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THE NATURAL OPERATORS $T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$ AND $T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2 T^*T^{r*}$

BY

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Abstract. Let r and n be natural numbers. For $n \geq 2$ all natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$ transforming vector fields on n-manifolds M to 1-forms on $T^{r*}M = J^r(M,\mathbb{R})_0$ are classified. For $n \geq 3$ all natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2 T^*T^{r*}$ transforming vector fields on n-manifolds M to 2-forms on $T^{r*}M$ are completely described.

0. Introduction. Let n and r be natural numbers. In this paper we study the problem how a vector field X on a n-dimensional manifold M can induce a 1-form $A(X)$ and a 2-form $B(X)$ on the r-cotangent bundle $T^{r*}M = J^r(M,\mathbb{R})_0$ of M. This problem is reflected in the concept of natural operators $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$ and $B : T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2 T^*T^{r*}$ in the sense of Kolář, Michor and Slovák [4].

The first main result of this paper is that for $n \geq 2$ the set of all natural operators $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$ is a free $2r$-dimensional $\mathcal{C}^\infty(\mathbb{R}^r)$-module, and we construct explicitly a basis of this module.

The second main result is that for $n \geq 3$ the set of all natural operators $B : T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2 T^*T^{r*}$ is a free $2r^2$-dimensional $\mathcal{C}^\infty(\mathbb{R}^r)$-module, and we also construct explicitly a basis of this module.

Some natural operators transforming functions, vector fields, forms (etc.) on some natural bundles F are used practically in all papers in which the problem of prolongation of geometric structures is considered. That is why such natural operators are studied. For $F = T^{r*}$ such natural operators are studied or classified in [2], [3], [5], [6], [8], [9], and for $F = T^{1*} = T^*$ in [1], [7], [11].

From now on $x^1, \ldots, x^n$ denote the usual coordinates on $\mathbb{R}^n$, and $\partial_i = \partial/\partial x^i$ for $i = 1, \ldots, n$ are the canonical vector fields on $\mathbb{R}^n$.

All manifolds are assumed to be finite-dimensional and smooth, i.e. of class $\mathcal{C}^\infty$. Maps between manifolds are assumed to be smooth.

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1. The natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$

1.1. *The r-cotangent bundle T^{r*}.* For every n-dimensional manifold M we have the vector bundle $T^{r*}M = J^r(M,\mathbb{R})_0$ over M with respect to the source projection $\pi : T^{r*}M \to M$. It is called the *r-cotangent bundle* of M. Every embedding $\varphi : M \to N$ of n-manifolds induces a vector bundle map $T^{r*}\varphi : T^{r*}M \to T^{r*}N$, $T^{r*}\varphi(j^r_x\gamma) = j^r_{\varphi(x)}(\gamma\circ\varphi^{-1})$, $\gamma : M \to \mathbb{R}$, $x \in M$, $\gamma(x) = 0$. The correspondence $T^{r*} : \mathcal{M}f_n \to \mathcal{VB}$ is a natural vector bundle over n-manifolds [4].

For $r = 1$ we have the natural equivalence $T^{1*}M \cong T^*M$, $j^1_x\gamma \cong d_x\gamma$.

1.2. *Examples of natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$*

Example 1. Let X be a vector field on an n-manifold M. For every $s = 1,\dots,r$ we have the map

$$\overset{(s)}{X} : T^{r*}M \to \mathbb{R}, \qquad \overset{(s)}{X}(j^r_x\gamma) := (X^s\gamma)(x),$$

$\gamma : M \to \mathbb{R}$, $x \in M$, $\gamma(x) = 0$, where $X^s = X\circ\ldots\circ X$ (s times). Then for every $s = 1,\dots,r$ we have the 1-form $d\overset{(s)}{X}$ on $T^{r*}M$. The correspondence $\overset{(s)}{A} : T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$, $X \mapsto d\overset{(s)}{X}$, is a natural operator.

Example 2. Let X be a vector field on an n-manifold M. For every $s = 1,\dots,r$ we have the 1-form $\overset{\langle s\rangle}{X} : TT^{r*}M \to \mathbb{R}$ on $T^{r*}M$,

$$\overset{\langle s\rangle}{X}(v) = \langle d_x(X^{s-1}\gamma), T\pi(v)\rangle, \qquad v \in (TT^{r*})_xM,$$

$x \in M$, $\gamma : M \to \mathbb{R}$, $\gamma(x) = 0$, $p^T(v) = j^r_x\gamma$, $p^T : TT^{r*}M \to T^{r*}M$ is the tangent bundle projection. The correspondence $\overset{\langle s\rangle}{A} : T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$, $X \mapsto \overset{\langle s\rangle}{X}$, is a natural operator.

1.3. *The $\mathcal{C}^\infty(\mathbb{R}^r)$-module of natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$.* The set of all natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$ is a module over the algebra $\mathcal{C}^\infty(\mathbb{R}^r)$. Indeed, if $f \in \mathcal{C}^\infty(\mathbb{R}^r)$ and $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$ is a natural operator, then $fA : T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$ is given by $(fA)(X) = f(\overset{(1)}{X},\dots,\overset{(r)}{X})A(X)$, $X \in \mathcal{X}(M)$, $M \in \mathrm{Obj}(\mathcal{M}f_n)$.

