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REMARKS AND EXAMPLES CONCERNING DISTANCE ELLIPSOIDS

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Abstract. We provide for every $2 \leq k \leq n$ an n-dimensional Banach space E with a unique distance ellipsoid $\mathcal{E}$ such that there are precisely k linearly independent contact points between $\mathcal{E}$ and B_E. The corresponding result holds for spaces with non-unique distance ellipsoids as well. We construct n-dimensional Banach spaces E such that one distance ellipsoid has precisely k linearly independent contact points and all other distance ellipsoids have less than $k-1$ such points.

1. Preliminaries & introduction. We consider finite-dimensional Banach spaces over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For technical reasons we always treat $\mathbb{K}^m$ for $m \leq n$ as a subspace of $\mathbb{K}^n$ embedded on the first m coordinates. Let $E = (\mathbb{K}^n, \|\cdot\|_E)$ be a Banach space and B_E be its (closed) unit ball. Given any compact, absolutely convex subset $B \subseteq \mathbb{K}^n$ having 0 as an interior point, the *Minkowski functional*

$$|x|_B := \inf\{\lambda > 0 \mid x \in \lambda B\} \quad (x \in \mathbb{K}^n)$$

is a norm on $\mathbb{K}^n$ with unit ball B, i.e., $B = B_E$ with $E = (\mathbb{K}^n, |\cdot|_B)$. For the $\mathbb{K}^n$ equipped with the ℓ_p-norm we denote the unit ball by $B_p^n := \{x \in \mathbb{K}^n \mid \|x\|_p \leq 1\}$. An *ellipsoid* $\mathcal{E}$ in $E = (\mathbb{K}^n, \|\cdot\|_E)$ is the image $u(B_2^n)$ of the Euclidean ball under an arbitrary isomorphism $u \in L(\ell_2^n, E)$. Thus, the Minkowski functional $|\cdot|_{\mathcal{E}}$ is a Hilbert norm and the corresponding scalar product is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{E}}$.

Well known and important examples of ellipsoids in Banach space theory are the *John ellipsoid* $\mathcal{D}_E^{\max}$ and the *Loewner ellipsoid* $\mathcal{D}_E^{\min}$ of a Banach space $E = (\mathbb{K}^n, \|\cdot\|_E)$ (see [DJT, T, Pi]): $\mathcal{D}_E^{\max}$ is the unique ellipsoid of maximal volume contained in the unit ball B_E of E and $\mathcal{D}_E^{\min}$ is the unique ellipsoid of minimal volume containing B_E, where we take the $2n$-dimensional Lebesgue measure in the case $\mathbb{K} = \mathbb{C}$. Both ellipsoids can be characterized geometrically: An ellipsoid $\mathcal{E}$ in E is the John ellipsoid $\mathcal{D}_E^{\max}$ (resp. the Loewner ellipsoid $\mathcal{D}_E^{\min}$) if and only if there are weights $d_1, \ldots, d_N > 0$ and

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so called *contact points* $x_1, \ldots, x_N \in E$ such that

(J1) $\sum_{j=1}^{N} d_j = n$,
(J2) $\|x_j\|_E = 1 = |x_j|_{\mathcal{E}}$ for all $1 \le j \le N$,
(J3) $x = \sum_{j=1}^{N} d_j \langle x, x_j \rangle_{\mathcal{E}} x_j$ for all $x \in E$,
(J4) $\mathcal{E} \subseteq B_E$ (resp. $B_E \subseteq \mathcal{E}$).

In this case one can choose $N \in \mathbb{N}$ with $N \le \frac{1}{2}n(n+1)$ for $\mathbb{K} = \mathbb{R}$ and $N \le n^2$ for $\mathbb{K} = \mathbb{C}$ and both estimates were shown to be sharp by Pełczyński and Tomczak-Jaegermann [PT].

Our aim was to study Banach spaces whose John and Loewner ellipsoids are homothetic, i.e., there is a scalar $d > 0$ with $\mathcal{D}_E^{\min} = d\,\mathcal{D}_E^{\max}$. In view of Lemma 2.3 this leads to the subsequently discussed distance ellipsoids.

Definition 1.1. For n-dimensional Banach spaces E and F we denote by

$$d(F, E) = \inf \|u\| \cdot \|u^{-1}\|$$

the *Banach–Mazur distance* where the infimum is taken over all isomorphisms $u \in L(F, E)$. By compactness the infimum is attained for some u with $\|u\| = 1$, $\|u^{-1}\| = d(F, E)$. In the case $F = \ell_2^n$ we write $d_E = d(\ell_2^n, E)$ for the *Euclidean* (*Banach–Mazur*) *distance*. One can describe d_E as the smallest positive d for which there exists an ellipsoid $\mathcal{E}$ with $\mathcal{E} \subseteq B_E \subseteq d\mathcal{E}$. Every ellipsoid for which both inclusions hold with $d = d_E$ is called a *distance ellipsoid*.

Clearly, for every n-dimensional Banach space E and every subspace F of E we have the lower estimate

$$d_F \le d_E. \tag{1.1}$$

The distance ellipsoid of the Hilbert space ℓ_2^n is obviously unique. Moreover, the aforementioned Lemma 2.3 shows that in case $\mathcal{D}_E^{\min} = d\,\mathcal{D}_E^{\max}$ the John ellipsoid is the unique distance ellipsoid and $d_E = d$. This holds in particular for spaces with enough symmetries [T, Sections 15, 16].

In general, distance ellipsoids are not unique. In Section 2 we will show how to construct spaces with non-unique distance ellipsoids. A theorem of Maurey [M] shows that such spaces contain proper subspaces with the same Euclidean distance. In Section 3 we construct a Banach space with a unique distance ellipsoid having this "Maurey property" and show that the spaces with non-unique distance ellipsoids are dense in the set of all such spaces.

