ALGEBRAIC AND TOPOLOGICAL PROPERTIES OF SOME SETS IN $\ell_1$

BY

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Abstract. For a sequence $x \in \ell_1 \setminus c_{00}$, one can consider the set $E(x)$ of all subsums of the series $\sum_{n=1}^{\infty} x(n)$. Guthrie and Nymann proved that $E(x)$ is one of the following types of sets: (I) a finite union of closed intervals; (C) homeomorphic to the Cantor set; (MC) homeomorphic to the set $T$ of subsums of $\sum_{n=1}^{\infty} b(n)$ where $b(2n-1) = 3/4^n$ and $b(2n) = 2/4^n$. Denote by $I$, $C$ and $MC$ the sets of all sequences $x \in \ell_1 \setminus c_{00}$ such that $E(x)$ has the property (I), (C) and (MC), respectively. We show that $I$ and $C$ are strongly $c$-algebrable and $MC$ is $c$-lineable. We also show that $C$ is a dense $G_\delta$-set in $\ell_1$ and $I$ is a true $F_\sigma$-set. Finally we show that $I$ is spaceable while $C$ is not.

1. Introduction

1.1. Subsums of series. Let $x \in \ell_1$. The set of all subsums of $\sum_{n=1}^{\infty} x(n)$, meaning the set of sums of all subseries of $\sum_{n=1}^{\infty} x(n)$, is defined by

$$E(x) = \{ a \in \mathbb{R} : \exists A \subset \mathbb{N} \sum_{n \in A} x(n) = a \}.$$

Some authors call it the achievement set of $x$. The following theorem is due to Kakeya.

**Theorem 1.1 ([Ka]).** Let $x \in \ell_1$.

1. If $x \notin c_{00}$, then $E(x)$ is a perfect compact set.
2. If

$$|x(n)| > \sum_{i>n} |x(i)|$$

for almost all $n$,

then $E(x)$ is homeomorphic to the Cantor set.
3. If

$$|x(n)| \leq \sum_{i>n} |x(i)|$$

for $n$ sufficiently large,

then $E(x)$ is a finite union of closed intervals. If $x$ is non-increasing, the converse also holds.

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Moreover, Kakeya conjectured that $E(x)$ is either nowhere dense or a finite union of intervals. Probably, the first counterexample to this conjecture was given (without proof) by Weinstein and Shapiro [WS] and, with a correct proof, by Ferens [F]. Guthrie and Nymann [GN] showed that, for the sequence $b$ given by the formulas $b(2n - 1) = 3/4^n$ and $b(2n) = 2/4^n$, the set $T = E(b)$ is not a finite union of intervals but it has nonempty interior. In the same paper they formulated the following theorem:

**Theorem 1.2 ([GN]).** Let $x \in \ell_1 \setminus c_{00}$. Then $E(x)$ is of one of the following types:

(i) a finite union of closed intervals;

(ii) homeomorphic to the Cantor set;

(iii) homeomorphic to the set $T$.

A correct proof of the Guthrie and Nymann trichotomy was given by Nymann and Sáenz [NS]. The sets homeomorphic to $T$ are called Cantorvals (more precisely: M-Cantorvals). Note that Theorem 1.2 can be formulated as follows: The space $\ell_1$ is a disjoint union of $c_{00}$, $I$, $C$ and $MC$ where $I$ consists of all sequences $x$ with $E(x)$ equal to a finite union of intervals, $C$ consists of all $x$ with $E(x)$ homeomorphic to the Cantor set, and $MC$ of all $x$ with $E(x)$ being an M-Cantorval.

For $x \in \ell_1$, let $x'$ be an arbitrary finite modification of $x$, and let $|x|$ denote the sequence $y \in \ell_1$ such that $y(n) = |x(n)|$. Then $x \in I \iff |x| \in I \iff x' \in I$. The same equivalences hold for $C$ and $MC$.

1.2. Lineability, algebraicity and spaceability. Having an algebra $A$ and its subset $E \subset A$ one can ask if $E \cup \{0\}$ contains a subalgebra $A'$ of $A$. Roughly speaking, if the answer is positive, then $E$ is algebraable. It is a recent trend in mathematical analysis to establish the algebraability of sets $E$ which are far from being linear, that is, $x, y \in E$ does not generally imply $x + y \in E$. Such algebraability results were obtained in sequence spaces (see [BG1], [BGP], [BG2]) and in function spaces (see [ACPS], [AS], [APGS], [GMS] and [GPS]).