1.4. *The classification theorem.* The first main result of this paper is the following classification theorem.

Theorem 1. *For a natural number $n \geq 2$ the $\mathcal{C}^\infty(\mathbb{R}^r)$-module of all natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$ is free and $2r$-dimensional. The natural operators $\overset{(s)}{A}$ and $\overset{\langle s\rangle}{A}$ for $s = 1,\dots,r$ form a basis of this module over $\mathcal{C}^\infty(\mathbb{R}^r)$.*

The proof of Theorem 1 will occupy the rest of this subsection.

Consider a natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$. Since $\overset{(1)}{A}, \dots, \overset{(r)}{A}$, $\overset{\langle 1\rangle}{A}, \dots, \overset{\langle r\rangle}{A}$ are $\mathcal{C}^\infty(\mathbb{R}^r)$-linearly independent, we need only prove that A is their linear combination with $\mathcal{C}^\infty(\mathbb{R}^r)$-coefficients.

The following lemma shows that A is uniquely determined by the restriction $A(\partial_1)|(TT^{r*})_0\mathbb{R}^n$.

Lemma 1. *If* $A(\partial_1)|(TT^{r*})_0\mathbb{R}^n = 0$, *then* $A = 0$.

Proof. The proof is standard. We use the naturality of A and the fact that any non-vanishing vector field is locally ∂_1. ∎

So, we will study the restriction $A(\partial_1)|(TT^{r*})_0\mathbb{R}^n$.

Lemma 2. *There are* $f_1, \dots, f_r \in \mathcal{C}^\infty(\mathbb{R}^r)$ *with*

$$\Big(A - \sum_{s=1}^{r} f_s \overset{(s)}{A}\Big)(\partial_1)\Big|(VT^{r*})_0\mathbb{R}^n = 0,$$

where $VT^{r*}M \subset TT^{r*}M$ *denotes the* π*-vertical subbundle.*

Proof. We have the usual identification $(VT^{r*})_0\mathbb{R}^n \cong T_0^{r*}\mathbb{R}^n \times T_0^{r*}\mathbb{R}^n$, $\frac{d}{dt}\big|_{t=0}(u+tw) \cong (u, w)$, $u, w \in T_0^{r*}\mathbb{R}^n$. For $s = 1, \dots, r$ we define $f_s : \mathbb{R}^r \to \mathbb{R}$ by

$$f_s(a) = A(\partial_1)\bigg(j_0^r\Big(\sum_{l=1}^{r} \frac{1}{l!}\, a_l (x^1)^l\Big), j_0^r\Big(\frac{1}{s!}\,(x^1)^s\Big)\bigg),$$

$a = (a_1, \dots, a_r) \in \mathbb{R}^r$. We prove that the f_s are as required.

For simplicity set $\widetilde{A} := A - \sum_{s=1}^r f_s \overset{(s)}{A}$. Consider $\gamma, \eta : \mathbb{R}^r \to \mathbb{R}$ with $\gamma(0) = \eta(0) = 0$. Define $a = (a_1, \dots, a_r) \in \mathbb{R}^r$ and $b = (b_1, \dots, b_r) \in \mathbb{R}^r$ by

$$j_0^r(\gamma(x^1, 0, \dots, 0)) = j_0^r\Big(\sum_{l=1}^{r} \frac{1}{l!}\, a_l (x^1)^l\Big),$$

$$j_0^r(\eta(x^1, 0, \dots, 0)) = j_0^r\Big(\sum_{l=1}^{r} \frac{1}{l!}\, b_l (x^1)^l\Big).$$

Using the naturality of $\widetilde{A}$ with respect to the homotheties $(x^1, tx^2, \dots, tx^n)$ for $t \neq 0$ and putting $t \to 0$ we get

$$\widetilde{A}(\partial_1)(j_0^r\gamma, j_0^r\eta) = \widetilde{A}(\partial_1)(j_0^r(\gamma(x^1, 0, \dots, 0)), j_0^r(\eta(x^1, 0, \dots, 0))).$$

Then $\widetilde{A}(\partial_1)(j_0^r\gamma, j_0^r\eta) = \sum_{s=1}^r b_s f_s(a) - \sum_{s=1}^r f_s(a) b_s = 0$. ∎

Proof of Theorem 1. Replacing A by $A - \sum_{s=1}^r f_s \overset{(s)}{A}$ we can assume that $A(\partial_1)|(VT^{r*})_0\mathbb{R}^n = 0$. It remains to show that there exist $g_1, \dots, g_r \in$

$\mathcal{C}^\infty(\mathbb{R}^r)$ with

$$A = \sum_{s=1}^{r} g_s \overset{\langle s\rangle}{A}.$$

For $s = 1, \ldots, r$ define $g_s : \mathbb{R}^r \to \mathbb{R}$,

$$g_s(a) = A(\partial_1)\bigg(T^{r*}\partial_2\bigg(j_0^r\bigg(\sum_{l=1}^{r}\frac{1}{l!}\,a_l(x^1)^l + \frac{1}{(s-1)!}\,(x^1)^{s-1}x^2\bigg)\bigg)\bigg),$$

$a = (a_1, \ldots, a_r) \in \mathbb{R}^r$, where $T^{r*}X$ denotes the complete lifting of a vector field $X \in \mathcal{X}(M)$ to $T^{r*}M$. We prove that the g_s are as required.