In connection with the John and Loewner ellipsoids, it seemed to be of interest to study the geometric properties of distance ellipsoids. A theorem of Lewis [L] implies that there are at least two linearly independent contact points of $\mathcal{E}$ and B_E, where $\mathcal{E}$ is an arbitrary distance ellipsoid of E. As our main result, we show in Section 4 that the geometric properties of distance ellipsoids are much worse than those of the John and Loewner ellipsoids. We construct Banach spaces $E = (\mathbb{K}^n, \|\cdot\|_E)$ such that the distance ellipsoids and the unit ball B_E have precisely k linearly independent contact points, where $2 \le k \le n$ is arbitrary. In particular, there are n-dimensional Banach spaces such that the *best* distance ellipsoid has only 2 linearly independent contact points with the unit ball, i.e., Lewis' theorem is sharp.

2. Spaces with non-unique distance ellipsoids

THEOREM 2.1 (Maurey [M]). *If an n-dimensional Banach space E has two different distance ellipsoids one can find two distance ellipsoids $\mathcal{E}_1, \mathcal{E}_2$ with $\mathcal{E}_1 \subsetneq \mathcal{E}_2$.* ∎

LEMMA 2.2. *Let $2 \le m < n$ be integers and F an m-dimensional Banach space with $d_F > 1$. Assume without loss of generality that the Euclidean ball B_2^m is a distance ellipsoid of F and define an n-dimensional Banach space E by $B_E := \operatorname{abs\,conv}(B_2^n, B_F)$. Then F is a subspace of E, $d_E = d_F$, and B_2^n and $\mathcal{E}$ are distance ellipsoids of E with $\mathcal{E} \subsetneq B_2^n$, where*

$$\mathcal{E} := \Big\{ x \in \mathbb{K}^n \;\Big|\; \sum_{j=1}^{m} |x_j^2| + d_F^2 \sum_{j=m+1}^{n} |x_j^2| \le 1 \Big\}. \tag{2.1}$$

Proof. Since B_2^m is a distance ellipsoid of F we have $B_2^m \subseteq B_F \subseteq d_F B_2^m$. We infer that

$$B_E \cap \mathbb{K}^m = \operatorname{abs\,conv}(B_2^m, B_F) = B_F, \tag{2.2}$$

i.e., F is a subspace of E. In particular, we get $d_F \le d_E$ and the other estimate follows from $B_2^n \subseteq B_E \subseteq d_F B_2^n$, i.e., B_2^n is a distance ellipsoid of E. Because $\mathcal{E} \subsetneq B_2^n$ it remains to prove $B_E \subseteq d_F\mathcal{E}$. But this holds since $B_F \subseteq d_F B_2^m \subseteq d_F \mathcal{E}$ and $B_2^n \subseteq d_F\mathcal{E}$. ∎

LEMMA 2.3. *Let E be an n-dimensional Banach space whose John and Loewner ellipsoids are homothetic, i.e., $\mathcal{D}_E^{\min} = d\,\mathcal{D}_E^{\max}$ with a scalar $d \ge 1$. Then the John ellipsoid is the unique distance ellipsoid and the Euclidean distance is given by $d_E = d$.*

Proof. From the inclusions $\mathcal{E} \subseteq B_E \subseteq d_E\mathcal{E}$ we get

$$\begin{aligned} d^n \operatorname{vol}(\mathcal{E}) &\le d^n \operatorname{vol}(\mathcal{D}_E^{\max}) = \operatorname{vol}(d\mathcal{D}_E^{\max}) = \operatorname{vol}(\mathcal{D}_E^{\min}) \\ &\le \operatorname{vol}(d_E\mathcal{E}) = d_E^n \operatorname{vol}(\mathcal{E}), \end{aligned} \tag{2.3}$$

hence $d \le d_E$ (for $\mathbb{K} = \mathbb{R}$), and $\mathcal{D}_E^{\max} \subseteq B_E \subseteq \mathcal{D}_E^{\min} = d\mathcal{D}_E^{\max}$ gives the reverse estimate. Now, $d = d_E$ and (2.3) implies the equality $\operatorname{vol}(\mathcal{E}) = \operatorname{vol}(\mathcal{D}_E^{\max})$. Due to the uniqueness of the John ellipsoid, we get $\mathcal{E} = \mathcal{D}_E^{\max}$. Note that in the case $\mathbb{K} = \mathbb{C}$ formula (2.3) holds with exponent $2n$ instead of n. ∎

EXAMPLE 2.4. *Let $3 \le n$ be an integer, $1 < d \le \sqrt{2}$, and $E = (\mathbb{K}^n, \|\cdot\|_E)$ given by $B_E := \operatorname{abs\,conv}(B_2^n, \pm d e_1, \pm d e_2)$, where $e_j \in \mathbb{K}^n$ denote the unit vectors. Let $F := (\mathbb{K}^2, \|\cdot\|_E)$ denote the canonical 2-dimensional subspace and define the ellipsoids*

$$\mathcal{D} := d\,\mathcal{E}, \qquad \mathcal{E} := \Big\{ x \in \mathbb{K}^n \;\Big|\; |x_1^2| + |x_2^2| + d^2 \sum_{j=3}^{n} |x_j^2| \le 1 \Big\}. \tag{2.4}$$

Then the Banach space E has the following properties:

(i) $d_E = d = d_F$,

(ii) $\mathcal{E} \subsetneq B_2^n \subseteq B_E \subseteq d\mathcal{E} \subsetneq dB_2^n$, *i.e., $\mathcal{E}$ and the Euclidean ball B_2^n are distance ellipsoids*,

(iii) $\mathcal{D}_F^{\max} = B_2^2$, $\mathcal{D}_E^{\max} = B_2^n$,

(iv) $\mathcal{D}_F^{\min} = dB_2^2$, $\mathcal{D}_E^{\min} = \mathcal{D}$,

(v) $\dim_{\mathbb{K}} \operatorname{span}_{\mathbb{K}}\{x \in \mathbb{K}^n \mid |x|_{\mathcal{E}} = 1 = \|x\|_E\} = 2$,

(vi) $\dim_{\mathbb{K}} \operatorname{span}_{\mathbb{K}}\{x \in \mathbb{K}^n \mid \|x\|_2 = 1 = \|x\|_E\} = n$.

