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ALGEBRAIC AND TOPOLOGICAL PROPERTIES OF SOME SETS IN ℓ_1

BY

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Abstract. For a sequence $x \in \ell_1 \setminus c_{00}$, one can consider the set $E(x)$ of all subsums of the series $\sum_{n=1}^{\infty} x(n)$. Guthrie and Nymann proved that $E(x)$ is one of the following types of sets: ($\mathcal{I}$) a finite union of closed intervals; ($\mathcal{C}$) homeomorphic to the Cantor set; ($\mathcal{MC}$) homeomorphic to the set T of subsums of $\sum_{n=1}^{\infty} b(n)$ where $b(2n-1) = 3/4^n$ and $b(2n) = 2/4^n$. Denote by $\mathcal{I}$, $\mathcal{C}$ and $\mathcal{MC}$ the sets of all sequences $x \in \ell_1 \setminus c_{00}$ such that $E(x)$ has the property ($\mathcal{I}$), ($\mathcal{C}$) and ($\mathcal{MC}$), respectively. We show that $\mathcal{I}$ and $\mathcal{C}$ are strongly $\mathfrak{c}$-algebrable and $\mathcal{MC}$ is $\mathfrak{c}$-lineable. We also show that $\mathcal{C}$ is a dense $\mathcal{G}_\delta$-set in ℓ_1 and $\mathcal{I}$ is a true $\mathcal{F}_\sigma$-set. Finally we show that $\mathcal{I}$ is spaceable while $\mathcal{C}$ is not.

1. Introduction

1.1. Subsums of series. Let $x \in \ell_1$. The set of all subsums of $\sum_{n=1}^{\infty} x(n)$, meaning the set of sums of all subseries of $\sum_{n=1}^{\infty} x(n)$, is defined by

$$E(x) = \Big\{a \in \mathbb{R} : \exists A \subset \mathbb{N} \ \sum_{n\in A} x(n) = a\Big\}.$$

Some authors call it the *achievement set* of x. The following theorem is due to Kakeya.

THEOREM 1.1 ([Ka]). *Let $x \in \ell_1$.*

(1) *If $x \notin c_{00}$, then $E(x)$ is a perfect compact set.*
(2) *If*
$$|x(n)| > \sum_{i>n} |x(i)| \quad \text{for almost all } n,$$
then $E(x)$ is homeomorphic to the Cantor set.
(3) *If*
$$|x(n)| \le \sum_{i>n} |x(i)| \quad \text{for } n \text{ sufficiently large,}$$
then $E(x)$ is a finite union of closed intervals. If x is non-increasing, the converse also holds.

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Moreover, Kakeya conjectured that $E(x)$ is either nowhere dense or a finite union of intervals. Probably, the first counterexample to this conjecture was given (without proof) by Weinstein and Shapiro [WS] and, with a correct proof, by Ferens [F]. Guthrie and Nymann [GN] showed that, for the sequence b given by the formulas $b(2n-1) = 3/4^n$ and $b(2n) = 2/4^n$, the set $T = E(b)$ is not a finite union of intervals but it has nonempty interior. In the same paper they formulated the following theorem:

THEOREM 1.2 ([GN]). *Let $x \in \ell_1 \setminus c_{00}$. Then $E(x)$ is of one of the following types:*

(i) *a finite union of closed intervals;*
(ii) *homeomorphic to the Cantor set;*
(iii) *homeomorphic to the set T.*

A correct proof of the Guthrie and Nymann trichotomy was given by Nymann and Sáenz [NS]. The sets homeomorphic to T are called *Cantorvals* (more precisely: *M-Cantorvals*). Note that Theorem 1.2 can be formulated as follows: The space ℓ_1 is a disjoint union of c_{00}, $\mathcal{I}$, $\mathcal{C}$ and $\mathcal{MC}$ where $\mathcal{I}$ consists of all sequences x with $E(x)$ equal to a finite union of intervals, $\mathcal{C}$ consists of all x with $E(x)$ homeomorphic to the Cantor set, and $\mathcal{MC}$ of all x with $E(x)$ being an M-Cantorval.