By Lemma 1 and $A(\partial_1)|(VT^{r*})_0\mathbb{R}^n = 0$ it is sufficient to show

$$A(\partial_1)(T^{r*}\partial(j_0^r\gamma)) = \Big(\sum_{s=1}^{r} g_s \overset{\langle s\rangle}{A}\Big)(\partial_1)(T^{r*}\partial(j_0^r\gamma))$$

for any $\gamma : \mathbb{R}^n \to \mathbb{R}$, $\gamma(0) = 0$ and any constant vector field ∂ on $\mathbb{R}^n$ such that ∂_1 and ∂ are linearly independent. Using the naturality of A and $\sum_{s=1}^{r} g_s \overset{\langle s\rangle}{A}$ with respect to linear isomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ preserving ∂_1 we can assume $\partial = \partial_2$. For simplicity set $\widetilde{A} = \sum_{s=1}^{r} g_s \overset{\langle s\rangle}{A}$.

Consider $\gamma : \mathbb{R}^n \to \mathbb{R}$, $\gamma(0) = 0$. Define $a = (a_1, \ldots, a_r) \in \mathbb{R}^r$ by $a_s = \partial_1^s\gamma(0)$ and $b_s = (\partial_2\partial_1^{s-1}\gamma)(0)$ for $s = 1, \ldots, r$. Using the naturality of A with respect to the homotheties $(x^1, tx^2, \tau x^3 \ldots, \tau x^n)$ for $t, \tau \neq 0$ we get the homogeneity condition

$$\begin{aligned} &tA(\partial_1)(T^{r*}\partial_2(j_0^r\gamma(x^1, x^2, \ldots, x^n))) \\ &\qquad = A(\partial_1)(T^{r*}\partial_2(j_0^r\gamma(x^1, tx^2, \tau x^3, \ldots, \tau x^n))). \end{aligned}$$

This type of homogeneity gives $A(\partial_1)(T^{r*}\partial_2(j_0^r\gamma)) = \sum_{s=1}^{r} g_s(a)b_s$ by the homogeneous function theorem [4]. On the other hand $\widetilde{A}(\partial_1)(T^{r*}\partial_2(j_0^r\gamma)) = \sum_{s=1}^{r} g_s(a)b_s$. Then

$$A(\partial_1)(T^{r*}\partial(j_0^r\gamma)) = \Big(\sum_{s=1}^{r} g_s \overset{\langle s\rangle}{A}\Big)(\partial_1)(T^{r*}\partial(j_0^r\gamma)).$$

So, $A = \sum_{s=1}^{r} g_s \overset{\langle s\rangle}{A}$. ∎

1.5. *Corollaries.* Using the homogeneous function theorem, we have the following corollary of Theorem 1.

COROLLARY 1. *Let $n \geq 2$ be a natural number.*

(i) *If $r \geq 2$, then for every linear natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$ there exist real numbers α, β, γ such that*

$$A = \alpha \overset{(1)}{A} + \beta\,\mathrm{pr}_1 \overset{\langle 1\rangle}{A} + \gamma \overset{\langle 2\rangle}{A},$$

where $\mathrm{pr}_1 \in \mathcal{C}^\infty(\mathbb{R}^r)$ is the projection $\mathbb{R}^r \to \mathbb{R}$ on the first factor.

(ii) *If* $r=1$, *then for every linear natural operator* $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{1*}$ *there exist real numbers* α, β *such that*

$$A = \alpha \overset{(1)}{A} + \beta\, \mathrm{id}_{\mathbb{R}} \overset{\langle 1\rangle}{A},$$

where $\mathrm{id}_{\mathbb{R}} \in \mathcal{C}^\infty(\mathbb{R})$ *is the identity map.*

The operator $\overset{\langle 1\rangle}{A}$ can be considered as the well-known canonical 1-form λ^r on T^{r*}, the pull-back $(\pi^r_1)^*\lambda$ of the Liouville 1-form λ on $T^* \cong T^{1*}$ with respect to the jet projection $\pi^r_1 : T^{r*} \to T^{1*}$. Considering the values of natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$ at $X=0$ we obtain another corollary of Theorem 1. (For $r=1$ we recover the result of [1].)

