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EXISTENCE AND INTEGRAL REPRESENTATION OF REGULAR EXTENSIONS OF MEASURES

BY

WERNER RINKEWITZ (München)

Abstract. Let $\mathcal{L}$ be a δ-lattice in a set X, and let ν be a measure on a sub-σ-algebra of $\sigma(\mathcal{L})$. It is shown that ν extends to an $\mathcal{L}$-regular measure on $\sigma(\mathcal{L})$ provided $\nu^*|\mathcal{L}$ is σ-smooth at $\emptyset$ and $\nu^*(L) = \inf\{\nu^*(U) \mid X \setminus U \in \mathcal{L},\ U \supset L\}$ for all $L \in \mathcal{L}$. Moreover, a Choquet type representation theorem is proved for the set of all such extensions.

1. Introduction. Problems concerning the convex set $E_\sigma(\nu, \mathcal{A})$ of all measures μ on $\mathcal{A}$ that extend a given measure ν on a σ-algebra $\mathcal{B} \subset \mathcal{A}$ are of general interest within measure theory. It is well known that $E_\sigma(\nu, \mathcal{A})$ may be empty; moreover, even if there exists a measure extension of ν, it may happen that $E_\sigma(\nu, \mathcal{A})$ has no extreme points ([Pla], [Wz]). In order to obtain (extremal) measure extensions, it is therefore necessary to impose certain conditions on ν and $\mathcal{A}$. In §3, the main section of our paper, we will give such a condition. Namely, there we prove: If $\mathcal{A}$ is generated by a δ-lattice $\mathcal{L}$ in a set X, and if $\nu^*|\mathcal{L}$ is σ-smooth at $\emptyset$ and satisfies $\nu^*(L) = \inf\{\nu^*(U) \mid X \setminus U \in \mathcal{L},\ U \supset L\}$ for all $L \in \mathcal{L}$, then there exists an extremal $\mathcal{L}$-regular extension μ of ν (Theorem 3.2(b)). Furthermore, we show that under these assumptions the set of all $\mathcal{L}$-regular extensions of ν contains sufficiently many extreme points to yield a Choquet type integral representation theorem (Theorem 3.3(b); in 3.2(a) and 3.3(a) it is shown that quite similar results hold for finitely additive measures). Finally, we give an application concerning the extension of Baire to Borel measures (Corollary 3.5).

Our approach is based on previous work: we use results from [Le] and [Pla] in the proofs concerning measure extensions, and classical Choquet theory ([Ph]) in connection with integral representation. For a discussion of our theorems we refer the reader to Remark 3.4.

2. Preliminaries. In this section, we recall some basic facts from the theory of additive set functions; moreover, we present Lemma 2.1, an important tool for our proof of Proposition 3.1.

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Let $\mathcal{A}$ be an algebra of subsets of some set X, and let $\mathrm{ba}(\mathcal{A})$ be the family of all additive real-valued bounded functions on $\mathcal{A}$. Equipped with the topology generated by the evaluations $f_A : \mathrm{ba}(\mathcal{A}) \ni \varrho \mapsto f_A(\varrho) := \varrho(A) \in \mathbb{R}$, $A \in \mathcal{A}$, and the usual linear structure, $\mathrm{ba}(\mathcal{A})$ is a locally convex linear topological Hausdorff space. If $H \subset \mathrm{ba}(\mathcal{A})$ is convex, we write $\mathrm{ex}\, H$ for the set of extreme points of H. The family of all *contents on* $\mathcal{A}$, i.e. the family $\{\varrho \in \mathrm{ba}(\mathcal{A}) \mid \varrho \geq 0\}$, is denoted by $M(\mathcal{A})$.

Let $\mathcal{L} \subset \mathcal{A}$ be a *lattice*, i.e., $\emptyset \in \mathcal{L}$ and $\mathcal{L}$ is closed under finite unions and finite intersections. We call $\varrho \in M(\mathcal{A})$ ($\mathcal{L}$-)*regular* provided $\varrho(A) = \sup\{\varrho(L) \mid L \in \mathcal{L},\ L \subset A\}$ holds for all $A \in \mathcal{A}$; the corresponding class of contents is denoted by $M_{\mathrm{r}}(\mathcal{A})$ (1). For $\varrho \in M(\alpha(\mathcal{L}))$, where $\alpha(\mathcal{L})$ is the algebra generated by $\mathcal{L}$, we put

$$N(\varrho) := \{\gamma \in M(\alpha(\mathcal{L})) \mid \gamma(L) \geq \varrho(L) \text{ for all } L \in \mathcal{L},\ \gamma(X) = \varrho(X)\}.$$

Observe that $N(\varrho)$ is a convex compact subset of $\mathrm{ba}(\alpha(\mathcal{L}))$.