Assume that $V$ is a linear space (resp. an algebra). A subset $E \subset V$ is called lineable (resp. algebraable) whenever $E \cup \{0\}$ contains an infinite-dimensional linear space (infinitely generated algebra, respectively) (see [AGS], [B] and [GQ]). For a cardinal $\kappa > \omega$, the set $E$ is $\kappa$-algebraable (i.e. it contains a $\kappa$-generated algebra) if and only if it contains an algebra which is a $\kappa$-dimensional linear space (see [BG1]). Moreover, we say that a subset $E$ of a commutative algebra $V$ is strongly $\kappa$-algebraable ([BG1]) if there exists a $\kappa$-generated free algebra $A$ contained in $E \cup \{0\}$.

Note that $X = \{x_\alpha : \alpha < \kappa\} \subset E$ is a set of free generators of a free algebra $A \subset E$ if and only if the set $X'$ of elements of the form $x_{\alpha_1}^{k_1} \ldots x_{\alpha_n}^{k_n}$
is linearly independent and all linear combinations of elements from $X'$ are in $E \cup \{0\}$. It is easy to see that free algebras have no divisors of zero.

In practice, to prove $\kappa$-algebrability of a set $E \subset V$ we have to find $X \subseteq E$ of cardinality $\kappa$ such that for any polynomial $P$ in $n$ variables and any distinct $x_1, \ldots, x_n \in X$ we have either $P(x_1, \ldots, x_n) \in E$ or $P(x_1, \ldots, x_n) = 0$. To prove the strong $\kappa$-algebrability of $E$ we have to find $X \subseteq E$, $|X| = \kappa$, such that for any non-zero polynomial $P$ and distinct $x_1, \ldots, x_n \in X$ we have $P(x_1, \ldots, x_n) \in E$.

In general, there are subsets of algebras which are algebrable but not strongly algebrable. Let $c_{00}$ be the subset of $c_0$ consisting of all sequences with real terms that are eventually zero. Then the set $c_{00}$ is algebrable in $c_0$ but is not strongly $1$-algebrable [BG1].

Let $X$ be a Banach space. A subset $M$ of $X$ is spaceable if $M \cup \{0\}$ contains an infinite-dimensional closed subspace $Y$ of $X$. Since every infinite-dimensional Banach space contains a linearly independent set of cardinality continuum, spaceability implies $c$-lineability. However, spaceability is a much stronger property. The notions of spaceability and $c$-algebrability are incomparable. We will show that even $c$-algebraeble dense $G_\delta$-sets in $\ell_1$ may not be spaceable. On the other hand, there are sets in $c_0$ which are spaceable but not $1$-algebraeble (see [BG1]).

2. Algebraic substructures in $C$, $I$ and $MC$. In a very nice paper [J] Jones gives the following example. Let $x(n) = 1/2^n$ and $y(n) = 1/3^n$. Then clearly $x \in I$ and $y \in C$. Moreover, $x + y \in C$ and $x - y \in I$. Since $x = (x + y) - y$ and $y = -(x - y) + x$, neither $I$ nor $C$ is closed under pointwise addition. However, in the present paper we show that the sets $C$, $I$ and $MC$ each contain large ($c$-generated) algebraic structures. To prove the strong $c$-algebraeble of $C$ and $I$, we will combine Theorem 1.1 and the method of linearly independent exponents, which was successful in [BGP] and [BG1]. In the next theorem we construct generators as powers of one geometric series $x_q$ ($x_q(n) = q^n$) for $0 < q < 1/2$. Clearly, $x_q \in C$ by Theorem 1.1.

**Theorem 2.1.** $C$ is strongly $c$-algebraeble.

**Proof.** Fix $q \in (0, 1/2)$. Let $\{r_\alpha : \alpha < c\}$ be a linearly independent (over the rationals) set of reals greater than $1$. Let $x_\alpha(n) = q^{r_\alpha n}$. We will show that the set $\{x_\alpha : \alpha < c\}$ generates a free algebra $A$ which, except for the null sequence, is contained in $C$.