Proof. (iii) We use the fact that the John ellipsoid of ℓ_1^n is $(1/\sqrt{n})B_2^n$. From $B_2^n \subseteq B_E \subseteq \sqrt{n}\,B_1^n$ we infer that $\mathcal{D}_E^{\max} = B_2^n$. Also, direct calculation shows that $\mathcal{B} := \{w_1, w_2, e_3, \dots, e_n\}$ is a B_2^n-orthonormal basis satisfying $\|e\|_E = 1$ for $e \in \mathcal{B}$ where $w_j := (1/\sqrt{2})(e_1 \pm e_2)$ and e_j denote the standard unit vectors. Now one derives $B_2^n = \mathcal{D}_E^{\max}$ from the geometric characterization (J1)–(J4) of the John ellipsoid. The same argument shows $B_2^2 = \mathcal{D}_F^{\max}$.

(iv) The ellipsoid $\mathcal{D}$ is associated with the scalar product

$$\langle x, y\rangle_{\mathcal{D}} = \frac{x_1\overline{y}_1}{d^2} + \frac{x_2\overline{y}_2}{d^2} + \sum_{j=3}^{n} x_j\overline{y}_j \qquad (x, y \in \mathbb{K}^n). \tag{2.5}$$

Hence, $\mathcal{B} := \{de_1, de_2, e_3, \dots, e_4\} \subseteq B_E$ defines a $\mathcal{D}$-orthonormal basis. From $\mathcal{B} \subseteq B_E \subseteq \mathcal{D}$ we infer that $1 = |e|_{\mathcal{D}} \le \|e\|_E \le 1$ for all $e \in \mathcal{B}$. The geometric characterization of the Loewner ellipsoid implies $\mathcal{D} = \mathcal{D}_E^{\min}$ and analogously $dB_2^2 = \mathcal{D}_F^{\min}$.

(i) Lemma 2.3 implies $d = d_F$ and Lemma 2.2 shows (i) and (ii).

(v) Let $x \in \mathbb{K}^n$ with $|x|_{\mathcal{E}} = 1 = \|x\|_E$. From $\mathcal{E} \subsetneq B_2^n \subseteq B_E$ we infer that $\|x\|_2 = 1$, i.e.,

$$0 = |x|_{\mathcal{E}}^2 - \|x\|_2^2 = (d^2 - 1)\sum_{j=3}^{n} |x_j^2|, \tag{2.6}$$

whence $x \in \mathbb{K}^2$. The lower estimate of (v) and the equality in (vi) follow immediately from (iii) and the geometric characterization of the John ellipsoid. ∎

3. The Maurey property

DEFINITION 3.1. An n-dimensional Banach space E is said to have the *Maurey property* if it contains an $(n-1)$-dimensional subspace F with the same Euclidean distance, i.e., $d_E = d_F \leq \sqrt{n-1}$.

THEOREM 3.1 (Maurey [M]). *Every n-dimensional Banach space E with non-unique distance ellipsoids has the Maurey property.* ■

COROLLARY 3.2. (i) *The distance ellipsoid of an n-dimensional Banach space E is unique if $d_E > \sqrt{n-1}$. In particular, the distance ellipsoid of a 2-dimensional Banach space is unique.*

(ii) *If E is of maximal distance, i.e., $d_E = \sqrt{n}$, then the John and Loewner ellipsoids are homothetic, i.e., $\mathcal{D}_E^{\min} = \sqrt{n}\,\mathcal{D}_E^{\max}$, since both ellipsoids give the Euclidean distance.* ■

The converse of Maurey's theorem is obviously false, e.g., the n-dimensional Hilbert space $E = \ell_2^n$ has the Maurey property. Less trivial examples are given below by *almost rotation invariant spaces*. The next result shows that the Banach spaces with non-unique distance ellipsoids are dense in the Banach–Mazur distance in the set of spaces with the Maurey property.

PROPOSITION 3.3. *Let E be an n-dimensional Banach space with the Maurey property, $n \geq 3$. For every $\lambda > 1$ we can find a Banach space E' with non-unique distance ellipsoids and $d(E, E') \leq \lambda$. For $d_E > 1$, the construction leads to $d_E = d_{E'}$.*

Proof. We may assume that the Euclidean ball B_2^n is a distance ellipsoid of E and the subspace $F = (\mathbb{K}^{n-1}, \|\cdot\|_E)$ satisfies $d_E = d_F$. The case $d_E = 1$ is already proved in Example 2.4. We consider the case $d_E > 1$ and assume $1 < \lambda \leq d_E$. Define E' by

$$B_{E'} := \operatorname{abs\,conv}(B_2^n, B_F, (1/\lambda)B_E). \tag{3.1}$$

Obviously, we have $B_{E'} \subseteq B_E \subseteq \lambda B_{E'}$, hence $d(E, E') \leq \lambda$. Further, the construction implies $B_2^n \subseteq B_{E'} \subseteq d_E B_2^n$, whence $d_{E'} \leq d_E$, since the Euclidean balls are distance ellipsoids of E and F. Also, $B_{E'} \cap \mathbb{K}^{n-1} = B_F$ shows that F is a subspace of E' and Lemma 2.2 leads to $d_E = d_F \leq d_{E'}$. In particular, the Euclidean ball B_2^n is a distance ellipsoid of E'. Define another ellipsoid $\mathcal{E} \subsetneq B_2^n$ by

$$\mathcal{E} := \left\{ x \in \mathbb{K}^n \;\middle|\; \sum_{j=1}^{n-1} |x_j^2| + \lambda |x_n^2| \leq 1 \right\}. \tag{3.2}$$

To see that $\mathcal{E}$ is a distance ellipsoid of E' it remains to prove the inclusion $B_{E'} \subseteq d_E\mathcal{E}$. But this follows immediately from $B_F \subseteq d_E B_2^{n-1} \subseteq d_E\mathcal{E}$, and

$$B_2^n \subseteq d_E\mathcal{E} \quad \text{and} \quad (1/\lambda)B_E \subseteq (d_E/\lambda)B_2^n \subseteq d_E\mathcal{E}, \tag{3.3}$$

where we have used $\lambda/d_E \leq 1$. ■

DEFINITION 3.2. (i) An n-dimensional Banach space E is *sign invariant* if the diagonal operators $\varDelta := \operatorname{diag}(\varepsilon)$, $\varepsilon \in \{\pm 1\}^n$, are isometries of E.