For $x \in \ell_1$, let x' be an arbitrary finite modification of x, and let $|x|$ denote the sequence $y \in \ell_1$ such that $y(n) = |x(n)|$. Then $x \in \mathcal{I} \Leftrightarrow |x| \in \mathcal{I} \Leftrightarrow x' \in \mathcal{I}$. The same equivalences hold for $\mathcal{C}$ and $\mathcal{MC}$.

1.2. Lineability, algebrability and spaceability. Having an algebra A and its subset $E \subset A$ one can ask if $E \cup \{0\}$ contains a subalgebra A' of A. Roughly speaking, if the answer is positive, then E is algebrable. It is a recent trend in mathematical analysis to establish the algebrability of sets E which are far from being linear, that is, $x, y \in E$ does not generally imply $x + y \in E$. Such algebrability results were obtained in sequence spaces (see [BG1], [BGP] and [BG2]) and in function spaces (see [ACPS], [AS], [APGS], [GMS] and [GPS]).

Assume that V is a linear space (resp. an algebra). A subset $E \subset V$ is called *lineable* (resp. *algebrable*) whenever $E \cup \{0\}$ contains an infinite-dimensional linear space (infinitely generated algebra, respectively) (see [AGS], [B] and [GQ]). For a cardinal $\kappa > \omega$, the set E is *κ-algebrable* (i.e. it contains a κ-generated algebra) if and only if it contains an algebra which is a κ-dimensional linear space (see [BG1]). Moreover, we say that a subset E of a commutative algebra V is *strongly κ-algebrable* ([BG1]) if there exists a κ-generated free algebra A contained in $E \cup \{0\}$.

Note that $X = \{x_\alpha : \alpha < \kappa\} \subset E$ is a set of *free generators* of a *free algebra* $A \subset E$ if and only if the set X' of elements of the form $x_{\alpha_1}^{k_1} \dots x_{\alpha_n}^{k_n}$

is linearly independent and all linear combinations of elements from X' are in $E \cup \{0\}$. It is easy to see that free algebras have no divisors of zero.

In practice, to prove κ-algebrability of a set $E \subset V$ we have to find $X \subseteq E$ of cardinality κ such that for any polynomial P in n variables and any distinct $x_1, \ldots, x_n \in X$ we have either $P(x_1, \ldots, x_n) \in E$ or $P(x_1, \ldots, x_n) = 0$. To prove the strong κ-algebrability of E we have to find $X \subset E$, $|X| = \kappa$, such that for any non-zero polynomial P and distinct $x_1, \ldots, x_n \in X$ we have $P(x_1, \ldots, x_n) \in E$.

In general, there are subsets of algebras which are algebrable but not strongly algebrable. Let c_{00} be the subset of c_0 consisting of all sequences with real terms that are eventually zero. Then the set c_{00} is algebrable in c_0 but is not strongly 1-algebrable [BG1].

Let X be a Banach space. A subset M of X is *spaceable* if $M \cup \{0\}$ contains an infinite-dimensional closed subspace Y of X. Since every infinite-dimensional Banach space contains a linearly independent set of cardinality continuum, spaceability implies $\mathfrak{c}$-lineability. However, spaceability is a much stronger property. The notions of spaceability and $\mathfrak{c}$-algebrability are incomparable. We will show that even $\mathfrak{c}$-algebrable dense $\mathcal{G}_\delta$-sets in ℓ_1 may not be spaceable. On the other hand, there are sets in c_0 which are spaceable but not 1-algebrable (see [BG1]).

