COLLOQUIUM MATHEMATICUM

VOL. 129 2012 NO. 1

POSNER'S SECOND THEOREM AND ANNIHILATOR CONDITIONS WITH GENERALIZED SKEW DERIVATIONS

BY

VINCENZO DE FILIPPIS (Messina) and FENG WEI (Beijing)

Abstract. Let $\mathcal{R}$ be a prime ring of characteristic different from 2, $\mathcal{Q}_r$ be its right Martindale quotient ring and $\mathcal{C}$ be its extended centroid. Suppose that $\mathcal{G}$ is a non-zero generalized skew derivation of $\mathcal{R}$ and $f(x_1,\ldots,x_n)$ is a non-central multilinear polynomial over $\mathcal{C}$ with n non-commuting variables. If there exists a non-zero element a of $\mathcal{R}$ such that $a[\mathcal{G}(f(r_1,\ldots,r_n)),f(r_1,\ldots,r_n)]=0$ for all $r_1,\ldots,r_n\in\mathcal{R}$, then one of the following holds:

(a) there exists $\lambda\in\mathcal{C}$ such that $\mathcal{G}(x)=\lambda x$ for all $x\in\mathcal{R}$;
(b) there exist $q\in\mathcal{Q}_r$ and $\lambda\in\mathcal{C}$ such that $\mathcal{G}(x)=(q+\lambda)x+xq$ for all $x\in\mathcal{R}$ and $f(x_1,\ldots,x_n)^2$ is central-valued on $\mathcal{R}$.

1. Introduction. Let $\mathcal{R}$ be a prime ring with center $\mathcal{Z}(\mathcal{R})$ and d be a non-zero derivation of $\mathcal{R}$. The well-known theorem of Posner [P] states that if $[d(x),x]\in\mathcal{Z}(\mathcal{R})$ for all $x\in\mathcal{R}$, then $\mathcal{R}$ must be commutative. Starting from this result, several authors studied the relationship between the structure of prime ring $\mathcal{R}$ and the behavior of an additive mapping f which satisfies the Engel-type condition $[f(x),x]_k=0$. The Engel condition is defined by $[f(x),x]_k=[[f(x),x]_{k-1},x]$ for all $x\in\mathcal{R}$ and all $k>1$.

In [Lan], Lanski showed that if d is a derivation of $\mathcal{R}$ such that $[d(x),x]_k=0$ for all x in a Lie ideal $\mathcal{L}$ of $\mathcal{R}$, then either $\mathcal{L}$ is central in $\mathcal{R}$ or $\mathrm{char}(\mathcal{R})=2$ and $\mathcal{R}$ satisfies the standard polynomial identity $S_4(x_1,\ldots,x_4)$ of degree 4.

On the other hand, for a prime ring $\mathcal{R}$ of characteristic different from 2, any non-central Lie ideal contains the set $\{[x_1,x_2]:x_1,x_2\in\mathcal{I}\}$ of all evaluations of the polynomial $[x_1,x_2]$ in a two-sided ideal $\mathcal{I}$ of $\mathcal{R}$. For this reason, many researchers in this area analyzed in detail the case when the Lie ideal is replaced by the set of all evaluations of a polynomial $f(x_1,\ldots,x_n)$ and $[d(f(x_1,\ldots,x_n)),f(x_1,\ldots,x_n)]_k$ is a differential identity for a certain ideal of $\mathcal{R}$.

2010 *Mathematics Subject Classification*: 16W25, 16N60.
Key words and phrases: polynomial identity, generalized skew derivation, prime ring.

DOI: 10.4064/cm129-1-4

In particular, we refer the reader to the results obtained by P.-H. Lee and T.-K. Lee in [L2] and [LL]. They proved that if $f(x_1,\ldots,x_n)$ is a multilinear polynomial, then it must be central-valued in $\mathcal{R}$ unless $\mathrm{char}(\mathcal{R})=2$ and $\mathcal{R}$ satisfies $S_4(x_1,\ldots,x_4)$.

In a recent paper [DD], another related generalization is considered by the first author and Di Vincenzo. They describe what happens if the derivation d is replaced by an additive mapping δ satisfying the condition $\delta(xy)=\delta(x)y+xg(y)$ for all $x,y\in\mathcal{R}$ and for some derivation g of $\mathcal{R}$. Such a mapping δ is called a *generalized derivation* of $\mathcal{R}$ with associated derivation d. Obviously, any derivation of $\mathcal{R}$ and any mapping of $\mathcal{R}$ of the form $f(x)=ax+xb$, for some $a,b\in\mathcal{R}$, are generalized derivations. The latter are usually called *inner generalized derivations* and play a leading role in the development of the theory of generalized derivations.

Basing on these definitions, the first author obtained in [D1] a related result with a specific annihilator condition on a generalized derivation acting on a multilinear polynomial. Let $\mathcal{R}$ be a prime ring of characteristic different from 2, $\mathcal{U}$ be its symmetric Utumi quotient ring and $\mathcal{C}$ be its extended centroid. Let $f(x_1,\ldots,x_n)$ be a non-central multilinear polynomial over $\mathcal{C}$ with n non-commuting variables and $0\neq a\in\mathcal{R}$. Suppose that $\mathcal{G}:\mathcal{R}\to\mathcal{R}$ is a non-zero generalized derivation satisfying the condition

$$a[\mathcal{G}(f(r_1,\ldots,r_n)),f(r_1,\ldots,r_n)]=0 \quad \text{for all } r_1,\ldots,r_n\in\mathcal{R}.$$

Then either there exists $\lambda\in\mathcal{C}$ such that $\mathcal{G}(x)=\lambda x$ for all $x\in\mathcal{R}$, or there exist $q\in\mathcal{U}$ and $\lambda\in\mathcal{C}$ such that $\mathcal{G}(x)=(q+\lambda)x+xq$ for all $x\in\mathcal{R}$ and $f(x_1,\ldots,x_n)^2$ is central-valued on $\mathcal{R}$. Furthermore, the first author also addressed in [D2] the question of when the composition of two generalized derivations can be a generalized derivation. He described the forms of two generalized derivations $\mathcal{F}$ and $\mathcal{G}$ of a prime ring $\mathcal{R}$, in the case when $\mathcal{FG}$ acts as a generalized derivation on the elements of the subset $f(\mathcal{R})$, where $f(\mathcal{R})$ is the set of all evaluations in $\mathcal{R}$ of a non-central polynomial $f(x_1,\ldots,x_n)$ over $\mathcal{C}$ with n non-commuting variables.

In the current paper we continue the study of the set

$$\mathcal{S}=\{[\mathcal{G}(f(x_1,\ldots,x_n)),f(x_1,\ldots,x_n)]\mid x_1,\ldots,x_n\in\mathcal{R}\}$$

for a generalized skew derivation $\mathcal{G}$ of $\mathcal{R}$ instead of a generalized derivation.