(ii) Further, E is *almost rotation invariant* if the norm $\|\cdot\|_E$ is invariant in the last $n-1$ components under unitary matrices, i.e.,

$$\|x\|_E = \|Ux\|_E \quad \text{for all } x \in \mathbb{K}^n \text{ and unitary matrices } U \in \mathbb{K}^{n\times n} \text{ with } Ue_1 = e_1. \tag{3.4}$$

Geometrically, the ball B_E is determined by its 2-dimensional section $B_F = B_E \cap \mathbb{K}^2$ and given by rotation around the x_1-axis.

PROPOSITION 3.4. *The distance ellipsoid $\mathcal{E}$ of an almost rotation invariant Banach space $E = (\mathbb{K}^n, \|\cdot\|_E)$ is unique. Let $F' = (\mathbb{K}^2, \|\cdot\|_E)$ and $F = (\mathbb{R}^2, \|\cdot\|_E)$. Then the unique distance ellipsoid $\mathcal{E}_F$ of F is given by*

$$\mathcal{E}_F = \{x \in \mathbb{R}^2 \mid x_1^2/a_1^2 + x_2^2/a_2^2 \le 1\} \tag{3.5}$$

for some scalars $a_1, a_2 > 0$, the unique distance ellipsoid of E is

$$\mathcal{E} = \Big\{x \in \mathbb{K}^n \,\Big|\, |x_1^2|/a_1^2 + \sum_{j=2}^{n} |x_j^2|/a_2^2 \le 1\Big\}, \tag{3.6}$$

and the unique distance ellipsoid of F' is just the intersection $\mathcal{E} \cap \mathbb{K}^2$. Moreover, the Euclidean distances of E, F' and the (real) Euclidean distance of F are equal, i.e., $d_E = d_{F'} = d_F$. In particular, E has the Maurey property.

The proof uses the following lemma, whose elementary proof is left to the reader.

LEMMA 3.5. *Let $\mathcal{E} \subseteq \mathbb{K}^n$ be an ellipsoid which is sign invariant, i.e., $\varDelta(\mathcal{E}) = \mathcal{E}$ for all $\varepsilon \in \{\pm 1\}^n$, $\varDelta := \operatorname{diag}(\varepsilon)$. Then $\mathcal{E} = \{x \in \mathbb{K}^n \mid \sum_{j=1}^n |x_j^2|/a_j^2 \le 1\}$ for some suitable $a_j > 0$.* ■

Proof of Proposition 3.4. Notice that the norm $\|\cdot\|_E$ is sign invariant. By Lemma 3.5, the unique distance ellipsoid of the 2-dimensional subspace F' is given by

$$\mathcal{E}_{F'} = \{x \in \mathbb{K}^2 \mid |x_1^2|/b_1^2 + |x_2^2|/b_2^2 \le 1\} \tag{3.7}$$

for some $b_j > 0$. The same argument shows that the unique distance ellipsoid $\mathcal{E}_F$ of F is as in (3.5). First, we see that $\mathcal{D} := \mathcal{E}_{F'} \cap \mathbb{R}^2$ is an ellipsoid in $\mathbb{R}^2$ with $\mathcal{D} \subseteq B_F \subseteq d_{F'}\mathcal{D}$, hence $d_F \le d_{F'}$. Further, we define an ellipsoid $\mathcal{D}'$ in $\mathbb{K}^2$ extending $\mathcal{E}_F$ and get $\mathcal{D}' \subseteq B_{F'} \subseteq d_F\mathcal{D}'$. Thus, $d_F = d_{F'}$ and the uniqueness of the distance ellipsoid yields $\mathcal{E}_F = \mathcal{D}$ and $\mathcal{E}_{F'} = \mathcal{D}'$, i.e., $a_j = b_j$. Since F' is a subspace of E we get $d_F = d_{F'} \le d_E$. On the other hand, $\mathcal{E}$ from (3.6) satisfies $\mathcal{E} \subseteq B_E \subseteq d_{F'}\mathcal{E}$ since E and $\mathcal{E}$ are almost rotation invariant. Hence, $d_E = d_F$ and $\mathcal{E}$ is a distance ellipsoid of E.

To see that $\mathcal{E}$ is unique we fix a distance ellipsoid $\mathcal{E}'$ of E. For every unitary matrix $U \in \mathbb{K}^{n\times n}$ with $Ue_1 = e_1$ we consider the 2-dimensional subspace $F'_U := \mathrm{span}_{\mathbb{K}}\{e_1, Ue_2\}$. Since E is almost rotation invariant, U is an isometry from F' onto F'_U. Thus, $U(\mathcal{E}_{F'}) = \mathcal{E} \cap F'_U = \mathcal{E}' \cap F'_U$ since all give the unique distance ellipsoid of F'_U. This leads to $\mathcal{E} = \mathcal{E}'$. ∎

EXAMPLE 3.6. *Let* $n \geq 2$, $1 \leq \lambda$, *and* $E = (\mathbb{K}^n, \|\cdot\|_E)$ *with* $B_E := \mathrm{abs\,conv}(B_2^n, \pm\lambda e_1)$. *Then E is an almost rotation invariant Banach space with John ellipsoid* $\mathcal{D}_E^{\max} = B_2^n$ *and Loewner ellipsoid* $\mathcal{D}_E^{\min} = \{x \in \mathbb{K}^n \mid |x_1^2|/\lambda^2 + \sum_{j=2}^n |x_j^2| \leq 1\}$. *The Euclidean distance is* $d_E = \sqrt{2 - 1/\lambda^2}$ *and* $(1/d_E)\mathcal{D}_E^{\min}$ *is the unique distance ellipsoid.*