Corollary 2. *For a natural number* $n \geq 2$ *every canonical* 1*-form on* T^{r*} *is a constant multiple of* λ^r*.*

On T^*M we have the canonical Liouville 1-form λ and the canonical symplectic 2-form $\omega = d\lambda$. Under the natural equivalence $T^{1*}M \cong T^*M$ we have $\lambda = \overset{\langle 1\rangle}{A}$, $\overset{(1)}{A}(X) = i_{T^*X}\omega$, the inner differentiation, and $\overset{(1)}{X}(j^1_x\gamma) = \langle d_x\gamma, X_x\rangle$, $X \in \mathcal{X}(M)$, $x \in M$, $\gamma : M \to \mathbb{R}$, $\gamma(x) = 0$, where T^*X denotes the complete lifting of X to T^*M. Thus we have one more corollary of Theorem 1.

Corollary 3. *Let* $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^*$ *be a natural operator* $(n \geq 2)$*. Then there exist maps* $f, g : \mathbb{R} \to \mathbb{R}$ *such that*

$$A(X)_\eta = f(\langle \eta, X_x\rangle)\lambda_\eta + g(\langle \eta, X_x\rangle)(i_{T^*X}\omega)_\eta,$$

where M *is an* n*-manifold,* $X \in \mathcal{X}(M)$, $x \in M$, $\eta \in T^*_xM$*.*

2. The natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2T^*T^{r*}$

2.1. *Examples of natural operators* $T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2T^*T^{r*}$

Example 3. Let X be a vector field on an n-manifold M. For every $s_1, s_2 = 1, \ldots, r$ with $s_1 < s_2$ we have the 2-form $\overset{(s_1)}{A}(X) \wedge \overset{(s_2)}{A}(X)$ on $T^{r*}M$. The correspondence $\overset{(s_1)}{A} \wedge \overset{(s_2)}{A} : T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2T^*T^{r*}$, $X \mapsto \overset{(s_1)}{A}(X) \wedge \overset{(s_2)}{A}(X)$, is a natural operator.

Example 4. Let X be a vector field on an n-manifold M. For every $s_1, s_2 = 1, \ldots, r$ we have the 2-form $\overset{(s_1)}{A}(X) \wedge \overset{\langle s_2\rangle}{A}(X)$ on $T^{r*}M$. The correspondence $\overset{(s_1)}{A} \wedge \overset{\langle s_2\rangle}{A} : T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2T^*T^{r*}$, $X \mapsto \overset{(s_1)}{A}(X) \wedge \overset{\langle s_2\rangle}{A}(X)$, is a natural operator.

Example 5. Let X be a vector field on an n-manifold M. For every $s_1, s_2 = 1, \ldots, r$ with $s_1 < s_2$ we have the 2-form $\overset{\langle s_1\rangle}{A}(X) \wedge \overset{\langle s_2\rangle}{A}(X)$ on $T^{r*}M$.

The correspondence $\overset{\langle s_1\rangle}{A} \wedge \overset{\langle s_2\rangle}{A} : T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2 T^* T^{r*}$, $X \mapsto \overset{\langle s_1\rangle}{A}(X) \wedge \overset{\langle s_2\rangle}{A}(X)$, is a natural operator.

Example 6. Let X be a vector field on an n-manifold M. For every $s = 1, \ldots, r$ we have the 2-form $d(\overset{\langle s\rangle}{A}(X))$ on $T^{r*}M$. The correspondence $d\overset{\langle s\rangle}{A} : T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2 T^* T^{r*}$, $X \mapsto d(\overset{\langle s\rangle}{A}(X))$, is a natural operator.

2.2. *The classification theorem.* As in Subsection 1.3 the set of all natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2 T^* T^{r*}$ is a module over the algebra $\mathcal{C}^\infty(\mathbb{R}^r)$.

The second main result of this paper is the following classification theorem.

Theorem 2. *For a natural number $n \geq 3$ the $\mathcal{C}^\infty(\mathbb{R}^r)$-module of all natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2 T^* T^{r*}$ is free and $2r^2$-dimensional. The collection of natural operators of Examples* 3–6 (*i.e. the collection consisting of $\overset{(s_1)}{A} \wedge \overset{(s_2)}{A}$ for $s_1, s_2 = 1, \ldots, r$ with $s_1 < s_2$ and $\overset{(s_1)}{A} \wedge \overset{\langle s_2\rangle}{A}$ for $s_1, s_2 = 1, \ldots, r$ and $\overset{\langle s_1\rangle}{A} \wedge \overset{\langle s_2\rangle}{A}$ for $s_1, s_2 = 1, \ldots, r$ with $s_1 < s_2$ and $d\overset{\langle s\rangle}{A}$ for $s = 1, \ldots, r$*) *is a basis of this module over $\mathcal{C}^\infty(\mathbb{R}^r)$.*

The proof of Theorem 2 will occupy the rest of this subsection.

Consider a natural operator $B : T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2 T^* T^{r*}$. Since the collection of natural operators listed in the statement of the theorem is $\mathcal{C}^\infty(\mathbb{R}^r)$-linearly independent, we need only prove that B is their linear combination with $\mathcal{C}^\infty(\mathbb{R}^r)$-coefficients.