To do this, we will show that for any $\beta_1, \ldots, \beta_m \in \mathbb{R} \setminus \{0\}$, any matrix $[k_{il}]_{i \leq m, l \leq j}$ of natural numbers with nonzero distinct rows, and any $\alpha_1 < \cdots < \alpha_j < c$, the sequence $x$ given by

$$x(n) = P(x_{\alpha_1}, \ldots, x_{\alpha_j})(n),$$
where 
\[ P(z_1, \ldots, z_j) = \beta_1 z_1^{k_{11}} \cdots z_j^{k_{1j}} + \cdots + \beta_m z_1^{k_{m1}} \cdots z_j^{k_{mj}}, \]

is in \( C \). We have
\[ x(n) = \beta_1 q^{n(r_{\alpha_1}k_{11} + \cdots + r_{\alpha_j}k_{1j})} + \cdots + \beta_m q^{n(r_{\alpha_1}k_{m1} + \cdots + r_{\alpha_j}k_{mj})}. \]

Since \( r_{\alpha_1}, \ldots, r_{\alpha_j} \) are linearly independent and the rows of \( [k_{il}]_{i \leq m, l \leq j} \) are distinct, the numbers \( r_1 := r_{\alpha_1}k_{11} + \cdots + r_{\alpha_j}k_{1j}, \ldots, r_m := r_{\alpha_1}k_{m1} + \cdots + r_{\alpha_j}k_{mj} \) are distinct. We may assume that \( r_1 < \cdots < r_m \). Then
\[
\frac{|x(n)|}{\sum_{i>n} |x(i)|} = \frac{|\beta_1 q^{nr_1} + \cdots + \beta_m q^{nr_m}|}{\sum_{i>n} (|\beta_1 q^{ir_1} + \cdots + |\beta_m q^{ir_m}|)} \geq \frac{|\beta_1 q^{nr_1} + \cdots + \beta_m q^{nr_m}|}{\sum_{i>n} (|\beta_1 q^{ir_1} + \cdots + |\beta_m q^{ir_m}|)} = \frac{|\beta_1 q^{(n+1)r_1}|}{1-q^1} + \cdots + \frac{|\beta_m q^{(n+1)r_m}|}{1-q^1} \to \frac{1-q^1}{q^1} > 1.
\]

Therefore there is \( n_0 \) such that \( |x(n)| > \sum_{i>n} |x(i)| \) for all \( n \geq n_0 \). Hence, by Theorem 1.1 we conclude that \( x \in C \). 

It is obvious that the geometric sequence \( x_q \), even for \( q > 1/2 \), is not useful to construct the generators of an algebra contained in \( \mathcal{I} \). Indeed, for a sufficiently large exponent \( k \), the sequence \( x_q^k \) belongs to \( C \). So, in the next theorem we use the harmonic series.

**Theorem 2.2.** \( \mathcal{I} \) is strongly \( \mathcal{C} \)-algebraable.

**Proof.** Let \( K \) be a linearly independent subset of \((1, \infty)\) of cardinality \( \mathfrak{c} \). For \( \alpha \in K \), let \( x_\alpha \) be the sequence given by \( x_\alpha(n) = 1/n^\alpha \). We will show that the set \( \{x_\alpha : \alpha \in K\} \) generates a free algebra \( \mathcal{A} \) which is contained in \( \mathcal{I} \cup \{0\} \). To do this, we will show that for any \( \beta_1, \ldots, \beta_m \in \mathbb{R} \setminus \{0\} \), any matrix \( [k_{il}]_{i \leq m, l \leq j} \) of natural numbers with nonzero distinct rows, and any \( \alpha_1 < \cdots < \alpha_j \), the sequence \( x \) defined by
\[
x = P(x_{\alpha_1}, \ldots, x_{\alpha_j})
\]
\[= \beta_1 x_{\alpha_1}^{k_{11}} \cdots x_{\alpha_j}^{k_{1j}} + \beta_2 x_{\alpha_1}^{k_{21}} \cdots x_{\alpha_j}^{k_{2j}} + \cdots + \beta_m x_{\alpha_1}^{k_{m1}} \cdots x_{\alpha_j}^{k_{mj}}
\]
belongs to \( \mathcal{I} \). We have
\[
x(n) = P(x_{\alpha_1}, \ldots, x_{\alpha_j})(n)
\]
\[= \frac{1}{n^{\alpha_1 k_{11} + \cdots + \alpha_j k_{1j}}} + \cdots + \frac{1}{n^{\alpha_1 k_{m1} + \cdots + \alpha_j k_{mj}}}
\]
\[= \frac{1}{n^{\beta_1}} + \cdots + \beta_j \frac{1}{n^{\beta_m}}.
\]
Note that \( p_1, \ldots, p_m \) are distinct. Assume that \( p_1 < \cdots < p_m \). We have

