}g_{0k}, \\ S\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}\right) = YXX^i_{kij} + XXX^i_{kij} = 2R_{jk} + \frac{(n-1)(u'-v)}{u+2tv}R_{0j0k}, \\ S\left(\frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^k}\right) = S\left(\frac{\delta}{\delta x^k}, \frac{\partial}{\partial y^j}\right) = 0, \end{cases} \tag{3}$$

where R_{jk} denote the local components of the Ricci tensor field of g.

By (3), the pseudo-Riemannian manifold (TM, G) is Ricci flat (i.e. Einstein and with vanishing scalar curvature) if and only if the following conditions are satisfied:

$$\text{(i) } v = u', \quad \text{(ii) } \alpha = 0, \quad \text{(iii) } R_{jk} = 0. \tag{4}$$

From (2) we see that (4)(i) implies (4)(ii). The condition (4)(iii) says that the Riemannian manifold (M, g) is Ricci flat.

In the following, we give a geometric interpretation of the condition (4)(i) by using a Lagrangian function $L : TM \to \mathbb{R}$,

$$L = \int u(t)\, dt, \tag{5}$$

where $u : [0, \infty) \to \mathbb{R}$ is a smooth function such that $u(t) > 0$ for all $t \geq 0$ (see [11]). Usually in Lagrange geometry (see [2], [7], [8]), the symmetric M-tensor field of type $(0, 2)$ on TM is defined by the components

$$G_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j} = u g_{ij} + u' g_{0i} g_{0j}. \tag{6}$$

Remarks. (i) In [11], V. Oproiu and the present author have proved that the usual nonlinear connection determined by the Euler–Lagrange equations associated with the Lagrangian L defined by (5) coincides with the nonlinear connection defined by the Levi-Civita connection $\dot{\nabla}$ of g (see Proposition 1 of [11]).

(ii) Taking into account remark (i), it follows that the condition (4)(i) in the expression (1) of G is equivalent to the fact that the pseudo-Riemannian metric G defined on TM coincides with the pseudo-Riemannian metric h^c, where h^c is the complete lift of the quadratic form $h = G_{ij}(x, y) dx^i dx^j$ (see [8]).

(iii) By using the results obtained by V. Oproiu [8] and the above remarks, the pseudo-Riemannian metric $G = 2G_{ij} \dot{\nabla} y^i dx^j$, where G_{ij} are defined by (6) (i.e. G is the complete lift of the quadratic form $h = G_{ij} dx^i dx^j$), we see that (TM, G) is Ricci flat if and only if the base manifold (M, g) is Ricci flat.

Thus, we obtain the main result of this paper

Theorem 2. *Consider the pseudo-Riemannian manifold* (TM, G), *where G is given by* (1). *Then the following three assertions are equivalent*:

(i) *The pseudo-Riemannian manifold* (TM, G) *is Ricci flat.*

(ii) *The Riemannian manifold* (M, g) *is Ricci flat and the functions* $u(t)$ *and* $v(t)$ *are related by the condition* $v = u'$.

(iii) *The base manifold* (M, g) *is Ricci flat and the pseudo-Riemannian metric G is the complete lift of the quadratic form* $h = G_{ij}(x, y) dx^i dx^j$, *where the components* $G_{ij}(x, y)$ *are defined as in the usual Lagrange geometry by* (6), *considering on the base manifold the Lagrangian L defined by* (5).

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