We now recall the relevant definition. Let $\mathcal{R}$ be an associative ring and α be an automorphism of $\mathcal{R}$. An additive mapping $d:\mathcal{R}\to\mathcal{R}$ is called a *skew derivation* of $\mathcal{R}$ if

$$d(xy)=d(x)y+\alpha(x)d(y)$$

for all $x,y\in\mathcal{R}$; then α is called the *associated automorphism* of d. An additive mapping $\mathcal{G}:\mathcal{R}\to\mathcal{R}$ is said to be a *generalized skew derivation* of $\mathcal{R}$ if there exists a skew derivation d of $\mathcal{R}$ with associated automorphism α

such that

$$\mathcal{G}(xy) = \mathcal{G}(x)y + \alpha(x)d(y)$$

for all $x, y \in \mathcal{R}$; d is said to be the *associated skew derivation* of $\mathcal{G}$ and α is the *associated automorphism* of $\mathcal{G}$. This definition unifies the notions of skew derivation and generalized derivation, which are considered as classical additive mappings of non-associative algebras, and have been investigated by many researchers from various points of view (see [Cha1]–[Cha4], [CW], [L3], [Liu]).

One standard approach in studying the aforementioned set $\mathcal{S}$ is to examine its size. For this, it is reasonable to study its left annihilator in $\mathcal{R}$. In fact we will prove:

Main Theorem 1.1. *Let $\mathcal{R}$ be a prime ring of characteristic different from 2, $\mathcal{Q}_r$ be its right Martindale quotient ring and $\mathcal{C}$ be its extended centroid. Suppose that $\mathcal{G}$ is a non-zero generalized skew derivation of $\mathcal{R}$ and $f(x_1, \ldots, x_n)$ is a non-central multilinear polynomial over $\mathcal{C}$ with n non-commuting variables. If there exists a non-zero element a of $\mathcal{R}$ such that $a[\mathcal{G}(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)] = 0$ for all $r_1, \ldots, r_n \in \mathcal{R}$, then one of the following holds:*

(a) *there exists $\lambda \in \mathcal{C}$ such that $\mathcal{G}(x) = \lambda x$ for all $x \in \mathcal{R}$;*
(b) *there exist $q \in \mathcal{Q}_r$ and $\lambda \in \mathcal{C}$ such that*

$$\mathcal{G}(x) = (q + \lambda)x + xq \quad \text{for all } x \in \mathcal{R}$$

and $f(x_1, \ldots, x_n)^2$ is central-valued on $\mathcal{R}$.

We should remark that in case $\mathcal{G}$ is a usual (non-skew) derivation, the conclusion of Theorem 1 follows directly from the results of [DD] (where $\mathcal{G}$ is an ordinary derivation) and [D1] (where $\mathcal{G}$ is a generalized derivation).

In what follows, let $\mathcal{Q}_r$ be the right Martindale quotient ring of $\mathcal{R}$, $\mathcal{Q}$ be the two-sided Martindale quotient ring of $\mathcal{R}$ and $\mathcal{C} = \mathcal{Z}(\mathcal{Q}) = \mathcal{Z}(\mathcal{Q}_r)$ the center of $\mathcal{Q}$ and $\mathcal{Q}_r$; $\mathcal{C}$ is usually called the *extended centroid* of $\mathcal{R}$ and is a field when $\mathcal{R}$ is a prime ring. It should be remarked that $\mathcal{Q}$ is a centrally closed prime $\mathcal{C}$-algebra. We refer the reader to [BMM] for the definitions and the related properties of these objects.

It is well known that automorphisms, derivations and skew derivations of $\mathcal{R}$ can be extended to both $\mathcal{Q}$ and $\mathcal{Q}_r$. Chang [Cha1] extended the definition of generalized skew derivation to the right Martindale quotient ring $\mathcal{Q}_r$ of $\mathcal{R}$ as follows: by a (right) generalized skew derivation we mean an additive mapping $\mathcal{G} : \mathcal{Q}_r \to \mathcal{Q}_r$ such that $\mathcal{G}(xy) = \mathcal{G}(x)y + \alpha(x)d(y)$ for all $x, y \in \mathcal{Q}$, where d is a skew derivation of $\mathcal{R}$ and α is an automorphism of $\mathcal{R}$. Moreover, there exists $\mathcal{G}(1) = a \in \mathcal{Q}_r$ such that $\mathcal{G}(x) = ax + d(x)$ for all $x \in \mathcal{R}$. Furthermore, if $\mathcal{G}(1) \in \mathcal{Q}$, then $\mathcal{G}$ can be extended to $\mathcal{Q}$. We will adopt the following notation:

$$f(x_1,\ldots,x_n)=x_1\ldots x_n+\sum_{\sigma\in S_n,\,\sigma\neq \mathrm{id}}\alpha_\sigma x_{\sigma(1)}\ldots x_{\sigma(n)}$$

for some $\alpha_\sigma\in\mathcal{C}$. The polynomial $f(x_1,\ldots,x_n)\in\mathcal{C}\langle x_1,\ldots,x_n\rangle$ is said to be *central-valued* on $\mathcal{R}$ if $f(x_1,\ldots,x_n)\in\mathcal{Z}(\mathcal{R})$ for all $x_1,\ldots,x_n\in\mathcal{R}$. The polynomial $f(x_1,\ldots,x_n)\in\mathcal{C}\langle x_1,\ldots,x_n\rangle$ is called *non-central* if it is not central-valued on $\mathcal{R}$ (or equivalently on the central closure $\mathcal{CR}$ of $\mathcal{R}$). We always suppose that $\mathrm{char}(\mathcal{R})\neq 2$ and $f(x_1,\ldots,x_n)$ is non-central-valued on $\mathcal{R}$.

2. The case of inner generalized skew derivations. Throughout this section we always denote the ring of $m\times m$ matrices over an algebraic set $\mathcal{A}$ by $\mathcal{M}_m(\mathcal{A})$. Here $\mathcal{A}$ may be a field, a ring or an algebra in different contexts.