2. Algebraic substructures in $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{MC}$. In a very nice paper [J] Jones gives the following example. Let $x(n) = 1/2^n$ and $y(n) = 1/3^n$. Then clearly $x \in \mathcal{I}$ and $y \in \mathcal{C}$. Moreover, $x + y \in \mathcal{C}$ and $x - y \in \mathcal{I}$. Since $x = (x+y)-y$ and $y = -(x-y)+x$, neither $\mathcal{I}$ nor $\mathcal{C}$ is closed under pointwise addition. However, in the present paper we show that the sets $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{MC}$ each contain large ($\mathfrak{c}$-generated) algebraic structures. To prove the strong $\mathfrak{c}$-algebrability of $\mathcal{C}$ and $\mathcal{I}$, we will combine Theorem 1.1 and the method of linearly independent exponents, which was successful in [BGP] and [BG1]. In the next theorem we construct generators as powers of one geometric series x_q ($x_q(n) = q^n$) for $0 < q < 1/2$. Clearly, $x_q \in \mathcal{C}$ by Theorem 1.1.

THEOREM 2.1. *$\mathcal{C}$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Fix $q \in (0, 1/2)$. Let $\{r_\alpha : \alpha < \mathfrak{c}\}$ be a linearly independent (over the rationals) set of reals greater than 1. Let $x_\alpha(n) = q^{r_\alpha n}$. We will show that the set $\{x_\alpha : \alpha < \mathfrak{c}\}$ generates a free algebra $\mathcal{A}$ which, except for the null sequence, is contained in $\mathcal{C}$.

To do this, we will show that for any $\beta_1, \ldots, \beta_m \in \mathbb{R} \setminus \{0\}$, any matrix $[k_{il}]_{i \le m, l \le j}$ of natural numbers with nonzero distinct rows, and any $\alpha_1 < \cdots < \alpha_j < \mathfrak{c}$, the sequence x given by

$$x(n) = P(x_{\alpha_1}, \ldots, x_{\alpha_j})(n),$$

where

$$P(z_1,\dots,z_j)=\beta_1 z_1^{k_{11}}\dots z_j^{k_{1j}}+\dots+\beta_m z_1^{k_{m1}}\dots z_j^{k_{mj}},$$

is in $\mathcal{C}$. We have

$$x(n)=\beta_1 q^{n(r_{\alpha_1}k_{11}+\dots+r_{\alpha_j}k_{1j})}+\dots+\beta_m q^{n(r_{\alpha_1}k_{m1}+\dots+r_{\alpha_j}k_{mj})}.$$

Since $r_{\alpha_1},\dots,r_{\alpha_j}$ are linearly independent and the rows of $[k_{il}]_{i\le m,\,l\le j}$ are distinct, the numbers $r_1:=r_{\alpha_1}k_{11}+\dots+r_{\alpha_j}k_{1j},\dots,r_m:=r_{\alpha_1}k_{m1}+\dots+r_{\alpha_j}k_{mj}$ are distinct. We may assume that $r_1<\dots<r_m$. Then

$$\begin{aligned}\frac{|x(n)|}{\sum_{i>n}|x(i)|}&=\frac{|\beta_1 q^{nr_1}+\dots+\beta_m q^{nr_m}|}{\sum_{i>n}|\beta_1 q^{ir_1}+\dots+\beta_m q^{ir_m}|}\\&\ge\frac{|\beta_1 q^{nr_1}+\dots+\beta_m q^{nr_m}|}{\sum_{i>n}(|\beta_1|q^{ir_1}+\dots+|\beta_m|q^{ir_m})}=\frac{|\beta_1 q^{nr_1}+\dots+\beta_m q^{nr_m}|}{\frac{|\beta_1|q^{(n+1)r_1}}{1-q^{r_1}}+\dots+\frac{|\beta_m|q^{(n+1)r_m}}{1-q^{r_m}}}\\&\hspace{20em}\to\frac{1-q^{r_1}}{q^{r_1}}>1.\end{aligned}$$

Therefore there is n_0 such that $|x(n)|>\sum_{i>n}|x(i)|$ for all $n\ge n_0$. Hence, by Theorem 1.1, we conclude that $x\in\mathcal{C}$. ∎

It is obvious that the geometric sequence x_q, even for $q>1/2$, is not useful to construct the generators of an algebra contained in $\mathcal{I}$. Indeed, for a sufficiently large exponent k, the sequence x_q^k belongs to $\mathcal{C}$. So, in the next theorem we use the harmonic series.