\[
\frac{|x(n)|}{\sum_{k>n} |x(k)|} = \frac{\left| \beta_1 \frac{1}{np_1} + \beta_2 \frac{1}{np_2} + \cdots + \beta_m \frac{1}{np_m} \right|}{\sum_{k>n} \left| \beta_1 \frac{1}{kp_1} + \beta_2 \frac{1}{kp_2} + \cdots + \beta_m \frac{1}{kp_m} \right|} \\
\leq \frac{\sum_{k>n} \left( \left| \beta_1 \frac{1}{kp_1} \right| - \left| \beta_2 \frac{1}{kp_2} \right| - \cdots - \left| \beta_m \frac{1}{kp_m} \right| \right)}{\beta_1 \frac{1}{np_1} + \beta_2 \frac{1}{np_2} + \cdots + \beta_m \frac{1}{np_m}} \\
= \frac{\beta_1 \int_{n+1}^{\infty} \frac{1}{xp_1} \, dx - \beta_2 \int_{n}^{\infty} \frac{1}{xp_2} \, dx - \cdots - \beta_m \int_{n}^{\infty} \frac{1}{xp_m} \, dx}{n \left[ \beta_1 \frac{np_1 - 1}{p_1-1} \left( \frac{1}{np_1} \right) - \beta_2 \frac{np_2 - 1}{p_2-1} \left( \frac{1}{np_2} \right) - \cdots - \beta_m \frac{np_m - 1}{p_m-1} \left( \frac{1}{np_m} \right) \right]} \\
\quad \quad \quad \xrightarrow{n \to \infty} 0 < 1.
\]

Observe that the first inequality holds for \( n \) large enough. Therefore there is \( n_0 \) such that \( |x(n)| \leq \sum_{i>n} |x(i)| \) for any \( n \geq n_0 \). Hence, by Theorem 1.1 we conclude that \( x \in \mathcal{I} \). \( \blacksquare \)

The method described in the next lemma belongs to the mathematical folklore and was used to construct sequences \( x \) with \( E(x) \) being Cantorvals. We present its proof since we have not found it explicitly formulated in the literature.

**Lemma 2.3.** Let \( x \in \ell_1 \) be such that

(i) \( E(x) \) contains an interval;
(ii) \( |x(n)| > \sum_{i>n} |x(i)| \) for infinitely many \( n \);
(iii) \( |x_n| \geq |x_{n+1}| \) for almost all \( n \).

Then \( x \in \mathcal{MC} \).

**Proof.** By (ii)–(iii), the point \( x \) does not belong to \( \mathcal{I} \). By (i), the point \( x \) does not belong to \( \mathcal{C} \). Hence, by Theorem 1.2 we get \( x \in \mathcal{MC} \). \( \blacksquare \)

Until quite recently, only a few examples were known of sequences belonging to \( \mathcal{MC} \). These examples were not very useful to construct a large number of linearly independent sequences. Recently, Jones \( \llbracket \) has constructed a one-parameter family of sequences in \( \mathcal{MC} \). We shall use a modification of his example in the proof of our next theorem.

**Theorem 2.4.** \( \mathcal{MC} \) is \( \mathfrak{c} \)-lineable.

**Proof.** Let

\[
x_q = (4, 3, 2, 4q, 3q, 2q, 4q^2, 3q^2, 2q^2, 4q^3, \ldots)
\]

and

\[
y_q = (1, 1, 1, 1, q, q, q, q, q^2, q^2, q^2, q^2, q^2, q^2, q^2, q^2, q^2, \ldots)
\]
for $q \in [1/6, 2/11)$. Observe that the sequences $x_q$, $q \in [1/6, 2/11)$, are linearly independent. We need to show that each non-zero linear combination of these sequences $x_q$ satisfies assumptions (i)–(iii) of Lemma 2.3 and therefore it is in $\mathcal{M}C$. To prove this, let us fix $q_1 > \cdots > q_m \in [1/6, 2/11)$, $\beta_1, \ldots, \beta_m \in \mathbb{R}$ and define sequences $x$ and $y$ by