In this section we will deal with the case when $\mathcal{G}$ is an inner generalized skew derivation induced by elements $b,c\in\mathcal{R}$ and $\alpha\in\mathrm{Aut}(\mathcal{R})$, that is, $\mathcal{G}(x)=bx+\alpha(x)c$ for all $x\in\mathcal{R}$. Our aim is to prove the following:

PROPOSITION 2.1. *Let $\mathcal{R}$ be a prime ring of characteristic different from 2 and $f(x_1,\ldots,x_n)$ be a non-central multilinear polynomial over $\mathcal{C}$ with n non-commuting variables. Let $a,b,c\in\mathcal{R}$ with $a\neq 0$ and $\alpha\in\mathrm{Aut}(\mathcal{R})$ such that $\mathcal{G}(x)=bx+\alpha(x)c$ for all $x\in\mathcal{R}$. If*

$$a[bf(r_1,\ldots,r_n)+\alpha(f(r_1,\ldots,r_n))c,f(r_1,\ldots,r_n)]=0$$

for all $r_1,\ldots,r_n\in\mathcal{R}$, then one of the following holds:

(a) *there exists $\lambda\in\mathcal{C}$ such that $\mathcal{G}(x)=\lambda x$ for all $x\in\mathcal{R}$;*
(b) *$c-b\in\mathcal{C}$, $\mathcal{G}(x)=bx+xc$ for all $x\in\mathcal{R}$, and $f(x_1,\ldots,x_n)^2$ is central-valued on $\mathcal{R}$.*

2.1. The matrix case. Let us first consider the case when $\mathcal{R}=\mathcal{M}_m(\mathcal{K})$, where $\mathcal{K}$ is a field of characteristic different from 2. Note that the set $f(\mathcal{R})=\{f(r_1,\ \ldots,r_n)\mid r_1,\ldots,r_n\in\mathcal{R}\}$ is invariant under the action of all inner automorphisms of $\mathcal{R}$. Let us write $r=(r_1,\ldots,r_n)\in\mathcal{R}\times\cdots\times\mathcal{R}=\mathcal{R}^n$. Then for any inner automorphism φ of $\mathcal{M}_m(\mathcal{K})$, we get $\underline{r}=(\varphi(r_1),\ldots,\varphi(r_n))\in\mathcal{R}^n$ and $\varphi(f(r))=f(\underline{r})\in f(\mathcal{R})$. As usual, we denote by e_{ij} the matrix unit having 1 in the (i,j)-entry and zero elsewhere.

Let us recall some results from [L1] and [Ler]. Let $\mathcal{T}$ be a ring with 1 and let $e_{ij}\in\mathcal{M}_m(\mathcal{T})$ $(i,j=1,\ldots,m)$ be the matrix units. For a sequence $u=(\mathcal{A}_1,\ldots,\mathcal{A}_n)$ in $\mathcal{M}_m(\mathcal{T})$, the value of u is defined to be the product $|u|=\mathcal{A}_1\cdots\mathcal{A}_n$ and u is non-vanishing if $|u|\neq 0$. For a permutation σ of $\{1,\ldots,n\}$, we write $u^\sigma=(\mathcal{A}_{\sigma(1)},\ldots,\mathcal{A}_{\sigma(n)})$. We call u *simple* if it is of the form $u=(a_1e_{i_1j_1},\ldots,a_ne_{i_nj_n})$, where $a_i\in\mathcal{T}$. A simple sequence u is called *even* if for some σ, $|u^\sigma| = be_{ii} \neq 0$, and *odd* if for some σ, $|u^\sigma| = be_{ij} \neq 0$, where $i \neq j$. We have:

Fact 2.2 ([L1, Lemma]). *Let $\mathcal{T}$ be a $\mathcal{K}$-algebra with* 1 *and let $\mathcal{R} = \mathcal{M}_m(\mathcal{T})$, $m \geq 2$. Suppose that $g(x_1, \dots, x_n)$ is a multilinear polynomial over $\mathcal{K}$ such that $g(u) = 0$ for all odd simple sequences u. Then $g(x_1, \dots, x_n)$ is central-valued on $\mathcal{R}$.*

Fact 2.3 ([Ler, Lemma 2]). *Let $\mathcal{T}$ be a $\mathcal{K}$-algebra with* 1 *and let $\mathcal{R} = \mathcal{M}_m(\mathcal{T})$, $m \geq 2$. Suppose that $g(x_1, \dots, x_n)$ is a multilinear polynomial over $\mathcal{K}$. Let $u = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ be a simple sequence from $\mathcal{R}$.*

(1) *If u is even, then $g(u)$ is a diagonal matrix.*
(2) *If u is odd, then $g(u) = ae_{pq}$ for some $a \in \mathcal{T}$ and $p \neq q$.*

Remark 2.4. Since $f(x_1, \dots, x_n)$ is not central-valued on $\mathcal{R}$, by Fact 2.2 there exists an odd simple sequence $r = (r_1, \dots, r_n)$ from $\mathcal{R}$ such that $f(r) = f(r_1, \dots, r_n) \neq 0$. By Fact 2.3, $f(r) = \beta e_{pq}$, where $0 \neq \beta \in \mathcal{C}$ and $p \neq q$. Since $f(x_1, \dots, x_n)$ is a multilinear polynomial and $\mathcal{C}$ is a field, we may assume that $\beta = 1$. Now, for distinct i, j, let $\sigma \in S_n$ be such that $\sigma(p) = i$ and $\sigma(q) = j$, and let ψ be the automorphism of $\mathcal{R}$ defined by $\psi(\sum_{s,t} \xi_{st} e_{st}) = \sum_{s,t} \xi_{st} e_{\sigma(s)\sigma(t)}$. Then $f(\psi(r)) = f(\psi(r_1), \dots, \psi(r_n)) = \psi(f(r)) = \beta e_{ij} = e_{ij}$.

Let us recall several known results:

Lemma 2.5 (Proposition 1 in [D1]). *Let $\mathcal{R}$ be a prime ring of characteristic different from* 2*, $f(x_1, \dots, x_n)$ be a non-central multilinear polynomial over $\mathcal{C}$ with n non-commuting variables and $a, b, c \in \mathcal{R}$, $a \neq 0$. If $a[bf(r_1, \dots, r_n) + f(r_1, \dots, r_n)c, f(r_1, \dots, r_n)] = 0$ for all $r_1, \dots, r_n \in \mathcal{R}$, then one of the following holds:*

(a) *$b, c \in \mathcal{C}$;*
(b) *$c - b \in \mathcal{C}$, and $f(x_1, \dots, x_n)^2$ is central-valued on $\mathcal{R}$.*

Lemma 2.6 ([Cha2, Lemma 2]). *Let $\mathcal{R}$ be a dense subring of the ring of linear transformations of a vector space $\mathcal{V}$ over a division ring $\mathcal{D}$ with* $\dim_{\mathcal{D}} \mathcal{V} \geq 2$ *and suppose $\mathcal{R}$ contains some non-zero linear transformations of finite rank. Let α be an automorphism of $\mathcal{R}$ and $a, b, c \in \mathcal{R}$. Suppose that*

$$\mathcal{G} : \mathcal{R} \to \mathcal{R}, \qquad x \mapsto bx + \alpha(x)c,$$

is a mapping from $\mathcal{R}$ into itself satisfying the condition $a[\mathcal{G}(x), x]_k = 0$ for all $x \in \mathcal{R}$, where k is a fixed positive integer. Then either $a = 0$ or α is the identical mapping on $\mathcal{R}$ and $b, c, \in \mathcal{Z}(\mathcal{R})$ unless $\dim_{\mathcal{D}} \mathcal{V} = 2$ *and $\mathcal{D} = GF(2)$, the Galois field of two elements.*

We start with the following lemma:

Lemma 2.7. *Let $\mathcal{H}$ be an infinite field and $m \geq 2$ an integer. If $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are not scalar matrices in $\mathcal{M}_m(\mathcal{H})$, then there exists an invertible matrix $\mathcal{B} \in \mathcal{M}_m(\mathcal{H})$ such that each matrix $\mathcal{B}\mathcal{A}_1\mathcal{B}^{-1}, \ldots, \mathcal{B}\mathcal{A}_k\mathcal{B}^{-1}$ has all entries non-zero.*

Proof. Let us first show that if $\mathcal{A} \in \mathcal{M}_m(\mathcal{H})$ is not scalar, then there exists a conjugate $\mathcal{B}\mathcal{A}\mathcal{B}^{-1}$ having a non-zero entry in any particular position.