LEMMA 2.1. *Let $\mathcal{L}$ be a lattice with $X \in \mathcal{L}$, and let $\varrho \in M(\alpha(\mathcal{L}))$. Then the set $M_{\mathrm{r}}(\alpha(\mathcal{L})) \cap \mathrm{ex}\, N(\varrho)$ is not empty.*

Proof. Let κ be the cardinality of $\mathcal{L}$, and let $(L_\alpha)_{\alpha<\kappa}$ be an enumeration of $\mathcal{L}$. Define recursively sets N_α and real numbers r_α, $\alpha < \kappa$, by

$$\begin{aligned} N_\alpha &:= \{\gamma \in N(\varrho) \mid \gamma(L_\beta) = r_\beta \text{ for all } \beta < \alpha\}, \\ r_\alpha &:= \sup\{\gamma(L_\alpha) \mid \gamma \in N_\alpha\}. \end{aligned}$$

By induction we first show that $N_\alpha \neq \emptyset$ for all $\alpha < \kappa$; hence r_α is well defined. For $\alpha = 0$ we have $N_\alpha = N(\varrho)$. Now, let $\alpha = \delta + 1$ for some ordinal $\delta < \kappa$. Then

$$\begin{aligned} N_\alpha &= \{\gamma \in N(\varrho) \mid \gamma(L_\beta) = r_\beta \text{ for all } \beta \leq \delta\} \\ &= \{\gamma \in N_\delta \mid \gamma(L_\delta) = r_\delta\} \\ &= \bigcap_{n\in\mathbb{N}} \{\gamma \in N_\delta \mid r_\delta - 1//n \leq \gamma(L_\delta) \leq r_\delta\} =: \bigcap_{n\in\mathbb{N}} N_\delta^n. \end{aligned}$$

By hypothesis, $N_\delta \neq \emptyset$. Together with the definition of N_δ and r_δ this shows that $(N_\delta^n)_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty closed subsets of the compact set $N(\varrho)$. Consequently, $N_\alpha \neq \emptyset$. For a limit ordinal α we have $N_\alpha = \bigcap_{\beta<\alpha} N_\beta$. Again, we conclude $N_\alpha \neq \emptyset$.

Our definitions show therefore that $(N_\alpha)_{\alpha<\kappa}$ is a decreasing family of nonempty compact sets. We claim that any $\mu \in \bigcap_{\alpha<\kappa} N_\alpha \neq \emptyset$ is a content with the desired properties.

First, let us prove $\mu \in M_{\mathrm{r}}(\mathcal{A})$, where $\mathcal{A} := \alpha(\mathcal{L})$. The definition "$\gamma_1 \prec \gamma_2$ iff $\gamma_1(L) \leq \gamma_2(L)$ for all $L \in \mathcal{L}$ and $\gamma_1(X) = \gamma_2(X)$" gives us an order relation

(1) As in our context there is no risk of confusion, we omit an additional label $\mathcal{L}$ at $M_{\mathrm{r}}(\mathcal{A})$.

on $M(\mathcal{A})$ in the sense of [Le, 2.7] (cf. [Le, 2.2(i)]; here we use $X \in \mathcal{L}$). We claim that μ is maximal in $M(\mathcal{A})$ with respect to $\prec$. In view of [Le, 2.11], this proves $\mu \in M_{\mathrm{r}}(\mathcal{A})$ (see also [Ple, Theorem 2]). Let $\mu' \in M(\mathcal{A})$ with $\mu \prec \mu'$ and $\mu' \neq \mu$ be given. Then $\mu' \in N(\varrho)$. Pick the minimal $\alpha < \kappa$ such that $\mu'(L_\alpha) > \mu(L_\alpha) = r_\alpha$. We obtain $\mu' \in N_\alpha$, and therefore $r_\alpha \geq \mu'(L_\alpha)$, a contradiction.