$$x(n) = \beta_1 x_{q_1}(n) + \cdots + \beta_m x_{q_m}(n)$$

and

$$y(n) = \beta_1 y_{q_1}(n) + \cdots + \beta_m y_{q_m}(n).$$

First, we will check that for almost all $n$,

$$2 |\beta_1 q_1^n + \cdots + \beta_m q_m^n| > 9 \sum_{k>n} |\beta_1 q_1^k + \cdots + \beta_m q_m^k|.$$  

We have

$$\frac{2 |\beta_1 q_1^n + \cdots + \beta_m q_m^n|}{9 \sum_{k>n} (|\beta_1 q_1^k + \cdots + \beta_m q_m^k|)} \geq \frac{2 |\beta_1 q_1^n + \cdots + \beta_m q_m^n|}{9 \sum_{k>n} (|\beta_1 q_1^k| + \cdots + |\beta_m q_m^k|)}$$

$$= \frac{2 |\beta_1 q_1^n + \cdots + \beta_m q_m^n|}{9 (|\beta_1 (q_1^{n+1} - q_1^n) + \cdots + |\beta_m (q_m^{n+1} - q_m^n)|)} \rightarrow \frac{2}{9} \cdot \frac{1 - q_1}{q_1} \geq \frac{2}{9} \cdot \frac{1 - 2/11}{2/11} = 1.$$ 

Note that if $n$ is not divisible by $3$, then $|x(n)| \geq |x(n + 1)|$. On the other hand, if $n = 3l$, then

$$|x(n)| = 2 |\beta_1 q_1^l + \cdots + \beta_m q_m^l|$$

and

$$|x(n + 1)| = 3 |\beta_1 q_1^{l+1} + \cdots + \beta_m q_m^{l+1}| \leq 9 \sum_{k>l} |\beta_1 q_1^k + \cdots + \beta_m q_m^k|.$$ 

Hence by (2.1) we obtain $|x(n)| \geq |x(n + 1)|$ for almost all $n$. By (2.1) we also have $|x(n)| > \sum_{i>n} |x(i)|$ for infinitely many $n$.

Now we will show that

$$|\beta_1 q_1^n + \cdots + \beta_m q_m^n| \leq 5 \sum_{k>n} |\beta_1 q_1^k + \cdots + \beta_m q_m^k|.$$ 

We have

$$\frac{|\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n|}{5 \sum_{k>n} |\beta_1 q_1^k + \beta_2 q_2^k + \cdots + \beta_m q_m^k|} \leq \frac{|\beta_1 + \beta_2 (\frac{q_2}{q_1})^n + \cdots + \beta_m (\frac{q_m}{q_1})^n|}{5 |\sum_{i>0} q_1^i + \beta_2 (\frac{q_2}{q_1})^n \sum_{i>0} q_2^i + \cdots + \beta_m (\frac{q_m}{q_1})^n \sum_{i>0} q_m^i|}$$

$$= \frac{1}{5} \cdot \frac{1 - q_1}{q_1} \leq \frac{1}{5} \cdot \frac{1 - 1/6}{1/6} = 1.$$
By (2.2) we find that $|y(n)| \leq \sum_{k>n} |y(k)|$ for almost all $n$. Therefore by Theorem 1.1, the set $E(y)$ is a finite union of closed intervals. Thus $E(y)$ has non-empty interior.

To end the proof we need to show that $E(x)$ has non-empty interior. We will prove that

$$2 \sum_{n=0} (\beta_1 q_1^n + \cdots + \beta_m q_m^n) + E(y) \subseteq E(x).$$

Let

$$t \in 2 \sum_{n=0} (\beta_1 q_1^n + \cdots + \beta_m q_m^n) + E(y).$$

Note that any element $s$ of $E(y)$ is of the form

$$s = k_0(\beta_1 + \cdots + \beta_m) + k_1(\beta_1 q_1 + \cdots + \beta_m q_m)$$