Assume that $\mathcal{A}$ is not diagonal. Then for some $i \neq j$ the (i,j)-entry $\mathcal{A}_{ij}$ of $\mathcal{A}$ is non-zero. If $p \neq q$, then there exists a permutation $\sigma \in S_m$ such that $\sigma(i) = p$ and $\sigma(j) = q$. Consider the automorphism φ_σ on $\mathcal{M}_m(\mathcal{H})$ defined by $\varphi_\sigma(e_{rs}) = e_{\sigma(r)\sigma(s)}$ for all matrix unit e_{rs}. Let $\mathcal{B} \in \mathcal{M}_m(\mathcal{H})$ be the permutation matrix which induces the automorphism φ_σ in $\mathcal{M}_m(\mathcal{H})$. Thus the (p,q)-entry of $\mathcal{B}\mathcal{A}\mathcal{B}^{-1}$ is $\mathcal{A}_{ij}$. Assume now that $p = q$. By the previous argument, for $s \neq p$, some conjugate $\mathcal{A}'$ of $\mathcal{A}$ has non-zero (p,s)-entry. Let $\lambda \in \mathcal{H}$, and put $\mathcal{A}'_\lambda = (\mathcal{I} + \lambda e_{sp})\mathcal{A}'(\mathcal{I} - \lambda e_{sp})$. Then the (p,p)-entry of $\mathcal{A}'_\lambda$ is $\mathcal{A}'_{pp} - \lambda\mathcal{A}'_{ps}$. Of course, we can choose λ in $\mathcal{H}$ such that $\mathcal{A}'_{pp} - \lambda\mathcal{A}'_{ps}$ is not zero. This proves our claim in the case when $\mathcal{A}$ is not diagonal.

If $\mathcal{A}$ is a diagonal matrix which is not scalar, there exist $i \neq j$ such that $\mathcal{A}_{ii} \neq \mathcal{A}_{jj}$. The (i,j)-entry of the conjugate $\mathcal{A}'' = (\mathcal{I} + e_{ij})\mathcal{A}(\mathcal{I} - e_{ij})$ is $\mathcal{A}_{jj} - \mathcal{A}_{ii}$, which is not zero. Hence $\mathcal{A}''$ is not diagonal and by the previous case we are done.

Let us consider the set $\{x_{ij} : 1 \leq i,j \leq m\}$ of n^2 commutative indeterminates and let $\mathcal{M}_m(\mathcal{H}[x_{ij}])$ be the algebra of $m \times m$ matrices over the polynomial ring $\mathcal{H}[x_{ij}]$. Let $\mathcal{E} = \sum_{ij} x_{ij}e_{ij}$ be the generic matrix and consider $\mathcal{E}_l = \mathcal{E} \cdot \mathcal{A}_l \cdot \operatorname{adj}(\mathcal{E})$ for $l = 1, \ldots, k$. Any substitution of $c_{ij} \in \mathcal{H}$ for the indeterminates x_{ij} induces a homomorphism $\varphi : \mathcal{M}_m(\mathcal{H}[x_{ij}]) \to \mathcal{M}_m(\mathcal{H})$. If $\varphi(\mathcal{E})$ is an invertible matrix $\mathcal{B}$, then $\varphi(\mathcal{E}_l)$ is a non-zero scalar multiple of $\mathcal{B}\mathcal{A}_l\mathcal{B}^{-1}$. Clearly, any matrix $\mathcal{B} \in \mathcal{M}_m(\mathcal{H})$ is the image of $\mathcal{E}$ under the action of some such homomorphism. Since each entry of $\operatorname{adj}(\mathcal{E})$ is a homogeneous polynomial in $\{x_{ij}\}$, the entries of $\mathcal{E}_l$ are homogeneous polynomials in $\{x_{ij}\}$ without constant terms. None of these entries is zero by our observation above: in any particular position some conjugate of $\mathcal{A}_l$ has a non-zero entry. The determinant $\det(\mathcal{E})$ is a non-zero polynomial of $\mathcal{H}[x_{ij}]$. Let $\mathcal{W}(x_{ij})$ be the product of $\det(\mathcal{E})$ and all entries of $\mathcal{E}_l$ for $l = 1, \ldots, k$. It is not difficult to observe that $\mathcal{W}(x_{ij})$ is a non-zero polynomial. Since the field $\mathcal{H}$ is infinite, some evaluation of $\mathcal{W}(x_{ij})$ is not zero in $\mathcal{H}$. As above, let φ be the homomorphism induced by this evaluation, then $\mathcal{B} = \varphi(\mathcal{E})$ is invertible and $\mathcal{B}\mathcal{A}_l\mathcal{B}^{-1} = \frac{1}{\det(\mathcal{B})}\varphi(\mathcal{E}_l)$ is a matrix with all entries non-zero, for $l = 1, \ldots, k$. ∎

Lemma 2.8. *Let $\mathcal{H}$ be an infinite field, $m \geq 2$ an integer and $\mathcal{R} = \mathcal{M}_m(\mathcal{H})$. If there exist $b, c, q \in \mathcal{R}$ such that q is an invertible matrix and*

$[bu + quq^{-1}c, u] = 0$ *for all* $u \in f(\mathcal{R})$*, then one of the following holds:*

(a) $q^{-1}c, b + c \in \mathcal{Z}(\mathcal{R})$;

(b) $q, c - b \in \mathcal{Z}(\mathcal{R})$ *and* $u^2 \in \mathcal{Z}(\mathcal{R})$ *for all* $u \in f(\mathcal{R})$.