Theorem 2.2. *$\mathcal{I}$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Let K be a linearly independent subset of $(1,\infty)$ of cardinality $\mathfrak{c}$. For $\alpha\in K$, let x_α be the sequence given by $x_\alpha(n)=1/n^\alpha$. We will show that the set $\{x_\alpha:\alpha\in K\}$ generates a free algebra $\mathcal{A}$ which is contained in $\mathcal{I}\cup\{0\}$. To do this, we will show that for any $\beta_1,\dots,\beta_m\in\mathbb{R}\setminus\{0\}$, any matrix $[k_{il}]_{i\le m,\,l\le j}$ of natural numbers with nonzero distinct rows, and any $\alpha_1<\dots<\alpha_j$, the sequence x defined by

$$\begin{aligned}x&=P(x_{\alpha_1},\dots,x_{\alpha_j})\\&=\beta_1 x_{\alpha_1}^{k_{11}}\dots x_{\alpha_j}^{k_{1j}}+\beta_2 x_{\alpha_1}^{k_{21}}\dots x_{\alpha_j}^{k_{2j}}+\dots+\beta_m x_{\alpha_1}^{k_{m1}}\dots x_{\alpha_j}^{k_{mj}}\end{aligned}$$

belongs to $\mathcal{I}$. We have

$$\begin{aligned}x(n)&=P(x_{\alpha_1},\dots,x_{\alpha_j})(n)\\&=\beta_1\frac{1}{n^{\alpha_1k_{11}+\dots+\alpha_jk_{1j}}}+\dots+\beta_m\frac{1}{n^{\alpha_1k_{m1}+\dots+\alpha_jk_{mj}}}\\&=\beta_1\frac{1}{n^{p_1}}+\dots+\beta_j\frac{1}{n^{p_m}}.\end{aligned}$$

Note that $p_1, \dots, p_m$ are distinct. Assume that $p_1 < \dots < p_m$. We have

$$\begin{aligned}
\frac{|x(n)|}{\sum_{k>n}|x(k)|} &= \frac{\left|\beta_1 \frac{1}{n^{p_1}} + \beta_2 \frac{1}{n^{p_2}} + \dots + \beta_m \frac{1}{n^{p_m}}\right|}{\sum_{k>n}\left|\beta_1 \frac{1}{k^{p_1}} + \beta_2 \frac{1}{k^{p_2}} + \dots + \beta_m \frac{1}{k^{p_m}}\right|} \\
&\le \frac{\left|\beta_1 \frac{1}{n^{p_1}} + \beta_2 \frac{1}{n^{p_2}} + \dots + \beta_m \frac{1}{n^{p_m}}\right|}{\sum_{k>n}\left(\left|\beta_1 \frac{1}{k^{p_1}}\right| - \left|\beta_2 \frac{1}{k^{p_2}}\right| - \dots - \left|\beta_m \frac{1}{k^{p_m}}\right|\right)} \\
&\le \frac{\left|\beta_1 \frac{1}{n^{p_1}} + \beta_2 \frac{1}{n^{p_2}} + \dots + \beta_m \frac{1}{n^{p_m}}\right|}{|\beta_1| \int_{n+1}^{\infty} \frac{1}{x^{p_1}}\,dx - |\beta_2| \int_{n}^{\infty} \frac{1}{x^{p_2}}\,dx - \dots - |\beta_m| \int_{n}^{\infty} \frac{1}{x^{p_m}}\,dx} \\
&= \frac{\left|\beta_1 + \beta_2 \frac{n^{p_1}}{n^{p_2}} + \dots + \beta_m \frac{n^{p_1}}{n^{p_m}}\right|}{n\left[|\beta_1| \frac{1}{p_1-1} \frac{n^{p_1-1}}{(n+1)^{p_1-1}} - |\beta_2| \frac{1}{p_2-1} \frac{n^{p_1-1}}{n^{p_2-1}} - \dots - |\beta_m| \frac{1}{p_m-1} \frac{n^{p_1-1}}{n^{p_m-1}}\right]} \\
&\underset{n\to\infty}{\longrightarrow} 0 < 1.
\end{aligned}$$