$$+ k_2(\beta_1 q_1^2 + \cdots + \beta_m q_m^2) + \cdots$$

where $k_n \in \{0, 1, 2, 3, 4, 5\}$. Thus $t$ is of the form

$$t = 2 \sum_{n=0} (\beta_1 q_1^n + \cdots + \beta_m q_m^n)$$

$$+ [k_0(\beta_1 + \cdots + \beta_m) + k_1(\beta_1 q_1 + \cdots + \beta_m q_m)$$

$$+ k_2(\beta_1 q_1^2 + \cdots + \beta_m q_m^2) + \cdots]$$

$$= (2 + k_0)(\beta_1 + \cdots + \beta_m) + (2 + k_1)(\beta_1 q_1 + \cdots + \beta_m q_m)$$

$$+ (2 + k_2)(\beta_1 q_1^2 + \cdots + \beta_m q_m^2) + \cdots.$$

Note that each number from $\{2, 3, 4, 5, 6, 7\}$, that is, every number of the form $2 + k_n$, can be written as a sum of numbers $4, 3, 2$. Hence $t \in E(x)$ and $E(x)$ has non-empty interior. So $x \in \mathcal{MC}$.  

3. The topological size and Borel class of $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{MC}$. Let us observe that the sets $c_{00}$, $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{MC}$ are all dense in $\ell_1$. Moreover, $c_{00}$ is an $\mathcal{F}_\sigma$-set of the first category. We are interested in the topological size and Borel class of these sets. For this, let us consider the hyperspace $H(\mathbb{R})$ of all non-empty compact subsets of reals, equipped with the Vietoris topology (see [Ke 4F, pp. 24–28]). Recall that the Vietoris topology is generated by the subbase of sets of the form $\{K \in H(\mathbb{R}) : K \subset U\}$ and $\{K \in H(\mathbb{R}) : K \cap U \neq \emptyset\}$ for all open sets $U$ in $\mathbb{R}$. This topology is metrizable by the Hausdorff metric $d_H$ given by the formula

$$d_H(A, B) = \max \left\{ \max_{t \in A} d(t, B), \max_{s \in B} d(s, A) \right\}$$

where $d$ is the natural metric in $\mathbb{R}$. It is known that the set $N$ of all nowhere dense compact sets is a $G_\delta$-set in $H(\mathbb{R})$ and the set $F$ of all compact sets
with a finite number of connected components is an $F_\sigma$-set. To see this, it is enough to observe that

- $K$ is nowhere dense if and only if for any set $U_n$ from a fixed countable base of the natural topology in $\mathbb{R}$ there exists a set $U_m$ from this base such that $\text{cl}(U_m) \subset U_n$ and $K \subset (\text{cl}(U_m))^c$;
- $K$ has more than $k$ components if and only if there exist pairwise disjoint open intervals $J_1, \ldots, J_{k+1}$ such that $K \subset J_1 \cup \cdots \cup J_{k+1}$ and $K \cap J_i \neq \emptyset$ for $i = 1, \ldots, k+1$.

Now, let us observe that if we assign the set $E(x)$ to the sequence $x \in \ell_1$, we actually define a function $E : \ell_1 \to H(\mathbb{R})$.

**Lemma 3.1.** The function $E$ is Lipschitz with Lipschitz constant $L = 1$, hence it is continuous.

**Proof.** Let $t \in E(x)$. Then there exists a subset $A$ of $\mathbb{N}$ such that $t = \sum_{n \in A} x(n)$. We have

$$d(t, E(y)) \leq d\left(t, \sum_{n \in A} y(n)\right) = \left| \sum_{n \in A} (x(n) - y(n)) \right| \leq \sum_{n \in \mathbb{N}} |x(n) - y(n)|$$

where $\| \cdot \|_1$ denotes the norm in $\ell_1$. Hence, $d_H(E(x), E(y)) \leq \|x - y\|_1$. 

**Theorem 3.2.** The set $C$ is a dense $G_\delta$-set (and hence residual), $I$ is a true $F_\sigma$-set (i.e. it is $F_\sigma$ but not $G_\delta$) of the first category, and $MC$ is in the class $(F_\sigma \cap G_{\delta \sigma}) \setminus G_\delta$.