The following lemma shows that B is uniquely determined by the restriction $B(\partial_1)|(TT^{r*}\mathbb{R}^n \times_{T^{r*}\mathbb{R}^n} TT^{r*}\mathbb{R}^n)_0$.

Lemma 3. *If $B(\partial_1)|(TT^{r*}\mathbb{R}^n \times_{T^{r*}\mathbb{R}^n} TT^{r*}\mathbb{R}^n)_0 = 0$, then $B = 0$.*

Proof. The proof is similar to the proof of Lemma 1. ∎

So, we will study the restriction $B(\partial_1)|(TT^{r*}\mathbb{R}^n \times_{T^{r*}\mathbb{R}^n} TT^{r*}\mathbb{R}^n)_0$.

Lemma 4. *There are $f_{(s_1,s_2)} \in \mathcal{C}^\infty(\mathbb{R}^r)$ for $s_1, s_2 = 1, \ldots, r$ with $s_1 < s_2$ such that*

$$\Big(B - \sum_{1 \leq s_1 < s_2 \leq r} f_{(s_1,s_2)} \overset{(s_1)}{A} \wedge \overset{(s_2)}{A}\Big)(\partial_1)\Big|(VT^{r*}\mathbb{R}^n \times_{T^{r*}\mathbb{R}^n} VT^{r*}\mathbb{R}^n)_0 = 0.$$

Proof. The proof is similar to the one of Lemma 2. We have the identification

$$(VT^{r*}\mathbb{R}^n \times_{T^{r*}\mathbb{R}^n} VT^{r*}\mathbb{R}^n)_0 \cong T_0^{r*}\mathbb{R}^n \times T_0^{r*}\mathbb{R}^n \times T_0^{r*}\mathbb{R}^n,$$

$\big(\frac{d}{dt}\big|_{t=0}(u+tv), \frac{d}{dt}\big|_{t=0}(u+tw)\big) \cong (u,v,w)$, $u,v,w \in T^{r*}_0\mathbb{R}^n$. For $s_1, s_2 = 1,\dots,r$ with $s_1 < s_2$ we define $f_{(s_1,s_2)} : \mathbb{R}^r \to \mathbb{R}$ by

$$f_{(s_1,s_2)}(a) = B(\partial_1)\bigg(j^r_0\bigg(\sum_{l=1}^r \frac{1}{l!} a_l (x^1)^l\bigg), j^r_0\bigg(\frac{1}{(s_1)!}(x^1)^{s_1}\bigg), j^r_0\bigg(\frac{1}{(s_2)!}(x^1)^{s_2}\bigg)\bigg),$$

$a = (a_1,\dots,a_r) \in \mathbb{R}^r$. We prove that the $f_{(s_1,s_2)}$ are as required.

Set $\widetilde{B} := B - \sum_{1\le s_1<s_2\le r} f_{(s_1,s_2)} \overset{(s_1)}{A} \wedge \overset{(s_2)}{A}$. Consider $\gamma, \eta, \varrho : \mathbb{R}^n \to \mathbb{R}$ with $\gamma(0) = \eta(0) = \varrho(0) = 0$. Define $a = (a_1,\dots,a_r) \in \mathbb{R}^r$, $b = (b_1,\dots,b_r) \in \mathbb{R}^r$ and $c = (c_1,\dots,c_r) \in \mathbb{R}^r$ by

$$j^r_0(\gamma(x^1,0,\dots,0)) = j^r_0\bigg(\sum_{l=1}^r \frac{1}{l!}\, a_l (x^1)^l\bigg),$$

$$j^r_0(\eta(x^1,0,\dots,0)) = j^r_0\bigg(\sum_{l=1}^r \frac{1}{l!}\, b_l (x^1)^l\bigg),$$

$$j^r_0(\varrho(x^1,0,\dots,0)) = j^r_0\bigg(\sum_{l=1}^r \frac{1}{l!}\, c_l (x^1)^l\bigg).$$

Using the naturality of $\widetilde{B}$ with respect to the homotheties $(x^1, tx^2, \dots, tx^n)$ for $t \neq 0$ and putting $t \to 0$ we get

$$\begin{aligned}&\widetilde{B}(\partial_1)(j^r_0\gamma, j^r_0\eta, j^r_0\varrho)\\ &\quad = \widetilde{B}(\partial_1)(j^r_0(\gamma(x^1,0,\dots,0)), j^r_0(\eta(x^1,0,\dots,0)), j^r_0(\varrho(x^1,0,\dots,0))).\end{aligned}$$

Consequently,

$$\begin{aligned}\widetilde{B}(\partial_1)(j^r_0\gamma, j^r_0\eta, j^r_0\varrho) = &\sum_{1\le s_1<s_2\le r} (b_{s_1}c_{s_2} - b_{s_2}c_{s_1}) f_{(s_1,s_2)}(a)\\ &- \sum_{1\le s_1<s_2\le r} f_{(s_1,s_2)}(a)(b_{s_1}c_{s_2} - b_{s_2}c_{s_1}) = 0. \ \blacksquare\end{aligned}$$