Proof. If either $q^{-1}c \in \mathcal{Z}(\mathcal{R})$ or $q \in \mathcal{Z}(\mathcal{R})$, then the conclusion follows from Lemma 2.5. Thus we may assume that neither $q^{-1}c$ nor q is a scalar matrix and proceed to obtain a contradiction. By Lemma 2.7, there exists some invertible matrix $\mathcal{B} \in \mathcal{M}_m(\mathcal{H})$ such that each matrix $\mathcal{B}(q^{-1}c)\mathcal{B}^{-1}, \mathcal{B}q\mathcal{B}^{-1}$ has all entries non-zero. Denote by $\varphi(x) = \mathcal{B}x\mathcal{B}^{-1}$ the inner automorphism induced by $\mathcal{B}$. Since $f(\mathcal{R})$ is invariant under the action of all inner automorphisms of $\mathcal{R}$, we have $[\varphi(b)u + \varphi(q)u\varphi(q^{-1}c), u] = 0$ for all $u \in f(\mathcal{R})$. Let us write

$$\varphi(q) = \sum_{hl} q_{hl}e_{hl}, \qquad \varphi(q^{-1}c) = \sum_{hl} c_{hl}e_{hl} \qquad \text{for } 0 \neq q_{hl}, 0 \neq c_{hl} \in \mathcal{H}.$$

Since $e_{ij} \in f(\mathcal{R})$ for all $i \neq j$, for any $i \neq j$ we have

$$X = [\varphi(b)e_{ij} + \varphi(q)e_{ij}\varphi(q^{-1}c), e_{ij}]e_{ij} = 0.$$

In particular, the (i, j)-entry of X is $q_{ji}c_{ji} = 0$, which is a contradiction. ∎

Lemma 2.9. *Let $\mathcal{H}$ be an infinite field, $m \geq 2$ an integer and $\mathcal{R} = \mathcal{M}_m(\mathcal{H})$. If there exist $a, b, c, q \in \mathcal{R}$ with $a \neq 0$ such that q is an invertible matrix and $a[bu + quq^{-1}c, u] = 0$ for all $u \in f(\mathcal{R})$, then one of the following holds:*

(a) $q^{-1}c, b + c \in \mathcal{Z}(\mathcal{R})$;

(b) $q, c - b \in \mathcal{Z}(\mathcal{R})$ *and* $u^2 \in \mathcal{Z}(\mathcal{R})$ *for all* $u \in f(\mathcal{R})$.

Proof. Assume that $a \in \mathcal{Z}(\mathcal{R})$. Since $a \neq 0$, we get $[bu + quq^{-1}c, u] = 0$ for all $u \in f(\mathcal{R})$ and we are done by Lemma 2.8. Hence we may assume that a is not central and as above neither $q^{-1}c$ nor q is a scalar matrix. Again by Lemma 2.7, there exists some invertible matrix $\mathcal{B} \in M_m(\mathcal{H})$ such that each matrix $\mathcal{B}a\mathcal{B}^{-1}, \mathcal{B}(q^{-1}c)\mathcal{B}^{-1}, \mathcal{B}q\mathcal{B}^{-1}$ has all entries non-zero. Denote by $\varphi(x) = \mathcal{B}x\mathcal{B}^{-1}$ the inner automorphism induced by $\mathcal{B}$. Mimicking the above proof we will write $\varphi(a) = \sum_{hl} a_{hl}e_{hl}$, $\varphi(q) = \sum_{hl} q_{hl}e_{hl}$ and $\varphi(q^{-1}c) = \sum_{hl} c_{hl}e_{hl}$, for $0 \neq a_{hl}, 0 \neq q_{hl}, 0 \neq c_{hl} \in \mathcal{B}$. Moreover, for $e_{ij} \in f(\mathcal{R})$,

$$Y = \varphi(a)[\varphi(b)e_{ij} + \varphi(q)e_{ij}\varphi(q^{-1}c), e_{ij}]e_{ij} = \varphi(a)e_{ij}\varphi(q)e_{ij}\varphi(q^{-1}c)e_{ij} = 0.$$

In particular, the (j, j)-entry of Y is $a_{ji}q_{ji}c_{ji} = 0$, which is a contradiction.

Thus either $q^{-1}c \in \mathcal{Z}(\mathcal{R})$ and $a[(b + c)u, u] = 0$ for all $u \in f(\mathcal{R})$, or $q \in \mathcal{Z}(\mathcal{R})$ and $a[(b+c)u, u] = 0$ for all $u \in f(\mathcal{R})$. In both cases the conclusion follows from Lemma 2.5. ∎

Lemma 2.10. *Let $\mathcal{K}$ be a field of characteristic different from 2, $m \geq 2$ an integer and $\mathcal{R} = \mathcal{M}_m(\mathcal{K})$. If there exist $0 \neq a, b, c, q \in \mathcal{R}$ such that q is* *an invertible matrix and* $a[bu + quq^{-1}c, u] = 0$ *for all* $u \in f(\mathcal{R})$ *then one of the following holds:*

(1) $q^{-1}c, b + c \in \mathcal{Z}(\mathcal{R})$;
(2) $q, c - b \in \mathcal{Z}(\mathcal{R})$ *and* $u^2 \in \mathcal{Z}(\mathcal{R})$ *for all* $u \in f(\mathcal{R})$.

Proof. If one assumes that $\mathcal{K}$ is infinite, the conclusion is a consequence of Lemma 2.9.

Now let $\mathcal{H}$ be an infinite field which is an extension of the field $\mathcal{K}$ and let $\overline{\mathcal{R}} = \mathcal{M}_m(\mathcal{H}) \cong \mathcal{R} \otimes_{\mathcal{K}} \mathcal{H}$. Note that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central-valued on $\mathcal{R}$ if and only if it is central-valued on $\overline{\mathcal{R}}$. We observe that the generalized polynomial

$$\Phi(x_1, \ldots, x_n) = a[bf(x_1, \ldots, x_n) - qf(x_1, \ldots, x_n)q^{-1}b, f(x_1, \ldots, x_n)]$$

is a generalized polynomial identity for $\mathcal{R}$. Moreover, $\Phi(x_1, \ldots, x_n)$ is multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $x_1, \ldots, x_n$. On the other hand, the complete linearization of $\Phi(x_1, \ldots, x_{n+1})$ leads to a multilinear generalized polynomial $\Theta(x_1, \ldots, x_n, y_1, \ldots, y_n)$, which is of the form

$$\Theta(x_1, \ldots, x_n, x_1, \ldots, x_n) = 2^n P(x_1, \ldots, x_n).$$

Clearly, the multilinear polynomial $\Theta(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a generalized polynomial identity for $\mathcal{R}$ and $\overline{\mathcal{R}}$ too. Since $\mathrm{char}(\mathcal{K}) \neq 2$, we obtain $\Phi(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \overline{\mathcal{R}}$, and the conclusion follows from Lemma 2.9. ∎