Proof. To determine $\mathcal{D}_E^{\max}$ and $\mathcal{D}_E^{\min}$ one can use the same argument as in Example 2.4. To determine the distance ellipsoid $\mathcal{E}$ of E we use Proposition 3.4 and get (3.6). Moreover, $\lambda e_1, e_2, \ldots, e_n \in B_E \subseteq d_E\,\mathcal{E}$ yields $\mathcal{D}_E^{\min} \subseteq d_E\,\mathcal{E}$. Thus $(1/d_E)\mathcal{D}_E^{\min} \subseteq \mathcal{E} \subseteq B_E \subseteq \mathcal{D}_E^{\min}$ and this shows that $\mathcal{D}_E^{\min}$ is the unique distance ellipsoid of E. The calculation of d_E is elementary since we only have to consider the 2-dimensional *real* case. ∎

REMARK 3.1. Lemma 2.3 is optimal in the following sense: Example 2.4 gives a Banach space whose Euclidean distance is attained for the John *and* Loewner ellipsoids although the two ellipsoids are not homothetic, i.e.,

$$(1/d_E)\mathcal{D}_E^{\min} \subsetneq \mathcal{D}_E^{\max} \subseteq B_E \subseteq \mathcal{D}_E^{\min} \subsetneq d_E\mathcal{D}_E^{\max}. \tag{3.8}$$

Example 3.6 provides a Banach E space whose unique distance ellipsoid is the Loewner ellipsoid, although the John and Loewner ellipsoids are not homothetic. By duality the unique distance ellipsoid of the dual space E^* is its John ellipsoid and the two ellipsoids, i.e., the John and Loewner ellipsoids, are not homothetic.

REMARK 3.2. In general, a result analogous to Proposition 3.4 does not hold even for the John or Loewner ellipsoid: Consider the ellipsoid $\mathcal{D} := \{x \in \mathbb{R}^n \mid x_1^2/a_1^2 + \sum_{j=1}^n x_j^2/a_2^2 \leq 1\}$ and the real Banach space $E = (\mathbb{R}^n, \|\cdot\|_E)$ given by $B_E := \mathrm{abs\,conv}(B_2^n, \mathcal{D})$ with $a_1 := \sqrt{9/10}$, $a_2 := \sqrt{11/10}$, $n \geq 3$. An elementary calculation for the subspace $F = (\mathbb{R}^2, \|\cdot\|_E)$ shows $\mathcal{D}_F^{\max} = B_2^2$. In spite of this $\mathrm{vol}(\mathcal{D}) > \mathrm{vol}(B_2^n)$, hence $\mathcal{D}_E^{\max} \neq B_2^n$ although E is almost rotation invariant. A counterexample for the Loewner ellipsoid follows by duality.

4. Lewis' Theorem about contact points

THEOREM 4.1 (Lewis [L]). *Let E and F be n-dimensional Banach spaces and $u \in L(F, E)$ be an isomorphism with $\|u\| = 1$, $\|u^{-1}\| = d(F, E)$. Then*

$$\dim_{\mathbb{K}} \mathrm{span}_{\mathbb{K}}\{x \in \mathbb{K}^n \mid \|x\|_F = 1 = \|ux\|_E\} \geq 2. \tag{4.1}$$ ∎

COROLLARY 4.2. *For any n-dimensional Banach space E with distance ellipsoid $\mathcal{E}$, we have*

$$\dim_{\mathbb{K}} \operatorname{span}_{\mathbb{K}}\{x \in \mathbb{K}^n \mid |x|_{\mathcal{E}} = 1 = \|x\|_E\} \geq 2. \blacksquare$$

REMARK 4.1. Corollary 4.2 in combination with Theorem 2.1 of Maurey gives another proof that the distance ellipsoid of a 2-dimensional Banach space is unique.

THEOREM 4.3. *For every $2 \leq k < n$ and $1 < d \leq \sqrt{k}$, there are n-dimensional Banach spaces $E_k^n = (\mathbb{K}^n, \|\cdot\|_E)$, $\widehat{E}_k^n = (\mathbb{K}^n, \|\cdot\|_{\widehat{E}})$ such that*

(a) *the Euclidean ball B_2^n is the unique distance ellipsoid of E_k^n and*

$$\dim_{\mathbb{K}} \operatorname{span}_{\mathbb{K}}\{x \in \mathbb{K}^n \mid \|x\|_2 = 1 = \|x\|_E\} = k, \tag{4.2}$$

(b) *the distance ellipsoid of $\widehat{E}_k^n$ is not unique, B_2^n is a distance ellipsoid which satisfies*

$$\dim_{\mathbb{K}} \operatorname{span}_{\mathbb{K}}\{x \in \mathbb{K}^n \mid \|x\|_2 = 1 = \|x\|_{\widehat{E}}\} = k, \tag{4.3}$$

and $\mathcal{E} \subseteq B_2^n$ for any other distance ellipsoid of $\widehat{E}_k^n$. Therefore,

$$\dim_{\mathbb{K}} \operatorname{span}_{\mathbb{K}}\{x \in \mathbb{K}^n \mid |x|_{\mathcal{E}} = 1 = \|x\|_{\widehat{E}}\} < k \tag{4.4}$$

for every distance ellipsoid $\mathcal{E} \neq B_2^n$ of $\widehat{E}_k^n$. Also, the explicit construction of E_k^n and $\widehat{E}_k^n$ leads to the following properties:

(i) $d_{E_k^n} = d = d_{\widehat{E}_k^n}$,

(ii) E_k^n, $\widehat{E}_k^n$ *are sign invariant,*

(iii) E_k^n *and* $\widehat{E}_k^n$ *have the Maurey property.*

We give the proof by explicit construction of the unit balls of E_k^n and $\widehat{E}_k^n$. This is done in several steps.