Now, we show $\mu \in \operatorname{ex} N(\varrho)$. Let $\mu_1,\ \mu_2$ be members of the convex set $N(\varrho)$ with $\mu = \frac{1}{2}(\mu_1 + \mu_2)$. If $\mu_1 \neq \mu_2$, then $\mu_1|\mathcal{L} \neq \mu_2|\mathcal{L}$. Hence there exists a minimal α with $\mu_1(L_\alpha) \neq \mu_2(L_\alpha)$. We may assume $\mu_1(L_\alpha) > \mu_2(L_\alpha)$, and obtain $\mu_1(L_\alpha) > \mu(L_\alpha) = r_\alpha$. As $\mu_1 \in N_\alpha$, this is impossible. Therefore, $\mu_1 = \mu_2 = \mu$. ∎

Remark. (a) The recursive construction used in the proof of Lemma 2.1 is borrowed from [BiŠt, Theorem 4].

(b) The fact $M_{\mathrm{r}}(\alpha(\mathcal{L})) \cap N(\varrho) \neq \emptyset$ has already been shown by J. Lembcke ([Le, Korollar 2.12]). For our purposes, however, the existence of an $\mathcal{L}$-regular content $\mu \in \operatorname{ex} N(\varrho)$ is crucial.

3. The main results. This section contains the central results of the present paper, Theorem 3.2 and Theorem 3.3. For a discussion we refer the reader to Remark 3.4.

We introduce the (new) notation used in Proposition 3.1, a cornerstone in our proceeding. Let $\mathcal{L}$ be a lattice in a set X; we write $\mathcal{L}_{\mathrm{c}}$ for $\{X \setminus L \mid L \in \mathcal{L}\}$ and $\sigma(\mathcal{L})$ for the σ-algebra generated by $\mathcal{L}$. As usual, $\mathcal{L}$ is called a *δ-lattice* if it is closed under countable intersections. Now, let $\mathcal{A}, \mathcal{B}$ be algebras with $\mathcal{L} \subset \mathcal{A} \supset \mathcal{B}$. The convex set of all contents μ on $\mathcal{A}$ that extend a given $\nu \in M(\mathcal{B})$ is denoted by $E(\nu, \mathcal{A})$; moreover, we put $E_{\mathrm{r}}(\nu, \mathcal{A}) := E(\nu, \mathcal{A}) \cap M_{\mathrm{r}}(\mathcal{A})$. If $\mathcal{A}$ is a σ-algebra, $E_{\mathrm{r}\sigma}(\nu, \mathcal{A})$ is the set of all $\mathcal{L}$-regular measures extending ν. The outer content associated with $\nu \in M(\mathcal{B})$ is denoted by ν^*, i.e., $\nu^*(P) = \inf\{\nu(B) \mid B \in \mathcal{B},\ B \supset P\}$ for all $P \subset X$.

Proposition 3.1. *Let $\mathcal{L}$ be a lattice, and let $\nu \in M(\mathcal{B})$ be a content satisfying*

$$(*) \qquad \nu^*(L) = \inf\{\nu^*(U) \mid U \in \mathcal{L}_{\mathrm{c}},\ U \supset L\} \quad \text{for all } L \in \mathcal{L}.$$

Then

(a) *For every $\lambda \in [\mathrm{ex}]E(\nu, \alpha(\mathcal{L}))$ there exists $\mu \in [\mathrm{ex}]E_{\mathrm{r}}(\nu, \alpha(\mathcal{L}))$ with $\mu(L) \geq \lambda(L)$ for all $L \in \mathcal{L}$.*

(b) *For every $\lambda \in [\mathrm{ex}]E(\nu, \sigma(\mathcal{L}))$ there exists $\mu \in [\mathrm{ex}]E_{\mathrm{r}\sigma}(\nu, \sigma(\mathcal{L}))$ with $\mu(L) \geq \lambda(L)$ for all $L \in \mathcal{L}$ if $\mathcal{L}$ is a δ-lattice and $\nu^*|\mathcal{L}$ is σ-smooth at $\emptyset$* (2).