Observe that the first inequality holds for n large enough. Therefore there is n_0 such that $|x(n)| \le \sum_{i>n}|x(i)|$ for any $n \ge n_0$. Hence, by Theorem 1.1 we conclude that $x \in \mathcal{I}$. ■

The method described in the next lemma belongs to the mathematical folklore and was used to construct sequences x with $E(x)$ being Cantorvals. We present its proof since we have not found it explicitly formulated in the literature.

LEMMA 2.3. *Let $x \in \ell_1$ be such that*

(i) *$E(x)$ contains an interval;*
(ii) *$|x(n)| > \sum_{i>n}|x(i)|$ for infinitely many n;*
(iii) *$|x_n| \ge |x_{n+1}|$ for almost all n.*

Then $x \in \mathcal{MC}$.

Proof. By (ii)–(iii), the point x does not belong to $\mathcal{I}$. By (i), the point x does not belong to $\mathcal{C}$. Hence, by Theorem 1.2 we get $x \in \mathcal{MC}$. ■

Until quite recently, only a few examples were known of sequences belonging to $\mathcal{MC}$. These examples were not very useful to construct a large number of linearly independent sequences. Recently, Jones [J] has constructed a one-parameter family of sequences in $\mathcal{MC}$. We shall use a modification of his example in the proof of our next theorem.

THEOREM 2.4. *$\mathcal{MC}$ is $\mathfrak{c}$-lineable.*

Proof. Let

$$x_q = (4, 3, 2, 4q, 3q, 2q, 4q^2, 3q^2, 2q^2, 4q^3, \dots)$$

and

$$y_q = (1, 1, 1, 1, 1, q, q, q, q, q, q^2, q^2, q^2, q^2, q^2, q^3, \dots)$$

for $q \in [1/6, 2/11)$. Observe that the sequences x_q, $q \in [1/6, 2/11)$, are linearly independent. We need to show that each non-zero linear combination of these sequences x_q satisfies assumptions (i)–(iii) of Lemma 2.3 and therefore it is in $\mathcal{MC}$. To prove this, let us fix $q_1 > \cdots > q_m \in [1/6, 2/11)$, $\beta_1, \ldots, \beta_m \in \mathbb{R}$ and define sequences x and y by

$$x(n) = \beta_1 x_{q_1}(n) + \cdots + \beta_m x_{q_m}(n)$$

and

$$y(n) = \beta_1 y_{q_1}(n) + \cdots + \beta_m y_{q_m}(n).$$

First, we will check that for almost all n,

$$2|\beta_1 q_1^n + \cdots + \beta_m q_m^n| > 9 \sum_{k>n} |\beta_1 q_1^k + \cdots + \beta_m q_m^k|. \tag{2.1}$$

We have

$$\frac{2|\beta_1 q_1^n + \cdots + \beta_m q_m^n|}{9 \sum_{k>n}(|\beta_1 q_1^k + \cdots + \beta_m q_m^k|)} \geq \frac{2|\beta_1 q_1^n + \cdots + \beta_m q_m^n|}{9 \sum_{k>n}(|\beta_1 q_1^k| + \cdots + |\beta_m q_m^k|)}$$
$$= \frac{2|\beta_1 q_1^n + \cdots + \beta_m q_m^n|}{9\big(|\beta_1| \frac{q_1^{n+1}}{1-q_1} + \cdots + |\beta_m| \frac{q_m^{n+1}}{1-q_m}\big)} \underset{n\to\infty}{\longrightarrow} \frac{2}{9} \cdot \frac{1-q_1}{q_1} > \frac{2}{9} \cdot \frac{1-2/11}{2/11} = 1.$$