ASSERTION 1. *Define the concave function $f : [-1, 1] \to [0, 1]$, $f(t) := \sqrt{1 - t^2} + \alpha t^2$ with some fixed $\alpha > 0$ satisfying $1 + \alpha \leq \min\{\sqrt{2}, d\}$. Set*

$$A_0 := \operatorname{abs\,conv}\{(f(t), t), (-f(t), t) \mid -1 \leq t \leq 1\}.$$

Then $E_0 := (\mathbb{R}^2, |\cdot|_{A_0})$ defines a 2-dimensional sign invariant Banach space with

(i) $B_2^2 \subseteq A_0 \subseteq dB_2^2$, $A_0 \subseteq B_\infty^2$,

(ii) $\mathcal{K}_0 := \{x \in \mathbb{R}^2 \mid \|x\|_2 = 1 = |x|_{A_0}\} = \{\pm e_1, \pm e_2\}$,

(iii) *every ellipsoid $\mathcal{E} \subseteq A_0$ with $\pm e_1 \in \mathcal{E}$ can be written as*

$$\mathcal{E} = \mathcal{E}_\lambda := \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2/\lambda^2 \leq 1\} \tag{4.5}$$

for a suitable scalar $0 < \lambda \leq 1$,

(iv) *for $\beta := \min\{\sqrt{3/2}, \sqrt{(d^2+1)/2}\} > 1$, $\alpha = \beta - 1$, and $\lambda = 1/\beta$ we have*

$$\mathcal{E}_\lambda \subsetneq B_2^2 \subseteq A_0 \subseteq d\mathcal{E}_\lambda \subsetneq dB_2^2. \tag{4.6}$$

Proof. From $\alpha \le \sqrt{2}-1$ we infer that A_0 is closed and absolutely convex and (i)–(iv) follow by direct calculation using the fact that $\alpha \le d-1$. ∎

ASSERTION 2. *$A_{1,\beta} := \operatorname{abs\,conv}(A_0, \beta e_2)$, $1 < \beta \le d$, $E_{1,\beta} := (\mathbb{R}^2, |\cdot|_{A_{1,\beta}})$ defines a sign invariant Banach space which shares properties* (i), (iii), *and* (iv) *with E_0, but only $\pm e_1$ are contact points between $A_{1,\beta}$ and B_2^2, i.e., $\mathcal{K}_1 := \{x \in \mathbb{R}^2 \mid \|x\|_2 = 1 = |x|_{A_{1,\beta}}\} = \{\pm e_1\}$.* ∎

REMARK 4.2. For the construction of E_k^n we will use arbitrary $1 < 1+\alpha \le \min\{\sqrt{2}, d\}$ and $\beta = d$, whereas $\widehat{E}_k^n$ is constructed with $\beta := \min\{\sqrt{3/2}, \sqrt{(d^2+1)/2}\}$, $\alpha = \beta - 1$. See Assertions 6 and 7 below.

We shall use the following obvious lemma to obtain the next assertion.

LEMMA 4.4. *Let $F = (\mathbb{R}^2, \|\cdot\|_F)$ be a sign invariant real Banach space and define*

$$E := (\mathbb{K}^n, \|\cdot\|_E) \quad \textit{via} \quad \|x\|_E := \|(|x_1|, \|(x_j)_{j=2}^n\|_2)\|_F \quad (x \in \mathbb{K}^n). \tag{4.7}$$

Then:

(i) *E is an almost rotation invariant Banach space and the real subspace $\operatorname{span}_{\mathbb{R}}\{e_1, e\}$ with arbitrary $e \in \operatorname{span}_{\mathbb{R}}\{e_2, \dots, e_n\}$ is isometric to F (via rotation)*,

(ii) *if for some $\lambda > 0$ the ellipsoid $\mathcal{E}_\lambda$ from* (4.5) *satisfies $\mathcal{E}_\lambda \subseteq B_F \subseteq d\mathcal{E}_\lambda$, then $\mathcal{E}'_\lambda \subseteq B_E \subseteq d\mathcal{E}'_\lambda$, where*

$$\mathcal{E}'_\lambda := \left\{x \in \mathbb{K}^n \,\middle|\, |x_1^2| + \frac{1}{\lambda^2}\sum_{j=2}^n |x_j^2|\right\}. \tag{4.8}$$ ∎

ASSERTION 3. *Define $E_{2,\beta} = (\mathbb{K}^n, \|\cdot\|_{E_2})$ as in Lemma 4.4 with $F = E_{1,\beta}$. Denote by $A_{2,\beta}$ the ball of $E_{2,\beta}$. Then the Banach space $E_{2,\beta}$ is sign invariant and has the following properties*:

(i) $B_2^n \subseteq A_{2,\beta} \subseteq dB_2^n$,

(ii) $\mathcal{K}_2 := \{x \in \mathbb{K}^n \mid \|x\|_2 = 1 = |x|_{A_{2,\beta}}\} = \{\lambda e_1 \mid \lambda \in S_{\mathbb{K}}\}$,

(iii) *for any $e \in \operatorname{span}_{\mathbb{R}}\{e_2, \dots, e_n\}$, the real subspace $\operatorname{span}_{\mathbb{R}}\{e_1, e\}$ is isometric to $E_{1,\beta}$.* ∎

The next step will use the following simple interpolation lemma, whose proof is left to the reader.

LEMMA 4.5. *Let $E = (\mathbb{K}^m, \|\cdot\|_E)$ and $F = (\mathbb{K}^n, \|\cdot\|_F)$ be finite-dimensional Banach spaces with $m \le n$ and $X = (\mathbb{K}^m, \|\cdot\|_X)$ and $Y = (\mathbb{K}^n, \|\cdot\|_Y)$ defined by $B_X := B_E \cap B_F$ and $B_Y := \operatorname{conv}(B_E, B_F)$. Then the norms of X resp. Y are given by*

$$\|x\|_X = \max\{\|x\|_E, \|x\|_F\} \quad (x \in \mathbb{K}^m), \tag{4.9}$$

$$\|y\|_Y = \min\{\|e\|_E + \|f\|_F \mid e \in \mathbb{K}^m,\ f \in \mathbb{K}^n,\ y = e + f\} \quad (y \in \mathbb{K}^n). \tag{4.10}$$ ∎

Assertion 4. *$A_{3,\beta} := A_{2,\beta} \cap (B_2^k \times dB_2^{n-k})$ yields a sign invariant Banach space $E_{3,\beta} := (\mathbb{K}^n, |\cdot|_{A_{3,\beta}})$ with the following properties*:

(i) $B_2^n \subseteq A_{3,\beta} \subseteq dB_2^n$, $A_{3,\beta} \cap \mathbb{K}^k = B_2^k$,

(ii) $\mathcal{K}_3 := \{x \in \mathbb{K}^n \mid \|x\|_2 = 1 = |x|_{A_{3,\beta}}\} = B_2^k$,

(iii) *the real subspaces* $\operatorname{span}_{\mathbb{R}}\{e_1, e\}$ *with* $e \in \operatorname{span}_{\mathbb{R}}\{e_{k+1}, \dots, e_n\}$ *are isometric to* $E_{1,\beta}$.