(2) I.e., $\inf_{n\in\mathbb{N}} \nu^*(L_n) = 0$ for every sequence $(L_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ with $L_n \downarrow \emptyset$.

Proof. As the proofs are quite similar, we only show (b). Moreover, we assume $X \in \mathcal{L}$; otherwise, introduce the δ-lattice $\mathcal{L}' := \mathcal{L} \cup \{X\}$ and observe that (due to $(*)$) the set of $\mathcal{L}'$-regular measure extensions of ν coincides with the set of $\mathcal{L}$-regular ones.

1. Let $\lambda \in E(\nu, \sigma(\mathcal{L}))$, and put $\varrho := \lambda|\mathcal{A}_0$, where $\mathcal{A}_0 := \alpha(\mathcal{L})$. According to Lemma 2.1, there exists an $\mathcal{L}$-regular content $\mu \in \operatorname{ex} N(\varrho)$ (3). Since $\mu(X) = \varrho(X)$, we have $\mu(U) \le \varrho(U) = \lambda(U)$ for all $U \in \mathcal{L}_{\mathrm{c}}$. Together with $\lambda|\mathcal{B} = \nu$, and the assumption $(*)$, this yields

$$\mu(L) \le \inf\{\lambda(U) \mid U \in \mathcal{L}_{\mathrm{c}},\, U \supset L\} \le \inf\{\nu^*(U) \mid U \in \mathcal{L}_{\mathrm{c}},\, U \supset L\} = \nu^*(L)$$

for all $L \in \mathcal{L}$. Therefore, $\mu|\mathcal{L}$ is σ-smooth at $\emptyset$. Since μ is $\mathcal{L}$-regular, it is also σ-smooth at $\emptyset$ on $\mathcal{A}_0$. Hence μ can be extended to an $\mathcal{L}$-regular measure on $\mathcal{A} := \sigma(\mathcal{L})$; this measure is denoted by μ again. Moreover, we infer

$$\mu(B) = \sup\{\mu(L) \mid L \in \mathcal{L},\, L \subset B\} \le \sup\{\nu^*(L) \mid L \in \mathcal{L},\, L \subset B\} \le \nu(B)$$

for all $B \in \mathcal{B}$. In view of $\mu(X) = \varrho(X) = \nu(X)$, this gives us $\mu(B) = \nu(B)$, and therefore $\mu \in E_{\mathrm{r}\sigma}(\nu, \mathcal{A})$. Hence μ has the desired properties.

2. Now, assume $\lambda \in \operatorname{ex} E(\nu, \mathcal{A})$. We will show that in this case $\mu \in \operatorname{ex} E_{\mathrm{r}\sigma}(\nu, \mathcal{A})$. Let $\mu_i \in E_{\mathrm{r}\sigma}(\nu, \mathcal{A})$, $i = 1, 2$, with $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ be given. We claim that

$$\text{(!)} \qquad \mu_i|\mathcal{A}_0 \in N(\varrho).$$

In view of $\mu|\mathcal{A}_0 \in \operatorname{ex} N(\varrho)$, this proves $\mu_i|\mathcal{A}_0 = \mu|\mathcal{A}_0$, and we are done.

Ad (!). Of course, $\mu_i(X) = \varrho(X)$. Now, let us prove $\mu_i(U) \le \varrho(U)$ for all $U \in \mathcal{L}_{\mathrm{c}}$ (the dual condition). Fix $U \in \mathcal{L}_{\mathrm{c}}$ and $\varepsilon > 0$. As λ is extremal, there exists, according to [Pla, Theorem 1], a set $B \in \mathcal{B}$ with

(1) $\lambda(U \triangle B) \le \varepsilon$.