**Proof.** Let us observe that $C \cup c_{00} = E^{-1}[N]$ and $I \cup c_{00} = E^{-1}[F]$ where $N$, $F$, $E$ are defined as before. Hence $C \cup c_{00}$ is $G_\delta$ and $I \cup c_{00}$ is $F_\sigma$. Thus $C$ is $G_\delta$ (because $c_{00}$ is $F_\sigma$) and $I \cup MC$ is $F_\sigma$. Moreover, $I = (I \cup c_{00}) \cap (I \cup MC)$ is $F_\sigma$, too. By the density of $C$, $C$ is residual. Since $I$ is dense of the first category, it cannot be $G_\delta$. For the same reason, $MC$ cannot be $G_\delta$. Since $MC$ is a difference of two $F_\sigma$-sets, it is in the class $F_{\sigma \delta} \cap G_{\delta \sigma}$.

**Remark 3.3.** In [BG1] the following similar result was shown by quite different methods: the set of bounded sequences, with the set of limit points homeomorphic to the Cantor set, is strongly $c$-algebrable and residual in $\ell_\infty$.

4. **Spaceability.** In this section we will show that $I$ is spaceable while $C$ is not. This shows that there is a subset $M$ of $\ell_1$ containing a dense $G_\delta$-subset and a linear subspace of dimension $c$, but $Y \setminus M \neq \emptyset$ for any infinite-dimensional closed subspace $Y$ of $\ell_1$.

**Theorem 4.1.** Let $I_1$ be the subset of $I$ which consists of those $x \in \ell_1$ for which $E(x)$ is an interval. Then $I_1$ is spaceable.
Proof. Let $A_1, A_2, \ldots$ be a partition of $\mathbb{N}$ into infinitely many infinite subsets. Let $A_n = \{k_1^n < k_2^n < \cdots \}$. Define $x_n \in \ell_1$ by $x_n(k_i^n) = 2^{-j}$ and $x_n(i) = 0$ if $i \notin A_n$. Then $\|x_n\|_1 = 1$ and $\{x_n : x \in \mathbb{N}\}$ forms a normalised basic sequence. Let $Y$ be a closed linear space generated by $\{x_n : x \in \mathbb{N}\}$. Then

$$y \in Y \iff \exists t \in \ell_1 \left( y = \sum_{n=1}^{\infty} t(n)x_n \right).$$

Since $E(x_n) = [0,1]$, we have $E(\sum_{n=1}^{\infty} t(n)x_n) = \bigcup_{n=1}^{\infty} I_n$ where $I_n$ is an interval with endpoints 0 and $t(n)$. Put $t^+(n) = \max\{t(n), 0\}$ and $t^-(n) = \min\{-t(n), 0\}$. Then $E(\sum_{n=1}^{\infty} t(n)x_n) = [\sum_{n=1}^{\infty} t^-(n), \sum_{n=1}^{\infty} t^+(n)]$ and the result follows. ■

Let us mention the very recent result by Bernal-González and Ordóñez Cabrera [BO, Theorem 2.2], who gave sufficient conditions for spaceability of sets in Banach spaces. Using that result, one can prove the spaceability of $\mathcal{I}$, but it cannot be used to prove Theorem 4.1, since the assumptions are not satisfied.

However, we do not know other results giving sufficient conditions for a set in a Banach space not to be spaceable. An interesting example of a non-spaceable set was given in the classical paper [G] by Gurariy where it was proved that the set of all differentiable functions from $C[0,1]$ is not spaceable. It is well known that the set of all differentiable functions in $C[0,1]$ is dense but meager. We will prove that even dense $G_\delta$-sets in Banach spaces may not be spaceable.

**Theorem 4.2.** Let $Y$ be an infinite-dimensional closed subspace of $\ell_1$. Then there is $y \in Y$ such that $E(y)$ contains an interval.

Proof. Let $\varepsilon_n \downarrow 0$. Let $x_1$ be any non-zero element of $Y$ with $\|x_1\|_1 = 1 + \varepsilon_1$. Since $x_1 \in \ell_1$, there is $n_1$ with $\sum_{n=n_1+1}^{\infty} |x_1(n)| \leq \varepsilon_1$. Let $E_1$ consist of all finite sums $\sum_{n=1}^{n_1} \delta_n x_1(n)$ where $\delta_n \in \{0,1\}$. Then $E_1$ is a finite set with $\min E_1 = \sum_{n=1}^{n_1} x_1^-(n)$, $\max E_1 = \sum_{n=1}^{n_1} x_1^+(n)$ and $1 \leq \max E_1 - \min E_1 \leq 1 + \varepsilon_1$.