So, replacing B by $B - \sum_{1\le s_1<s_2\le r} f_{(s_1,s_2)} \overset{(s_1)}{A} \wedge \overset{(s_2)}{A}$ we can assume that

$$(*) \qquad B(\partial_1)|(VT^{r*}\mathbb{R}^n \times_{T^{r*}\mathbb{R}^n} VT^{r*}\mathbb{R}^n)_0 = 0.$$

Lemma 5. *Under the assumption* $(*)$ *there exist* $g_{(s_1,s_2)} \in \mathcal{C}^\infty(\mathbb{R}^r)$ *for* $s_1, s_2 = 1,\dots,r$ *and* $h_s \in \mathcal{C}^\infty(\mathbb{R}^r)$ *for* $s = 1,\dots,r$ *such that*

$$\bigg(B - \sum_{s_1,s_2=1}^r g_{(s_1,s_2)} \overset{(s_1)}{A} \wedge \overset{\langle s_2\rangle}{A} - \sum_{s=1}^r h_s d\overset{\langle s\rangle}{A}\bigg)(\partial_1)\bigg|(VT^{r*}\mathbb{R}^n \times_{T^{r*}\mathbb{R}^n} TT^{r*}\mathbb{R}^n)_0 = 0.$$

Proof. For $s_1, s_2 = 1,\dots,r$ define $g_{(s_1,s_2)} : \mathbb{R}^r \to \mathbb{R}$ by

$$g_{(s_1,s_2)}(a) = B(\partial_1)\bigg(\bigg(j^r_0(\gamma_{a,s_1}), j^r_0\bigg(\frac{1}{(s_2)!}\,(x^1)^{s_2}\bigg)\bigg), T^{r*}\partial_2(j^r_0(\gamma_{a,s_1}))\bigg),$$

$a = (a_1, \ldots, a_r) \in \mathbb{R}^r$, where

$$\gamma_{a,s_1} = \sum_{l=1}^{r} \frac{1}{l!}\, a_l (x^1)^l + \frac{1}{(s_1-1)!}\, (x^1)^{s_1-1} x^2.$$

For $s = 1, \ldots, r$ define $h_s : \mathbb{R}^r \to \mathbb{R}$ by

$$h_s(a) = B(\partial_1)\Big(\Big(j_0^r(\gamma_a), j_0^r\Big(\frac{1}{(s-1)!}\,(x^1)^{s-1}x^2\Big)\Big), T^{r*}\partial_2(j_0^r(\gamma_a))\Big),$$

$a = (a_1, \ldots, a_r) \in \mathbb{R}^r$, where $\gamma_a = \sum_{l=1}^r (1/l!) a_l (x^1)^l$. We prove that the $g_{(s_1,s_2)}$ and h_s are as required.

Set

$$\widetilde{B} = B - \sum_{s_1,s_2=1}^{r} g_{(s_1,s_2)} \overset{(s_1)}{A} \wedge \overset{\langle s_2\rangle}{A} - \sum_{s=1}^{r} h_s d\overset{\langle s\rangle}{A}.$$

By $(*)$, $\widetilde{B}(\partial_1)|(VT^{r*}\mathbb{R}^n \times_{T^{r*}\mathbb{R}^n} VT^{r*}\mathbb{R}^n)_0 = 0$. So, it remains to show that $\widetilde{B}(\partial_1)((j_0^r\gamma, j_0^r\eta), T^{r*}\partial(j_0^r\gamma)) = 0$ for any $\gamma, \eta : \mathbb{R}^n \to \mathbb{R}$ with $\gamma(0) = \eta(0) = 0$ and any constant vector field ∂ on $\mathbb{R}^n$. Using the naturality of $\widetilde{B}$ with respect to the linear isomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ preserving ∂_1, we can assume that $\partial = \partial_2$.

Consider $\gamma, \eta : \mathbb{R}^n \to \mathbb{R}$ as above. Define $a_l = \partial_1^l \gamma(0)$, $b_l = \partial_1^l \eta(0)$, $c_l = \partial_2 \partial_1^{l-1}\gamma(0)$ and $d_l = \partial_2\partial_1^{l-1}\eta(0)$ for $l = 1, \ldots, r$. Let $a := (a_1, \ldots, a_r) \in \mathbb{R}^k$. Using the naturality of $\widetilde{B}$ with respect to the homotheties $a_{t,\tau} = (x^1, tx^2, \tau x^3, \ldots, \tau x^n)$ for $t, \tau \neq 0$ we obtain the homogeneity condition

$$\begin{aligned} &t\widetilde{B}(\partial_1)((j_0^r\gamma, j_0^r\eta), T^{r*}\partial_2(j_0^r\gamma)) \\ &\qquad = \widetilde{B}(\partial_1)((j_0^r(\gamma\circ a_{t,\tau}), j_0^r(\eta\circ a_{t,\tau})), T^{r*}\partial_2(j_0^r(\gamma\circ a_{t,\tau}))). \end{aligned}$$