For B we can find a set $V \in \mathcal{L}_{\mathrm{c}}$ with

(2) $\mu(B \setminus V) \le \varepsilon$ and
(3) $\lambda(V \setminus B) \le \varepsilon$

in the following way: Since μ is $\mathcal{L}$-regular, there exists $L \in \mathcal{L}$ with $L \subset B$ and $\mu(B \setminus L) \le \varepsilon$. Due to $(*)$, there are sets $V \in \mathcal{L}_{\mathrm{c}}$ and $B' \in \mathcal{B}$ such that $L \subset V \subset B'$ and $\nu(B') \le \nu^*(L) + \varepsilon$. Thus we obtain $\mu(B \setminus V) \le \varepsilon$ and

$$\lambda(V \setminus B) \le \lambda(B' \setminus B) = \nu(B' \setminus B) = \nu(B') - \nu(B' \cap B) \le \nu(B') - \nu^*(L) \le \varepsilon,$$

as desired. In view of (2), $\mu \in N(\varrho)$, (1), and (3), we have

$$\mu(U \cup B) \le \mu(U \cup V) + \varepsilon \le \lambda(U \cup V) + \varepsilon \le \lambda(B \cup V) + 2\varepsilon \le \lambda(B) + 3\varepsilon.$$

(3) For 1., an $\mathcal{L}$-regular $\mu \in N(\varrho)$ would be sufficient (cf. Remark (b) in §2).

Together with the relations $\mu = \frac{1}{2}(\mu_1+\mu_2)$, $\mu_2|\mathcal{B} = \lambda|\mathcal{B}$, and (1), this implies

$$\begin{aligned}\mu_1(U) &\le \mu_1(U \cup B) = 2\mu(U \cup B) - \mu_2(U \cup B)\\ &\le 2(\lambda(B) + 3\varepsilon) - \mu_2(B) = \lambda(B) + 6\varepsilon\\ &\le \lambda(U) + 7\varepsilon = \varrho(U) + 7\varepsilon.\end{aligned}$$

In the same way we obtain $\mu_2(U) \le \varrho(U) + 7\varepsilon$. This completes the proof of Proposition 3.1. ■

A first consequence of Proposition 3.1 is

THEOREM 3.2. *Let $\mathcal{L}$ and ν be as in Proposition* 3.1. *Then*

(a) $\operatorname{ex} E_{\mathrm{r}}(\nu, \alpha(\mathcal{L})) \neq \emptyset$*;*
(b) $\operatorname{ex} E_{\mathrm{r}\sigma}(\nu, \sigma(\mathcal{L})) \neq \emptyset$ *if $\mathcal{L}$ is a δ-lattice and $\nu^*|\mathcal{L}$ is σ-smooth at $\emptyset$.*

Proof. Due to [Pla, Corollary], the sets $\operatorname{ex} E(\nu, \alpha(\mathcal{L}))$, $\operatorname{ex} E(\nu, \sigma(\mathcal{L}))$ are not empty. Hence the statements (a), (b) follow from Proposition 3.1(a), (b), respectively. ■

The property "$\mu(L) \ge \lambda(L)$ for all $L \in \mathcal{L}$" of the extremal regular extension μ in Proposition 3.1 was irrelevant for the proof of Theorem 3.2; it will become important, however, in the proof of 3.3, our integral representation theorem. The formulation of this result requires some more definitions. (We use the terminology of [BiŠt].)

For $M \subset \mathrm{ba}(\mathcal{A})$ we denote by $\Sigma(M)$ the σ-algebra (over M) generated by the evaluations $f_A|M$, $A \in \mathcal{A}$ (see §2). We say that a convex set $H \subset \mathrm{ba}(\mathcal{A})$ has the *integral representation property* (*IRP*) if for every $\mu \in H$ there is a probability measure γ on $\Sigma(\operatorname{ex} H)$ such that

$$\mu(A) = \int\limits_{\operatorname{ex} H} \beta(A)\, \gamma(d\beta) \quad \text{for all } A \in \mathcal{A}.$$

In this case we say that γ *represents* μ.

THEOREM 3.3. *Let $\mathcal{L}$ and ν be as in Proposition* 3.1. *Then*

(a) $E_{\mathrm{r}}(\nu, \alpha(\mathcal{L}))$ *has IRP*;
(b) $E_{\mathrm{r}\sigma}(\nu, \sigma(\mathcal{L}))$ *has IRP if $\mathcal{L}$ is a δ-lattice and $\nu^*|\mathcal{L}$ is σ-smooth at $\emptyset$.*

Proof. Again, we only prove (b). Fix $\mu \in E_{\mathrm{r}\sigma}(\nu, \sigma(\mathcal{L}))$, and put $\mathcal{A} := \sigma(\mathcal{L})$, $E := E(\nu, \mathcal{A})$, and $E_{\mathrm{r}\sigma} := E_{\mathrm{r}\sigma}(\nu, \mathcal{A})$.