Proof. Obviously, $A_{3,\beta}$ is an absolutely convex, compact set which satisfies (i) and (iii). Thus, $|\cdot|_{A_{3,\beta}}$ is a norm and $B_2^k \subseteq \mathcal{K}_3$. Simple interpolation yields

$$|x|_{A_{3,\beta}} = \max\{|x|_{A_{2,\beta}}, \|(x_j)_{j=1}^k\|_2, \|x\|_\infty/d\} \quad (x \in \mathbb{K}^n). \tag{4.11}$$

To get the reverse inclusion in (ii) let $x \in \mathbb{K}^n$ with $|x|_{A_{3,\beta}} = 1 = \|x\|_2$. In particular, $|x_j| \leq 1$ for all $1 \leq j \leq n$, whence $\|x\|_\infty/d < 1$. We consider the other two cases. If $|x|_{A_{2,\beta}} = 1$ Assertion 3 implies $x = \lambda e_1$ with a suitable $\lambda \in S_{\mathbb{K}}$. Otherwise, $\|(x_j)_{j=1}^k\|_2 = 1$ leads to $x_j = 0$ for $k+1 \leq j \leq n$. ∎

Due to (i) the Banach space $E_{3,\beta}$ has Euclidean distance $d_{E_{3,\beta}} \leq d$. The easiest way to get equality is to embed a subspace F with distance $d_F = d$. This is the last step of our construction.

Assertion 5. *Define*

$$A_{4,\beta} := \operatorname{abs\,conv}(A_{3,\beta}, B) \quad \textit{with} \quad B := \operatorname{abs\,conv}\left(B_2^k, \frac{d}{\sqrt{k}}\, B_\infty^k\right).$$

Then $E_{k,\beta}^n := (\mathbb{K}^n, |\cdot|_{A_{4,\beta}})$ is a sign invariant Banach space with the following properties:

(i) $d_{E_{k,\beta}^n} = d$ *and the Euclidean ball is a distance ellipsoid of* $E_{k,\beta}^n$,

(ii) *every distance ellipsoid $\mathcal{E}$ is contained in B_2^n, i.e., $\mathcal{E} \subseteq B_2^n$, and satisfies $\mathcal{E} \cap \mathbb{K}^k = B_2^k$*,

(iii) $\mathcal{K}_4 := \{x \in \mathbb{K}^n \mid \|x\|_2 = 1 = |x|_{A_{4,\beta}}\} \subseteq B_2^k$, $\dim_{\mathbb{K}} \operatorname{span}_{\mathbb{K}} \mathcal{K}_4 = k$,

(iv) *the real subspace* $\operatorname{span}_{\mathbb{R}}\{e_1, e\}$ *with* $e \in \operatorname{span}_{\mathbb{R}}\{e_{k+1}, \dots, e_n\}$ *is isometric to* $E_{1,\beta}$,

(v) $E_{k,\beta}^n$ *has the Maurey property, since* $d_{E_{k,\beta}^n} = d = d_F$ *with* $F := (\mathbb{K}^k, \|\cdot\|_E) = (\mathbb{K}^k, |\cdot|_B)$.

Proof. For abbreviation we write $E = E_{k,\beta}^n$. Again, it is trivial that $B_E = A_{4,\beta}$ is a sign invariant Banach space with $B_2^n \subseteq B_E \subseteq dB_2^n$ and $B_E \cap \mathbb{K}^k = B \subseteq B_\infty^k$. In particular, $F := (\mathbb{K}^k, \|\cdot\|_E) = (\mathbb{K}^k, |\cdot|_B)$ has enough symmetries and $d_F \leq d_E \leq d$. $\mathcal{D}_F^{\max}$ is the unique distance ellipsoid of F. From $B_2^k \subseteq B \subseteq B_\infty^k$ we infer that $\mathcal{D}_F^{\max} = B_2^k$. Hence we deduce that $d_F = d$ and obtain the equality $d = d_F = d_E$. In particular, the uniqueness of the distance ellipsoid of F implies $\mathcal{E} \cap \mathbb{K}^k = B_2^k$ for every distance ellipsoid $\mathcal{E}$ of E. Up to now, (i), (v), and part of (ii) are shown. The geometric characterization of the John ellipsoid $\mathcal{D}_F^{\max}$ implies $\mathbb{K}^k \subseteq \operatorname{span}_{\mathbb{K}} \mathcal{K}_4$ and together with $B_2^n \subseteq A_{3,\beta} \subseteq B_E$, hence $\mathcal{K}_4 \subseteq \mathcal{K}_3 = B_2^k$, we obtain (iii).

For the remaining part of (ii) we apply induction on n and $2 \leq k < n$. To start the induction we consider arbitrary $n \geq 3$ and $k = n-1$. Let $\mathcal{E}$ be a distance ellipsoid of E. We have already seen that $\mathcal{E} \cap \mathbb{K}^{n-1} = B_2^{n-1}$. Thus, we can find a vector $w \in \mathbb{K}^n$ such that $\{e_1, \ldots, e_{n-1}, w\}$ is an $\mathcal{E}$-orthonormal basis. To see that w is a multiple of e_n we assume that $w_j \neq 0$ for some $1 \leq j \leq n-1$. Without loss of generality let $w_1 > 0$, $0 \leq \mu \leq 4w_1/(w_1^2+4)$, and $\nu := 1 - \mu w_1/2 > 1 - \mu w_1 > 0$. By the choice of μ we infer that

$$\nu^2 + \mu^2 = 1 - \mu w_1 + \frac{1}{4}\mu^2 w_1^2 + \mu^2 = 1 + \frac{\mu}{4}\underbrace{\left(-4w_1 + \mu(w_1^2+4)\right)}_{\leq 0} \leq 1. \tag{4.12}$$