This implies

$$\begin{aligned} \widetilde{B}(\partial_1)((j_0^r\gamma, j_0^r\eta), T^{r*}\partial_2(j_0^r\gamma)) &= \sum_{s_1,s_2=1}^{r} c_{s_1} b_{s_2} g_{(s_1,s_2)}(a) + \sum_{s=1}^{r} d_s h_s(a) \\ &\quad - \sum_{s_1,s_2=1}^{r} g_{(s_1,s_2)}(a) c_{s_1} b_{s_2} - \sum_{s=1}^{r} h_s(a) d_s \\ &= 0 \end{aligned}$$

by the homogeneous function theorem. ∎

Replacing B by $B - \sum_{s_1,s_2=1}^r g_{(s_1,s_2)} \overset{(s_1)}{A} \wedge \overset{\langle s_2\rangle}{A} - \sum_{s=1}^r h_s d\overset{\langle s\rangle}{A}$ we can assume that

$$(**) \qquad B(\partial_1)|(VT^{r*}\mathbb{R}^n \times_{T^{r*}\mathbb{R}^n} TT^{r*}\mathbb{R}^n)_0 = 0.$$

Proof of Theorem 2. For $s_1, s_2 = 1, \dots, r$ with $s_1 < s_2$ define $F_{(s_1,s_2)} : \mathbb{R}^r \to \mathbb{R}$ by

$$F_{(s_1,s_2)}(a) = B(\partial_1)(T^{r*}\partial_2(j^r_0(\gamma_{a,s_1,s_2})), T^{r*}\partial_3(j^r_0(\gamma_{a,s_1,s_2}))),$$

$a = (a_1, \dots, a_r) \in \mathbb{R}^r$, where

$$\gamma_{a,s_1,s_2} = \sum_{l=1}^{r} \frac{1}{l!}\, a_l (x^1)^l + \frac{1}{(s_1-1)!}\,(x^1)^{s_1-1}x^2 + \frac{1}{(s_2-1)!}\,(x^1)^{s_2-1}x^3.$$

Using $(**)$ we show that $B = \sum_{1\le s_1<s_2\le r} F_{(s_1,s_2)} \overset{\langle s_1\rangle}{A} \wedge \overset{\langle s_2\rangle}{A}$. (Then the proof will be complete.) For simplicity denote the last right-hand side by $\widetilde{B}$.

First for $k = 2, \dots, r$ define $G_k : \mathbb{R}^r \to \mathbb{R}$ by

$$G_k(a) = B(\partial_1)(T^{r*}\partial_2(j^r_0(\gamma_{\langle a,k\rangle})), T^{r*}\partial_3(j^r_0(\gamma_{\langle a,k\rangle}))),$$

$a = (a_1, \dots, a_r) \in \mathbb{R}^r$, where

$$\gamma_{\langle a,k\rangle} = \sum_{l=1}^{r} \frac{1}{l!}\, a_l (x^1)^l + \frac{1}{(k-2)!}\,(x^1)^{k-2}x^2x^3.$$

By the invariance of $B(\partial_1)$ with respect to the diffeomorphism replacing x^2 by x^3 and x^3 by x^2 and preserving the other coordinates we see that $G_k = -G_k$, i.e. $G_k = 0$ for $k = 2, \dots, r$.

By Lemma 3 and $(**)$ to prove $B = \widetilde{B}$ it is sufficient to show that

$$(***) \qquad B(\partial_1)(T^{r*}\partial(j^r_0\gamma), T^{r*}\widetilde{\partial}(j^r_0\gamma)) = \widetilde{B}(\partial_1)(T^{r*}\partial(j^r_0\gamma), T^{r*}\widetilde{\partial}(j^r_0\gamma))$$

for any $\gamma : \mathbb{R}^n \to \mathbb{R}$ with $\gamma(0) = 0$ and any constant vector fields ∂ and $\widetilde{\partial}$ on $\mathbb{R}^n$ such that ∂_1, ∂ and $\widetilde{\partial}$ are linearly independent. Using the naturality of B and $\widetilde{B}$ with respect to linear isomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ preserving ∂_1 we can assume $\partial = \partial_2$ and $\widetilde{\partial} = \partial_3$.

Consider $\gamma : \mathbb{R}^n \to \mathbb{R}$, $\gamma(0) = 0$. Define $a_l = \partial_1^l\gamma(0)$, $b_l = \partial_2\partial_1^{l-1}\gamma(0)$ and $c_l = \partial_3\partial_1^{l-1}\gamma(0)$ for $l = 1, \dots, r$. Define also $d_k = \partial_2\partial_3\partial^{k-2}\gamma(0)$ for $k = 2, \dots, r$. Set $a := (a_1, \dots, a_r) \in \mathbb{R}^r$. Using the naturality of B with respect to the homotheties $a_{t_1,t_2,\tau} = (x^1, t_1x^2, t_2x^3, \tau x^4, \dots, \tau x^n)$ for $t_1, t_2, \tau \neq 0$ we get the homogeneity condition

$$\begin{aligned} t_1t_2B(\partial_1)&(T^{r*}\partial_2(j^r_0\gamma), T^{r*}\partial_3(j^r_0\gamma)) \\ &= B(\partial_1)(T^{r*}\partial_2(j^r_0(\gamma\circ a_{t_1,t_2,\tau})), T^{r*}\partial_3(j^r_0\gamma\circ a_{t_1,t_2,\tau}))). \end{aligned}$$