1. The formula $E = \bigcap_{B \in \mathcal{B}} \{\varrho \in M(\mathcal{A}) \mid \varrho(B) = \nu(B)\}$ shows that the convex set $E \subset \mathrm{ba}(\mathcal{A})$ is compact. According to the theorem of Bishop and de Leeuw ([Ph, Section 4]), there exists therefore a probability measure η_0 on the σ-algebra $\mathcal{S}$ generated by $\operatorname{ex} E$ and the Baire sets in E such that

$$(+) \qquad \mu(A) = \int\limits_E \beta(A)\, \eta_0(d\beta) \quad \text{for all } A \in \mathcal{A} \quad \text{and} \quad \eta_0(\operatorname{ex} E) = 1.$$

Due to $\Sigma(\operatorname{ex} E) \subset \mathcal{S}$, we can define a measure η in the set $\operatorname{ex} E$ by $\eta(S) := \eta_0(S)$, $S \in \Sigma(\operatorname{ex} E)$, and (+) shows that η represents μ.

2. By Proposition 3.1(b), there is for every $\lambda \in \operatorname{ex} E$ a measure $h(\lambda) \in \operatorname{ex} E_{\mathrm{r}\sigma}$ with $h(\lambda)(L) \geq \lambda(L)$ for all $L \in \mathcal{L}$ and, consequently, $h(\lambda)(U) \leq \lambda(U)$ for all $U \in \mathcal{L}_{\mathrm{c}}$. In this part, we will show that the mapping

$$h : \operatorname{ex} E \ni \lambda \mapsto h(\lambda) \in \operatorname{ex} E_{\mathrm{r}\sigma}$$

is $(\Sigma(\operatorname{ex} E)_\eta, \Sigma(\operatorname{ex} E_{\mathrm{r}\sigma}))$-measurable, where $\Sigma(\operatorname{ex} E)_\eta$ denotes the completion of $\Sigma(\operatorname{ex} E)$ with respect to η. The sets $M_A^t := \{\beta \in \operatorname{ex} E_{\mathrm{r}\sigma} \mid \beta(A) > t\}$, $A \in \mathcal{A}$, $t \in \mathbb{R}$, generate the σ-algebra $\Sigma(\operatorname{ex} E_{\mathrm{r}\sigma})$. Therefore, fix $A \in \mathcal{A}$, $t \in \mathbb{R}$, and regard

$$S := h^{-1}[M_A^t] = \{\lambda \in \operatorname{ex} E \mid h(\lambda)(A) > t\}.$$

Since μ is regular, there exists an increasing sequence $(L_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ and a decreasing sequence $(U_n)_{n\in\mathbb{N}} \subset \mathcal{L}_{\mathrm{c}}$ with $L_n \subset A \subset U_n$, $n \in \mathbb{N}$, and $\inf_{n\in\mathbb{N}} \mu(U_n \setminus L_n) = 0$. We put

$$S_1 := \{\lambda \in \operatorname{ex} E \mid \sup_{n\in\mathbb{N}} \lambda(L_n) > t\}, \qquad S_2 := \{\lambda \in \operatorname{ex} E \mid \inf_{n\in\mathbb{N}} \lambda(U_n) > t\},$$

and

$$S_3 := \{\lambda \in \operatorname{ex} E \mid \inf_{n\in\mathbb{N}} \lambda(U_n \setminus L_n) > 0\}.$$

Then $S_1, S_2, S_3 \in \Sigma(\operatorname{ex} E)$ and $S_2 \setminus S_1 \subset S_3$. Moreover, we claim that

(1) $S_1 \subset S \subset S_2$, and
(2) $\eta(S_3) = 0$.

This proves $S \in \Sigma(\operatorname{ex} E_{\mathrm{r}\sigma})_\eta$ and, consequently, the measurability of h.

Ad (1). The relations are an easy consequence of

$$\sup_{n\in\mathbb{N}} \lambda(L_n) \leq \sup_{n\in\mathbb{N}} h(\lambda)(L_n) \leq h(\lambda)(A) \leq \inf_{n\in\mathbb{N}} h(\lambda)(U_n) \leq \inf_{n\in\mathbb{N}} \lambda(U_n),$$

where $\lambda \in \operatorname{ex} E$.