Note that if n is not divisible by 3, then $|x(n)| \geq |x(n+1)|$. On the other hand, if $n = 3l$, then

$$|x(n)| = 2|\beta_1 q_1^l + \cdots + \beta_m q_m^l|$$

and

$$|x(n+1)| = 3|\beta_1 q_1^{l+1} + \cdots + \beta_m q_m^{l+1}| \leq 9 \sum_{k>l} |\beta_1 q_1^k + \cdots + \beta_m q_m^k|.$$

Hence by (2.1) we obtain $|x(n)| \geq |x(n+1)|$ for almost all n. By (2.1) we also have $|x(n)| > \sum_{i>n} |x(i)|$ for infinitely many n.

Now we will show that

$$|\beta_1 q_1^n + \cdots + \beta_m q_m^n| \leq 5 \sum_{k>n} |\beta_1 q_1^k + \cdots + \beta_m q_m^k|. \tag{2.2}$$

We have

$$\frac{|\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n|}{5 \sum_{k>n} |\beta_1 q_1^k + \beta_2 q_2^k + \cdots + \beta_m q_m^k|}$$
$$\leq \frac{|\beta_1 q_1^n + \beta_2 q_2^n + \cdots + \beta_m q_m^n|}{5| \sum_{k>n} \beta_1 q_1^k + \beta_2 q_2^k + \cdots + \beta_m q_m^k|}$$
$$= \frac{\big|\beta_1 + \beta_2 \big(\frac{q_2}{q_1}\big)^n + \cdots + \beta_m \big(\frac{q_m}{q_1}\big)^n\big|}{5\big|\beta_1 \sum_{i>0} q_1^i + \beta_2 \big(\frac{q_2}{q_1}\big)^n \sum_{i>0} q_2^i + \cdots + \beta_m \big(\frac{q_m}{q_1}\big)^n \sum_{i>0} q_m^i\big|}$$
$$\underset{n\to\infty}{\longrightarrow} \frac{1}{5} \cdot \frac{1-q_1}{q_1} \leq \frac{1}{5} \cdot \frac{1-1/6}{1/6} = 1.$$

By (2.2) we find that $|y(n)| \le \sum_{k>n} |y(k)|$ for almost all n. Therefore by Theorem 1.1, the set $E(y)$ is a finite union of closed intervals. Thus $E(y)$ has non-empty interior.

To end the proof we need to show that $E(x)$ has non-empty interior. We will prove that

$$2\sum_{n=0}(\beta_1 q_1^n + \cdots + \beta_m q_m^n) + E(y) \subseteq E(x).$$

Let

$$t \in 2\sum_{n=0}(\beta_1 q_1^n + \cdots + \beta_m q_m^n) + E(y).$$

Note that any element s of $E(y)$ is of the form

$$\begin{aligned} s = k_0(\beta_1 + \cdots + \beta_m) &+ k_1(\beta_1 q_1 + \cdots + \beta_m q_m) \\ &+ k_2(\beta_1 q_1^2 + \cdots + \beta_m q_m^2) + \cdots \end{aligned}$$

where $k_n \in \{0, 1, 2, 3, 4, 5\}$. Thus t is of the form

$$\begin{aligned} t &= 2\sum_{n=0}(\beta_1 q_1^n + \cdots + \beta_m q_m^n) \\ &\quad + [k_0(\beta_1 + \cdots + \beta_m) + k_1(\beta_1 q_1 + \cdots + \beta_m q_m) \\ &\quad + k_2(\beta_1 q_1^2 + \cdots + \beta_m q_m^2) + \cdots] \\ &= (2 + k_0)(\beta_1 + \cdots + \beta_m) + (2 + k_1)(\beta_1 q_1 + \cdots + \beta_m q_m) \\ &\quad + (2 + k_2)(\beta_1 q_1^2 + \cdots + \beta_m q_m^2) + \cdots . \end{aligned}$$