This type of homogeneity gives

$$\begin{aligned} B(\partial_1)(T^{r*}\partial_2(j^r_0\gamma), T^{r*}\partial_3(j^r_0\gamma)) = &\sum_{1\le s_1<s_2\le r} F_{(s_1,s_2)}(a)(b_{s_1}c_{s_2} - b_{s_2}c_{s_1}) \\ &+ \sum_{k=2}^{r} G_k(a)d_k \end{aligned}$$

by the homogeneous function theorem. Then

$$B(\partial_1)(T^{r*}\partial_2(j^r_0\gamma), T^{r*}\partial_3(j^r_0\gamma)) = \sum_{1\le s_1<s_2\le r} F_{(s_1,s_2)}(a)(b_{s_1}c_{s_2} - b_{s_2}c_{s_1})$$

because $G_k = 0$ for $k = 2, \dots, r$. On the other hand

$$\widetilde{B}(\partial_1)(T^{r*}\partial_2(j^r_0\gamma), T^{r*}\partial_3(j^r_0\gamma)) = \sum_{1\le s_1<s_2\le r} F_{(s_1,s_2)}(a)(b_{s_1}c_{s_2} - b_{s_2}c_{s_1}).$$

Thus we have $(***)$. The proof of Theorem 2 is complete. ∎

2.3. *Corollaries.* Using the homogeneous function theorem, we obtain the following corollary of Theorem 2.

COROLLARY 4. *Let $n \ge 3$ be a natural number.*

(i) *If $r \ge 2$, then for every linear natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2 T^* T^{r*}$ there exist real numbers $\alpha, \beta, \gamma, \delta$ such that*

$$A = \alpha \overset{(1)}{A} \wedge \overset{\langle 1\rangle}{A} + \beta \overset{\langle 1\rangle}{A} \wedge \overset{\langle 2\rangle}{A} + \gamma\, \mathrm{pr}_1\, d\overset{\langle 1\rangle}{A} + \delta d\overset{\langle 2\rangle}{A},$$

where $\mathrm{pr}_1 \in \mathcal{C}^\infty(\mathbb{R}^r)$ is the projection $\mathbb{R}^r \to \mathbb{R}$ on the first factor.

(ii) *If $r = 1$, then for every linear natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^* T^{1*}$ there exist real numbers α, β such that*

$$A = \alpha \overset{(1)}{A} \wedge \overset{\langle 1\rangle}{A} + \beta\, \mathrm{id}_{\mathbb{R}}\, d\overset{\langle 1\rangle}{A},$$

where $\mathrm{id}_{\mathbb{R}} \in \mathcal{C}^\infty(\mathbb{R})$ is the identity map on $\mathbb{R}$.

Considering the values of natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2 T^* T^{r*}$ at $X = 0$ we obtain

COROLLARY 5. *For a natural number $n \ge 3$ every canonical 2-form on T^{r*} is a constant multiple of $d\lambda^r$, where λ^r is as in Corollary 2.*

For $r = 1$ we deduce from Theorem 2 the following result.

COROLLARY 6. *Let $B : T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^2 T^* T^*$ be a natural operator $(n \ge 3)$. Then there exist maps $f, g : \mathbb{R} \to \mathbb{R}$ such that*

$$B(X)_\eta = f(\langle \eta, X_x\rangle)(\lambda \wedge i_{T^*X}\omega)_\eta + g(\langle \eta, X_x\rangle)\omega_\eta$$

where M is an n-manifold, $X \in \mathcal{X}(M)$, $x \in M$, $\eta \in T^*_x M$. Here λ and ω are as in Corollary 3.

3. Remark. What about natural operators $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^* T^{(r)}$ and $B : T_{|Mf_n} \rightsquigarrow \Lambda^2 T^* T^{(r)}$, where $T^{(r)}M = (T^{r*}M)^*$ is the linear r-tangent bundle? It turns out that $A = 0$ and $B = 0$, as follows from the following general fact.

THEOREM 3. *If $F : \mathcal{M}f \to \mathcal{FM}$ is a bundle functor with the point property, then every natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow \Lambda^p T^* F$ for $n \geq 2$ and $p \geq 1$ is* 0.

Proof. We have $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}\widetilde{F}$, where $\widetilde{F} = \Lambda^p TF : \mathcal{M}f \to \mathcal{FM}$. Of course, $\widetilde{F}$ has the point property. So, by the result of [10], $A = \text{const} \in \mathbb{R}$. Since A is fibre linear, $A = 0$. ∎

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