Let $Y_1 = Y \cap \{x \in \ell_1 : x(n) = 0 \text{ for every } n \leq n_1\}$. As $\{x \in \ell_1 : x(n) = 0 \text{ for every } n \leq n_1\}$ has a finite codimension, $Y_1$ is infinite-dimensional. Let $x_2$ be any non-zero element of $Y_1$ with $\|x_2\|_1 = 1 + \varepsilon_2$. Since $x_2 \in \ell_1$, there is $n_2 > n_1$ with $\sum_{n=n_2+1}^{\infty} |x_2(n)| \leq \varepsilon_2$, $i = 1,2$. Let $E_2$ consist of all finite sums $\sum_{n=n_1+1}^{n_2} \delta_n x_2(n)$, where $\delta_i \in \{0,1\}$. Then $E_2$ is a finite set with $\min E_2 = \sum_{n=n_1+1}^{n_2} x_2^-(n)$, $\max E_2 = \sum_{n=n_1+1}^{n_2} x_2^+(n)$ and $1 \leq \max E_2 - \min E_2 \leq 1 + \varepsilon_2$.

Proceeding inductively, we define natural numbers $n_1 < n_2 < \cdots$ and infinite-dimensional closed spaces $Y \supset Y_1 \supset Y_2 \supset \cdots$ such that $Y_k = \{x \in Y : x(n) = 0 \text{ for every } n \leq n_k\}$, non-zero elements $x_k \in Y_{k-1}$ with $\|x_k\|_1 = 1 + \varepsilon_k$.
and \( \sum_{n=n_k+1}^{\infty} |x_i(n)| \leq \varepsilon_k, \) \( i = 1, \ldots, k, \) and finite sets \( E_k \) consisting of all sums \( \sum_{n=n_k-1}^{n_k} \delta_n x_k(n) \) where \( \delta_i \in \{0,1\}. \) Note that \( 1 \leq \text{diam}(E_k) \leq 1 + \varepsilon_k. \) Consider \( y = \sum_{k=1}^{\infty} x_k/2^k. \) We claim that \( E(y) \) contains the interval \( I := [\min E_1, \max E_1]. \)

Note that for any \( t \in I \) there is \( t_1 \in E_1 \) with \(|t - t_1| \leq (1 + \varepsilon_1)/2. \) Since \( 1 \leq \text{diam}(E_2) \leq 1 + \varepsilon_2, \) there is \( t_2 \in E_1 + \frac{1}{2} E_2 \) with \(|t - t_2| \leq (1 + \varepsilon_2)/2^2. \) Hence, there is \( t \in E(x_1 + x_2/2) \) with \(|t - \tilde{t}| \leq (1 + \varepsilon_2)/2^3 + \varepsilon_1. \) Since \( 1 \leq \text{diam}(E_k) \leq 1 + \varepsilon_k, \) inductively we can find \( t_k \in E_1 + \frac{1}{2} E_2 + \cdots + \frac{1}{2^k} E_k \) with \(|t - t_k| \leq (1 + \varepsilon_k)/2^k. \) Hence, there is \( \tilde{t} \in E(x_1 + x_2/2 + \cdots + x_k/2^{k-1}) \) with

\[
|t - \tilde{t}| \leq (1 + \varepsilon_k)/2^k + \varepsilon_{k-1}/2 + \cdots + \varepsilon_{k-1}/2^{k-1} \leq (1 + \varepsilon_k)/2^k + 2\varepsilon_{k-1}.
\]

Since \( E(y) \) is closed and contains \( E(x_1 + x_2/2 + \cdots + x_k/2^{k-1}) \), it follows that \( t \in E(y) \) and consequently \( I \subset E(y). \)

Immediately we get the following.

**Corollary 4.3.** The set \( C \) is not spaceable.

We end the paper with some open questions on the set \( MC. \)

**Problem 4.4.**

(i) Is \( MC \) \( \sigma \)-algebrable?

(ii) Is \( MC \) an \( F_\sigma \)-subset of \( \ell_1? \)

(iii) Is \( MC \) spaceable?

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**References**


SOME SETS IN $\ell_1$


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