Note that each number from $\{2, 3, 4, 5, 6, 7\}$, that is, every number of the form $2 + k_n$, can be written as a sum of numbers $4, 3, 2$. Hence $t \in E(x)$ and $E(x)$ has non-empty interior. So $x \in \mathcal{MC}$. ∎

3. The topological size and Borel class of $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{MC}$. Let us observe that the sets $c_{00}, \mathcal{C}, \mathcal{I}$ and $\mathcal{MC}$ are all dense in ℓ_1. Moreover, c_{00} is an $\mathcal{F}_\sigma$-set of the first category. We are interested in the topological size and Borel class of these sets. For this, let us consider the hyperspace $H(\mathbb{R})$ of all non-empty compact subsets of reals, equipped with the Vietoris topology (see [Ke, 4F, pp. 24–28]). Recall that the Vietoris topology is generated by the subbase of sets of the form $\{K \in H(\mathbb{R}) : K \subset U\}$ and $\{K \in H(\mathbb{R}) : K \cap U \neq \emptyset\}$ for all open sets U in $\mathbb{R}$. This topology is metrizable by the Hausdorff metric d_{H} given by the formula

$$d_{\mathrm{H}}(A, B) = \max\Big\{\max_{t\in A} d(t, B), \max_{s\in B} d(s, A)\Big\}$$

where d is the natural metric in $\mathbb{R}$. It is known that the set N of all nowhere dense compact sets is a $\mathcal{G}_\delta$-set in $H(\mathbb{R})$ and the set F of all compact sets with a finite number of connected components is an $\mathcal{F}_\sigma$-set. To see this, it is enough to observe that

- K is nowhere dense if and only if for any set U_n from a fixed countable base of the natural topology in $\mathbb{R}$ there exists a set U_m from this base such that $\mathrm{cl}(U_m) \subset U_n$ and $K \subset (\mathrm{cl}(U_m))^c$;
- K has more than k components if and only if there exist pairwise disjoint open intervals $J_1, \ldots, J_{k+1}$ such that $K \subset J_1 \cup \cdots \cup J_{k+1}$ and $K \cap J_i \neq \emptyset$ for $i = 1, \ldots, k+1$.

Now, let us observe that if we assign the set $E(x)$ to the sequence $x \in \ell_1$, we actually define a function $E : \ell_1 \to H(\mathbb{R})$.

LEMMA 3.1. *The function E is Lipschitz with Lipschitz constant $L = 1$, hence it is continuous.*

Proof. Let $t \in E(x)$. Then there exists a subset A of $\mathbb{N}$ such that $t = \sum_{n\in A} x(n)$. We have

$$\begin{aligned} d(t, E(y)) &\leq d\Big(t, \sum_{n\in A} y(n)\Big) = \Big|\sum_{n\in A}(x(n) - y(n))\Big| \leq \sum_{n\in\mathbb{N}} |x(n) - y(n)| \\ &= \|x - y\|_1 \end{aligned}$$

where $\|\cdot\|_1$ denotes the norm in ℓ_1. Hence, $d_{\mathrm{H}}(E(x), E(y)) \leq \|x - y\|_1$. ■

THEOREM 3.2. *The set $\mathcal{C}$ is a dense $\mathcal{G}_\delta$-set* (*and hence residual*), *$\mathcal{I}$ is a true $\mathcal{F}_\sigma$-set* (*i.e. it is $\mathcal{F}_\sigma$ but not $\mathcal{G}_\delta$*) *of the first category, and $\mathcal{MC}$ is in the class* $(\mathcal{F}_{\sigma\delta} \cap \mathcal{G}_{\delta\sigma}) \setminus \mathcal{G}_\delta$.