Ad (2). Since η represents μ, we have

$$\begin{aligned}\int_{\operatorname{ex} E} \inf_{n\in\mathbb{N}} \lambda(U_n \setminus L_n)\, \eta(d\lambda) &= \inf_{n\in\mathbb{N}} \int_{\operatorname{ex} E} \lambda(U_n \setminus L_n)\, \eta(d\lambda)\\ &= \inf_{n\in\mathbb{N}} \mu(U_n \setminus L_n) = 0.\end{aligned}$$

Therefore, $\eta(S_3) = 0$.

3. The image measure $\gamma := h(\eta)$ ([4]) is a probability on $\Sigma(\operatorname{ex} E_{\mathrm{r}\sigma})$. We claim that γ represents μ. Fix $A \in \mathcal{A}$ and $\varepsilon > 0$. Since μ is regular, there

([4]) We identify η with its completion on $\Sigma(\operatorname{ex} E)_\eta$.

are sets $L \in \mathcal{L}$ and $U \in \mathcal{L}_{\mathrm{c}}$ with $L \subset A \subset U$ and $\mu(U \setminus L) \le \varepsilon$. Then we get

$$\int\limits_{\operatorname{ex} E_{\mathrm{r}\sigma}} \beta(A)\,\gamma(d\beta) \ge \int\limits_{\operatorname{ex} E_{\mathrm{r}\sigma}} \beta(L)\,h(\eta)(d\beta) = \int\limits_{\operatorname{ex} E} h(\lambda)(L)\,\eta(d\lambda)$$
$$\ge \int\limits_{\operatorname{ex} E} \lambda(L)\,\eta(d\lambda) = \mu(L) \ge \mu(A) - \varepsilon$$

and, analogously, $\int_{\operatorname{ex} E_{\mathrm{r}\sigma}} \beta(A)\,\gamma(d\beta) \le \mu(U) \le \mu(A) + \varepsilon$. This completes the proof of Theorem 3.3. ∎

Relations between our results on measure extension and previous ones are discussed in the following

REMARK 3.4. (a) Condition $(*)$ in Proposition 3.1 is not new; it can be found, e.g., in a special situation in [Ad1, 3.14]. Therefore, Theorem 3.2(b) extends the first part of [Ad1, 3.14].

(b) Let $\mathcal{L}$ be a lattice, and let $\nu \in M(\mathcal{B})$. [Li, Theorem 1] states implicitly that there exists an extremal extension of ν to an $\mathcal{L}$-regular content on $\alpha(\mathcal{L} \cup \mathcal{B})$ provided ν is $\mathcal{L}$-*tight*, i.e.

$$\nu(B) = \sup\{\nu_*(L)\ (^5) \mid L \in \mathcal{L},\ L \subset B\} \quad \text{for all } B \in \mathcal{B}.$$

It is instructive to compare this result with Theorem 3.2(a):

Obviously, a content ν is $\mathcal{L}$-tight iff $\nu_*(P) = \sup\{\nu_*(L) \mid L \in \mathcal{L},\ L \subset P\}$ for all $P \subset X$ iff $\nu^*(P) = \inf\{\nu^*(U) \mid U \in \mathcal{L}_{\mathrm{c}},\ U \supset P\}$ for all $P \subset X$. Therefore, every $\mathcal{L}$-tight $\nu \in M(\mathcal{B})$ satisfies $(*)$, the general supposition of Theorem 3.2(a). On the other hand, 3.2(a) merely yields an $\mathcal{L}$-regular extension of ν provided $\mathcal{B} \subset \alpha(\mathcal{L})$.

Finally, let us note that for $\mathcal{B} \subset \alpha(\mathcal{L})$ a content on $\mathcal{B}$ with property $(*)$ need not be $\mathcal{L}$-tight. Example: Let $X = \{1,2,3\}$, $\mathcal{L} = \{\emptyset, X, \{1\}, \{1,3\}\}$, $\mathcal{B} = \{\emptyset, X, \{1,2\}, \{3\}\}$, and let ν be the Dirac measure concentrated at $1 \in X$ restricted to $\mathcal{B}$. Then $\mathcal{L}$ is a lattice with $\alpha(\mathcal{L}) \supset \mathcal{B}$, and ν satisfies $(*)$. Since we have $\nu(\{1,2\}) = 1 \ne 0 = \sup\{\nu_*(L) \mid L \in \mathcal{L}, L \subset \{1,2\}\}$, ν is not $\mathcal{L}$-tight.