2.2. The proof of Proposition 2.1. Suppose first that α is an X-inner automorphism of $\mathcal{R}$, that is, there exists an element $q \in \mathcal{Q}$ such that $\alpha(x) = qxq^{-1}$ for all $x \in \mathcal{R}$. It is not difficult to see that the generalized polynomial

$$\Phi(x_1, \ldots, x_n) = a[bf(x_1, \ldots, x_n) - qf(x_1, \ldots, x_n)q^{-1}c, f(x_1, \ldots, x_n)]$$

is a generalized polynomial identity for $\mathcal{R}$. If $\{1, q^{-1}c\}$ are $\mathcal{C}$-linearly independent, then $\Phi(x_1, \ldots, x_n)$ is a non-trivial generalized polynomial identity for $\mathcal{R}$. It follows from [Chu1] that $\Phi(x_1, \ldots, x_n)$ is a non-trivial generalized polynomial identity for $\mathcal{Q}$. By the well-known Martindale theorem [M], $\mathcal{Q}$ is a primitive ring having non-zero socle with the field $\mathcal{C}$ as its associated division ring. By [J, p. 75], $\mathcal{Q}$ is isomorphic to a dense subring of the ring of linear transformations of a vector space $\mathcal{V}$ over $\mathcal{C}$, containing some non-zero linear transformations of finite rank. Assume first that $\dim_{\mathcal{C}} \mathcal{V} = \infty$. As in Lemma 2 of [W], the set $f(\mathcal{R}) = \{f(r_1, \ldots, r_n) \mid r_i \in \mathcal{R}\}$ is dense in $\mathcal{R}$. Since $\Phi(r_1, \ldots, r_n) = 0$ is a generalized polynomial identity of $\mathcal{R}$, we know that $\mathcal{R}$ satisfies the generalized polynomial identity

$$a[bx_1 - qx_1q^{-1}c, x_1].$$

This implies that $a[\mathcal{G}(x),x]=0$ for all $x\in\mathcal{R}$. In this case, the desired conclusion is due to Lemma 2.6. On the other hand, if $\dim_{\mathcal{C}}\mathcal{V}=k\geq 2$ is a finite positive integer, then $\mathcal{Q}\cong\mathcal{M}_k(\mathcal{C})$ and the conclusion follows from Lemma 2.10.

In case $\{1,q^{-1}c\}$ are $\mathcal{C}$-linearly dependent, that is, $q^{-1}c\in\mathcal{C}$, the ring $\mathcal{R}$ satisfies

$$\Phi(x_1,\dots,x_n)=a[bf(x_1,\dots,x_n)-cf(x_1,\dots,x_n),f(x_1,\dots,x_n)]$$

and we are done by Lemma 2.5.

So we may assume that α is X-outer. In view of [Chu2] we know that $\mathcal{R}$ and $\mathcal{Q}$ satisfy the same generalized polynomial identities with automorphisms. Therefore

$$\Phi(x_1,\dots,x_n)=a[bf(x_1,\dots,x_n)+\alpha(f(x_1,\dots,x_n))c,f(x_1,\dots,x_n)]$$

is also satisfied by $\mathcal{Q}$. Moreover, $\mathcal{Q}$ is a centrally closed prime $\mathcal{C}$-algebra. Note that if $c=0$ we are done by Lemma 2.5. We now suppose that both $c\neq 0$ and $a\neq 0$. In this case, it follows from [Chu3, Main Theorem] that $\Phi(x_1,\dots,x_n)$ is a non-trivial generalized identity for $\mathcal{R}$ and for $\mathcal{Q}$. By [K, Theorem 1] we deduce that $\mathcal{RC}$ has non-zero socle and $\mathcal{Q}$ is primitive. Since α is an outer automorphism and any $(x_i)^{\alpha}$-word degree in $\Phi(x_1,\dots,x_n)$ is equal to 1, by [Chu3, Theorem 3], $\mathcal{Q}$ satisfies the generalized polynomial identity

$$a[bf(x_1,\dots,x_n)+f(y_1,\dots,y_n)c,f(x_1,\dots,x_n)].$$

In particular, $\mathcal{Q}$ (and so also $\mathcal{R}$) satisfies the generalized polynomial identity

$$a[bf(x_1,\dots,x_n)+f(x_1,\dots,x_n)c,f(x_1,\dots,x_n)].$$

In view of Lemma 2.5, we obtain the required results.

3. The proof of Main Theorem 1.1. Let us first recall the following:

FACT 3.1 ([D1, Theorem 1]). *Let $\mathcal{R}$ be a prime ring of characteristic different from 2, $\mathcal{U}$ be its two-sided Utumi quotient ring and $\mathcal{C}$ be its extended centroid. Let δ be a non-zero generalized derivation of $\mathcal{R}$ and $f(x_1,\dots,x_n)$ be a non-central multilinear polynomial over $\mathcal{C}$ with n non-commuting variables. If there exists an element $a\in\mathcal{R}$ such that $a[\delta(f(r_1,\dots,r_n)),f(r_1,\dots,r_n)]=0$ for all $r_1,\dots,r_n\in\mathcal{R}$, then one of the following holds:*

(a) $a=0$;
(b) *there exists $\lambda\in\mathcal{C}$ such that $\delta(x)=\lambda x$ for all $x\in\mathcal{R}$;*
(c) *there exist $q\in\mathcal{U}$ and $\lambda\in\mathcal{C}$ such that $\delta(x)=(q+\lambda)x+xq$ for all $x\in\mathcal{R}$ and $f(x_1,\dots,x_n)^2$ is central-valued on $\mathcal{R}$.*

FACT 3.2 ([CL2, Theorem 1]). *Let $\mathcal{R}$ be a prime ring, $\mathcal{D}$ be an X-outer skew derivation of $\mathcal{R}$ and α be an X-outer automorphism of $\mathcal{R}$. If $\Phi(x_i,\mathcal{D}(x_i),\alpha(x_i))$ is a generalized polynomial identity for $\mathcal{R}$, then $\mathcal{R}$ also* *satisfies the generalized polynomial identity $\Phi(x_i, y_i, z_i)$, where x_i, y_i and z_i are distinct indeterminates.*

3.1. The proof of Main Theorem 1.1. As remarked in the Introduction, we can write $\mathcal{G}(x) = bx + d(x)$ for all $x \in \mathcal{R}$, where $b \in \mathcal{Q}_r$ and d is a skew derivation of $\mathcal{R}$ (see [Cha1]). Let us put $f(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \gamma_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$, where $\gamma_\sigma \in \mathcal{C}$. By [CL2, Theorem 2] we know that $\mathcal{R}$ and $\mathcal{Q}_r$ satisfy the same generalized polynomial identities with a single skew derivation. Thus $\mathcal{Q}_r$ satisfies

$$\begin{aligned}&\Phi(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n))\\ &\qquad = a[bf(x_1, \ldots, x_n) + d(f(x_1, \ldots, x_n)), f(x_1, \ldots, x_n)].\end{aligned}$$

If d is X-inner, then there exist $c \in \mathcal{Q}_r$ and $\alpha \in \mathrm{Aut}(\mathcal{Q}_r)$ such that $d(x) = cx + \alpha(x)c$ for all $x \in \mathcal{R}$. In this case $\mathcal{G}(x) = (b + c)x + \alpha(x)c$ and by Proposition 2.1 either $\mathcal{G}(x) = \lambda x$ for some $\lambda \in \mathcal{C}$, or $f(x_1, \ldots, x_n)^2$ is central-valued on $\mathcal{R}$ and $\mathcal{G}(x) = (b + c)x + xc$ for all $x \in \mathcal{R}$, where $b \in \mathcal{C}$.