Hence, the vector $x := \nu e_1 + \mu w \in \mathbb{K}^n$ satisfies $|x|_{\mathcal{E}}^2 = \nu^2 + \mu^2 \leq 1$, i.e., $x \in \mathcal{E}$, and $x_1 = \nu + \mu w_1 > 1$. But our construction of $B_E = A_{4,\beta}$ leads to $B_E \cap \mathbb{K}^k \subseteq B_\infty^k$, whence $|x_j| \leq 1$. This contradiction implies $w = \lambda e_n$ for a suitable scalar $\lambda \neq 0$ and from Parseval's equality we infer that

$$\mathcal{E} = \{x \in \mathbb{K}^n \mid |x|_{\mathcal{E}}^2 \leq 1\} = \left\{x \in \mathbb{K}^3 \,\middle|\, |x_n^2|/|\lambda^2| + \sum_{j=1}^{n-1} |x_j^2| \leq 1\right\}. \tag{4.13}$$

We can assume that $\lambda > 0$. Thus, $\mathcal{E} \cap \operatorname{span}_{\mathbb{R}}\{e_1, e_n\}$ is an ellipsoid in the real space $\operatorname{span}_{\mathbb{R}}\{e_1, e_n\}$, which is isometric to $E_{1,\beta}$. Hence, λ must satisfy $0 < \lambda \leq 1$ by Assertion 2 and Assertion 1(iii). This leads to $\mathcal{E} \subseteq B_2^n$.

Now consider arbitrary $n \geq 3$ and $2 \leq k < n$. Since the case $k = n-1$ has already been shown, we may assume $2 \leq k \leq n-2$. Let $X := \operatorname{span}_{\mathbb{K}}\{e_{n-1}, e_n\}$ and $F_x := \operatorname{span}_{\mathbb{K}}\{e_1, \ldots, e_{n-2}, x\}$ for fixed $x \in X$. Notice that by construction of $B_E = A_{4,\beta}$ we have $F_{e_{n-1}} = E_{k,\beta}^{n-1}$ and that all subspaces F_x are isometric to $F_{e_{n-1}}$ via a rotation of the last two coordinates. Hence, the induction hypothesis shows that for fixed $x \in F_x$ every distance ellipsoid $\mathcal{E}$ of E satisfies

$$\mathcal{E} \cap F_x \subseteq B_2^n \cap F_x \subseteq B_2^n, \tag{4.14}$$

since $\mathcal{E} \cap F_x$ is a distance ellipsoid of F_x. Let $e \in \mathcal{E}$ be written as $e = \lambda x + \mu y$ with suitable $x \in X$, $y \in \mathbb{K}^{n-2}$ and scalars $\lambda, \mu \in \mathbb{K}$. Then $e \in \mathcal{E} \cap F_x \subseteq B_2^n$. ∎

Assertion 6. *For arbitrary* $1 < 1 + \alpha \leq \min\{\sqrt{2}, d\}$ *and* $\beta = d$ *the Banach space* $E_k^n = E_{k,\beta}^n$ *satisfies the claim of Theorem* 4.3.

Proof. It only remains to prove that the Euclidean ball B_2^n is the unique distance ellipsoid of E_k^n. Let $\mathcal{E}$ be another distance ellipsoid. We already know that $\mathcal{E} \subseteq B_2^n \subseteq B_{E_k^n} \subseteq d\mathcal{E}$ and $\mathcal{E} \cap \mathbb{K}^k = B_2^k$. Further, $de_j \in B_{E_k^n}$ for $k+1 \leq j \leq n$ by construction, whence $e_j \in \mathcal{E}$. Thus, the n linearly independent contact points of $\mathcal{E}$ and B_2^n together with the one-way inclusion imply the equality $\mathcal{E} = B_2^n$. ∎

ASSERTION 7. *For* $\beta := \min\{\sqrt{3/2}, \sqrt{(d^2+1)/2}\}$ *and* $\alpha = \beta - 1$ *the Banach space* $\widehat{E}_k^n := E_{k,\beta}^n$ *satisfies the second part of Theorem* 4.3.

Proof. Define the ellipsoid

$$\mathcal{E} := \left\{ x \in \mathbb{K}^n \;\middle|\; \sum_{j=1}^{k} |x_j^2| + \frac{1}{\lambda^2} \sum_{j=k+1}^{n} |x_j^2| \le 1 \right\} \tag{4.15}$$

with $\lambda = 1/\beta$. Since $\mathcal{E} \subsetneq B_2^n \subseteq B_{\widehat{E}_k^n}$ it only remains to show $B_{\widehat{E}_k^n} \subseteq d\mathcal{E}$. Assertion 2 states

$$A_{1,\beta} \subseteq d\,\{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2/\lambda^2 \le 1\}$$

and therefore via rotation

$$A_{2,\beta} \subseteq d \left\{ x \in \mathbb{K}^n \;\middle|\; |x_1^2| + \frac{1}{\lambda^2} \sum_{j=2}^{n} |x_j^2| \le 1 \right\} \subseteq d\mathcal{E}.$$

From $A_{3,\beta} = A_{2,\beta} \cap (B_2^k \times dB_\infty^{n-k})$ we infer that

$$A_{3,\beta} \subseteq A_{2,\beta} \subseteq d\mathcal{E}.$$

Due to $B \subseteq dB_2^k \subseteq d\mathcal{E}$ with B defined in Assertion 5, we conclude that

$$A_{4,\beta} = \operatorname{abs\,conv}(A_{3,\beta}, B) \subseteq d\mathcal{E}.$$

Thus, $\mathcal{E} \neq B_2^n$ is another distance ellipsoid of $\widehat{E}_k^n$. ∎

REMARK 4.3. Theorem 4.3 shows that Lewis' Theorem 4.1 is optimal. Moreover, for $k = n$ there are also spaces E_k^n resp. $\widehat{E}_k^n$ satisfying (4.2) resp. (4.3). Consider, e.g., Banach spaces with enough symmetries resp. the space provided by Example 2.4.

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