(c) It is well known that extension problems in topological measure theory can often be reduced to the following abstract situation (see, e.g., [Ad2] and the references given there): Let $\mathcal{K}$, $\mathcal{L}$ be lattices with $\mathcal{K} \subset \mathcal{L}$, and let $\nu \in M(\alpha(\mathcal{K}))$ be $\mathcal{K}$-regular. Then ν is obviously $\mathcal{L}$-tight, and we gather from (b) that in the described situation ν satisfies the general supposition $(*)$ of our theorems. According to 3.2(a), e.g., we deduce that ν admits an extremal extension to an $\mathcal{L}$-regular content on $\alpha(\mathcal{L})$. This is exactly [Ad2, Theorem 2.3].

(d) Under condition $(*)$, Theorem 3.3(a) gives an affirmative answer to a natural question concerning the set $E_{\mathrm{r}}(\nu, \alpha(\mathcal{L}))$. Even in the case mentioned

(^{5}) ν_* denotes the inner content associated with ν.

in (c) this result has not been known so far. Since in general $E_{\mathrm{r}}(\nu, \alpha(\mathcal{L}))$ is not closed in $\mathrm{ba}(\alpha(\mathcal{L}))$, we cannot obtain 3.3(a) by a direct application of a general Choquet integral representation theorem. (The same is true for 3.3(b) or for Corollary 3.5(b).)

(e) Regard the following problem concerning preimage measures: Let $(Y, \mathcal{B}, \nu)$ be a finite measure space, $\mathcal{L}$ be a δ-lattice, and $p : X \to Y$ be a $(\sigma(\mathcal{L}), \mathcal{B})$-measurable map. When does there exist an extremal $\mathcal{L}$-regular measure μ on $\sigma(\mathcal{L})$ with $p(\mu) = \nu$? A straightforward generalization of our procedure in the proofs of Proposition 3.1 and Theorem 3.2 shows that we obtain an affirmative answer under the assumptions

- $\nu^*(p[X]) = \nu(Y)$ ([6]),
- $\nu^*(p\,[L]) = \inf\{\nu^*(p[U]) \mid U \in \mathcal{L}_{\mathrm{c}},\ U \supset L\}$ for all $L \in \mathcal{L}$,
- $\inf_{n\in\mathbb{N}} \nu^*(p[L_n]) = 0$ for every sequence $(L_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ with $L_n \downarrow \emptyset$.

(This result extends [Ad2, 3.4].) Moreover, under these conditions the set of all $\mathcal{L}$-regular preimage measures μ of ν (with respect to p) has IRP.

It is usual to give some topological applications of abstract theorems like 3.2 or 3.3 (see, e.g., [Ad2, §3]). We restrict ourselves to Corollary 3.5. Recall that a topological space is said to be *Baire dominated* if for every sequence $(F_n)_{n\in\mathbb{N}}$ of closed sets with $F_n \downarrow \emptyset$, there exists a sequence of Baire sets $(C_n)_{n\in\mathbb{N}}$ such that $F_n \subset C_n$ for all $n \in \mathbb{N}$ and $C_n \downarrow \emptyset$.

COROLLARY 3.5. *Let X be a Baire-dominated topological space, and let ν be a finite Baire measure in X. Then*

(a) *there exists an extremal extension of ν to a regular Borel measure*;
(b) *the set of all regular Borel measures that extend ν has IRP.*

Proof. Since every zero-set is closed, and since a Baire measure is regular with respect to the lattice of zero-sets, Remark 3.4(c) shows that ν satisfies $(*)$ of Proposition 3.1. As X is Baire dominated, ν^* restricted to the closed sets is σ-smooth at $\emptyset$. Hence, the statements (a) and (b) follow from 3.2(b), 3.3(b), respectively. ∎

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Mathematisches Institut
Universität München
Theresienstraße 39
D-80333 München, Germany
E-mail: rinkewit@rz.mathematik.uni-muenchen.de

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