Suppose that d is X-outer and that $\alpha \in \mathrm{Aut}(\mathcal{Q}_r)$ is the associated automorphism of d. When α is the identity mapping on $\mathcal{R}$, then d is a usual derivation of $\mathcal{R}$. And hence $\mathcal{G}$ becomes a generalized derivation of $\mathcal{R}$. In this case, the required results are due to Fact 3.1. Hence in what follows we always assume that $1_{\mathcal{R}} \neq \alpha \in \mathrm{Aut}(\mathcal{R})$. We denote by $f^d(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient γ_σ with $d(\gamma_\sigma)$. It should be remarked that

$$\begin{aligned}d(\gamma_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}) &= d(\gamma_\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}\\ &\quad + \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \cdots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \cdots x_{\sigma(n)}.\end{aligned}$$

So we have

$$\begin{aligned}d(f(x_1, \ldots, x_n)) &= f^d(x_1, \ldots, x_n)\\ &\quad + \sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \cdots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \cdots x_{\sigma(n)}.\end{aligned}$$

Since $\mathcal{Q}_r$ satisfies $\Phi(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n))$, it also satisfies

$$\begin{aligned}&a[bf(x_1, \ldots, x_n) + f^d(x_1, \ldots, x_n), f(x_1, \ldots, x_n)\\ &+ a\Big[\sum_{\sigma \in S_n} \alpha(\gamma_\sigma) \sum_{j=0}^{n-1} \alpha(x_{\sigma(1)} \cdots x_{\sigma(j)}) d(x_{\sigma(j+1)}) x_{\sigma(j+2)} \cdots x_{\sigma(n)}, f(x_1, \ldots, x_n)\Big].\end{aligned}$$

By [CL2, Theorem 1] it follows that $\mathcal{Q}_r$ satisfies $\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n)$, that is,

$$a[bf(x_1,\dots,x_n)+f^d(x_1,\dots,x_n),f(x_1,\dots,x_n)]$$
$$+a\Big[\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\sum_{j=0}^{n-1}\alpha(x_{\sigma(1)}\cdots x_{\sigma(j)})y_{\sigma(j+1)}x_{\sigma(j+2)}\cdots x_{\sigma(n)},f(x_1,\dots,x_n)\Big].$$

In particular, for any $i=1,\dots,n$, $\mathcal{Q}_r$ satisfies

$$a\Big[\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)\alpha(x_{\sigma(1)}\cdots x_{\sigma(i-1)})y_{\sigma(i)}x_{\sigma(i+1)}\cdots x_{\sigma(n)},f(x_1,\dots,x_n)\Big]. \tag{3.1}$$

Here we divide the argument into two subcases. Let us first consider the case when α is an inner automorphism of $\mathcal{R}$. Then there exists an invertible element $q\in\mathcal{Q}$ such that $\alpha(x)=qxq^{-1}$ for all $x\in\mathcal{R}$. Since $1_{\mathcal{R}}\neq\alpha\in\mathrm{Aut}(\mathcal{R})$, we may assume that $q\notin\mathcal{C}$. Moreover, it is clear that $\alpha(\gamma_\sigma)=\gamma_\sigma$ for all coefficients involved in $f(x_1,\dots,x_n)$. Replacing each $y_{\sigma(i)}$ with $qx_{\sigma(i)}$ in (3.1), we find that $\mathcal{Q}_r$ satisfies

$$a\Big[q\sum_{\sigma\in S_n}\gamma_\sigma x_{\sigma(1)}\cdot x_{\sigma(2)}\cdots x_{\sigma(i-1)}x_{\sigma(i)}x_{\sigma(i+1)}\cdots x_{\sigma(n)},f(x_1,\dots,x_n)\Big],$$

that is,

$$a[qf(x_1,\dots,x_n),f(x_1,\dots,x_n)].$$

Note that $q\notin\mathcal{C}$ and $f(x_1,\dots,x_n)$ is not central-valued on $\mathcal{Q}_r$. Combining these facts with Fact 2.5 yields $a=0$. We now assume that α is X-outer. In light of Fact 3.2 and the relation (3.1), $\mathcal{Q}_r$ satisfies the generalized polynomial identity

$$a\Big[\sum_{\sigma\in S_n}\alpha(\gamma_\sigma)z_{\sigma(1)}\cdots z_{\sigma(i-1)})y_{\sigma(i)}x_{\sigma(i+1)}\cdots x_{\sigma(n)},f(x_1,\dots,x_n)\Big] \tag{3.2}$$

for all $i=1,\dots,n$. In particular, we choose:

- for all $i\geq 2$, $y_{\sigma(i)}=0$;
- for all $i\geq 2$, $z_{\sigma(i)}=0$.

Therefore by (3.2), $\mathcal{Q}_r$ satisfies the generalized polynomial identity

$$a\Big[y_1\sum_{\sigma\in S_{n-1}}\alpha(\gamma_\sigma)x_{\sigma(2)}\cdots x_{\sigma(n)},f(x_1,\dots,x_n)\Big]. \tag{3.3}$$

Let us write $\sum_{\sigma\in S_{n-1}}\alpha(\gamma_\sigma)x_{\sigma(2)}\cdots x_{\sigma(n)}=t_1(x_2,\dots,x_n)$. Then $\mathcal{Q}_r$ satisfies the generalized polynomial identity

$$a[y_1t_1(x_2,\dots,x_n),f(x_1,\dots,x_n)]. \tag{3.4}$$

Applying [CL1, Lemma 3] to (3.4) we see that

$$[y_1t_1(x_2,\dots,x_n),f(x_1,\dots,x_n)]$$

is a generalized polynomial identity for $\mathcal{Q}_r$. Therefore there exists a suitable field $\mathcal{K}$ and an integer $t \geq 1$ such that $\mathcal{Q}_r$ and the matrix ring $\mathcal{M}_t(\mathcal{K})$ satisfy the same polynomial identities. In particular, $\mathcal{M}_t(\mathcal{K})$ satisfies the generalized polynomial identity $[y_1 t_1(x_2, \ldots, x_n), f(x_1, \ldots, x_n)]$. Since $f(x_1, \ldots, x_n)$ is not central-valued on $\mathcal{Q}_r$, we may assume $t \geq 2$. In this situation, by Fact 2.2, Fact 2.3 and Remark 2.4, for all $i \neq j$, there exist $r_1 \ldots, r_n \in \mathcal{M}_t(\mathcal{K})$ such that $f(r_1, \ldots, r_n) = e_{ij} \neq 0$ and

$$[y_1 t_1(r_2, \ldots, r_n), e_{ij}] = 0 \tag{3.5}$$

for all $y_1 \in \mathcal{M}_t(\mathcal{K})$. Here we also denote by $f^{\alpha}(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ through replacing each coefficient γ_σ with $\alpha(\gamma_\sigma)$. Note that $f^{\alpha}(r_1, \ldots, r_n) \neq 0$. By (3.5), for $y_1 = e_{ii}X$ and for any $X \in \mathcal{M}_t(\mathcal{K})$, we have $e_{ii}X t_1(r_2, \ldots, r_n) e_{ij} = 0$, that is, $t_1(r_2, \ldots, r_n)e_{ij} = 0$. In view of (3.5) we get

$$0 = y_1 t_1(r_2, \ldots, r_n)e_{ij} - e_{ij} y_1 t_1(r_2, \ldots, r_n) = -e_{ij} y_1 t_1(r_2, \ldots, r_n),$$

which implies $t_1(r_2, \ldots, r_n) = 0$. Let us start again from (3.2) and fix an index $j \in \{1, \ldots, n\}$. We choose:

- for all $i \neq j$, $y_{\sigma(i)} = 0$;
- for all $i \neq j$, $z_{\sigma(i)} = 0$.

Therefore by (3.2) we deduce that $\mathcal{Q}_r$ satisfies the generalized polynomial identity

$$a\Big[y_j \sum_{\sigma \in S_{n-1}} \alpha(\gamma_\sigma) x_{\sigma(1)} \cdots x_{\sigma(j-1)} x_{\sigma(j+1)} \cdots x_{\sigma(n)}, f(x_1, \ldots, x_n)\Big]. \tag{3.6}$$

Let us adopt a new notation for later discussion:

$$\sum_{\sigma \in S_{n-1}} \alpha(\gamma_\sigma) x_{\sigma(1)} \cdots x_{\sigma(j-1)} x_{\sigma(j+1)} \cdots x_{\sigma(n)} = t_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n).$$

Thus $\mathcal{Q}_r$ satisfies the generalized polynomial identity

$$a[y_j t_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n), f(x_1, \ldots, x_n)].$$

Moreover, we know that there exist $r_1, \ldots, r_n \in \mathcal{M}_t(\mathcal{K})$ such that $f(r_1, \ldots, r_n) = e_{ij} \neq 0$, and using the above argument, $t_j(r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_n) = 0$. Finally notice that

$$f^{\alpha}(x_1, \ldots, x_n) = \sum_j x_j t_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n),$$

where each t_j is a multilinear polynomial of degree $n-1$ and x_j appears in no monomial of t_j. This leads to the contradiction $f^{\alpha}(r_1, \ldots, r_n) = 0$.

Acknowledgements. The authors heartily thank the referee for many relevant and insightful comments which significantly helped us improve the final presentation of this article. They are also indebted to the copy editor, Jerzy Trzeciak, for his careful work on editing the paper and his patience.

REFERENCES

[BMM] K. I. Beidar, W. S. Martindale III and A. V. Mikhalev, *Rings with Generalized Identities*, Pure Appl. Math., Dekker, New York, 1996.

[Cha1] J.-C. Chang, *On the identitity* $h(x) = af(x) + g(x)b$, Taiwanese J. Math. 7 (2003), 103–113.

[Cha2] J.-C. Chang, *Generalized skew derivations with annihilating Engel conditions*, Taiwanese J. Math. 12 (2009), 1641–1650.

[Cha3] J.-C. Chang, *Generalized skew derivations with nilpotent values on Lie ideals*, Monatsh. Math. 161 (2010), 155–160.

[Cha4] J.-C. Chang, *Generalized skew derivations with power central values on Lie ideals*, Comm. Algebra 39 (2011), 2241–2248.

[CW] H.-W. Cheng and F. Wei, *Generalized skew derivations of rings*, Adv. Math. (China) 35 (2006), 237–243.

[Chu1] C.-L. Chuang, *GPIs having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc. 103 (1988), 723–728.

[Chu2] C.-L. Chuang, *Differential identities with automorphisms and antiautomorphisms I*, J. Algebra 149 (1992), 371–404.

[Chu3] C.-L. Chuang, *Differential identities with automorphisms and antiautomorphisms II*, J. Algebra 160 (1993), 130–171.

[CL1] C.-L. Chuang and T.-K. Lee, *Rings with annihilator conditions on multilinear polynomials*, Chinese J. Math. 24 (1996), 177–185.

[CL2] C.-L. Chuang and T.-K. Lee, *Identities with a single skew derivation*, J. Algebra 288 (2005), 59–77.

[D1] V. De Filippis, *Posner's second theorem and an annihilator condition with generalized derivations*, Turkish J. Math. 32 (2008), 197–211.

[D2] V. De Filippis, *A product of two generalized derivations on polynomials in prime rings*, Collect. Math. 61 (2010), 303–322.

[DD] V. De Filippis and O. M. Di Vincenzo, *Posner's second theorem and an annihilator condition*, Math. Pannonica 12 (2001), 69–81.

[DW] V. De Filippis and F. Wei, *Posner's second theorem for skew derivations on multilinear polynomials on left ideals*, Houston J. Math. 38 (2012), 373–395.

[J] N. Jacobson, *Structure of Rings*, Amer. Math. Soc., Providence, RI, 1964.

[K] V. K. Kharchenko, *Generalized identities with automorphisms*, Algebra i Logika 14 (1975), 215–237 (in Russian); English transl.: Algebra and Logic 14 (1975), 132–148.

[Lan] C. Lanski, *An Engel condition with derivation*, Proc. Amer. Math. Soc. 118 (1993), 731–734.

[L1] T.-K. Lee, *Derivations with invertible values on a multilinear polynomial*, Proc. Amer. Math. Soc. 119 (1993), 1–5.

[L2] T.-K. Lee, *Derivations with Engel conditions on polynomials*, Algebra Colloq. 5 (1998), 13–24.

[L3] T.-K. Lee, *Generalized skew derivations characterized by acting on zero products*, Pacific J. Math. 216 (2004), 293–301.

[LL] P.-H. Lee and T.-K. Lee, *Derivations with Engel conditions on multilinear polynomials*, Proc. Amer. Math. Soc. 124 (1996), 2625–2629.

[Ler] U. Leron, *Nil and power central polynomials in rings*, Trans. Amer. Math. Soc. 202 (1975), 97–103.

[Liu] K.-S. Liu, *Differential identities and constants of algebraic automorphisms in prime rings*, Ph.D. Thesis, National Taiwan Univ., 2006.

[M] W. S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra 12 (1969), 576–584.

[P] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. 8 (1957), 1093–1100.

[W] T.-L. Wong, *Derivations with power central values on multilinear polynomials*, Algebra Colloq. 3 (1996), 369–378.

Vincenzo De Filippis
DI.S.I.A., Faculty of Engineering
University of Messina
Contrada Di Dio
98166 Messina, Italy
E-mail: defilippis@unime.it

Feng Wei (corresponding author)
School of Mathematics
Beijing Institute of Technology
Beijing, 100081, P.R. China
E-mail: daoshuo@hotmail.com

Received 14 August 2012;
revised 17 November 2012 (5735)