Proof. Let us observe that $\mathcal{C} \cup c_{00} = E^{-1}[N]$ and $\mathcal{I} \cup c_{00} = E^{-1}[F]$ where N, F, E are defined as before. Hence $\mathcal{C} \cup c_{00}$ is $\mathcal{G}_\delta$ and $\mathcal{I} \cup c_{00}$ is $\mathcal{F}_\sigma$. Thus $\mathcal{C}$ is $\mathcal{G}_\delta$ (because c_{00} is $\mathcal{F}_\sigma$) and $\mathcal{I} \cup \mathcal{MC}$ is $\mathcal{F}_\sigma$. Moreover, $\mathcal{I} = (\mathcal{I} \cup c_{00}) \cap (\mathcal{I} \cup \mathcal{MC})$ is $\mathcal{F}_\sigma$, too. By the density of $\mathcal{C}$, $\mathcal{C}$ is residual. Since $\mathcal{I}$ is dense of the first category, it cannot be $\mathcal{G}_\delta$. For the same reason, $\mathcal{MC}$ cannot be $\mathcal{G}_\delta$. Since $\mathcal{MC}$ is a difference of two $\mathcal{F}_\sigma$-sets, it is in the class $\mathcal{F}_{\sigma\delta} \cap \mathcal{G}_{\delta\sigma}$. ■

REMARK 3.3. In [BG1] the following similar result was shown by quite different methods: the set of bounded sequences, with the set of limit points homeomorphic to the Cantor set, is strongly $\mathfrak{c}$-algebrable and residual in ℓ_∞.

4. Spaceability. In this section we will show that $\mathcal{I}$ is spaceable while $\mathcal{C}$ is not. This shows that there is a subset M of ℓ_1 containing a dense $\mathcal{G}_\delta$-subset and a linear subspace of dimension $\mathfrak{c}$, but $Y \setminus M \neq \emptyset$ for any infinite-dimensional closed subspace Y of ℓ_1.

THEOREM 4.1. *Let $\mathcal{I}_1$ be the subset of $\mathcal{I}$ which consists of those $x \in \ell_1$ for which $E(x)$ is an interval. Then $\mathcal{I}_1$ is spaceable.*

Proof. Let $A_1, A_2, \dots$ be a partition of $\mathbb{N}$ into infinitely many infinite subsets. Let $A_n = \{k_n^1 < k_n^2 < \cdots\}$. Define $x_n \in \ell_1$ by $x_n(k_n^j) = 2^{-j}$ and $x_n(i) = 0$ if $i \notin A_n$. Then $\|x_n\|_1 = 1$ and $\{x_n : x \in \mathbb{N}\}$ forms a normalised basic sequence. Let Y be a closed linear space generated by $\{x_n : x \in \mathbb{N}\}$. Then

$$y \in Y \ \Leftrightarrow\ \exists t \in \ell_1 \ \Big(y = \sum_{n=1}^{\infty} t(n)x_n\Big).$$

Since $E(x_n) = [0,1]$, we have $E(\sum_{n=1}^{\infty} t(n)x_n) = \bigcup_{n=1}^{\infty} I_n$ where I_n is an interval with endpoints 0 and $t(n)$. Put $t^+(n) = \max\{t(n), 0\}$ and $t^-(n) = \min\{-t(n), 0\}$. Then $E(\sum_{n=1}^{\infty} t(n)x_n) = [\sum_{n=1}^{\infty} t^-(n), \sum_{n=1}^{\infty} t^+(n)]$ and the result follows. ∎

Let us mention the very recent result by Bernal-González and Ordóñez Cabrera [BO, Theorem 2.2], who gave sufficient conditions for spaceability of sets in Banach spaces. Using that result, one can prove the spaceability of $\mathcal{I}$, but it cannot be used to prove Theorem 4.1, since the assumptions are not satisfied.

However, we do not know other results giving sufficient conditions for a set in a Banach space not to be spaceable. An interesting example of a non-spaceable set was given in the classical paper [G] by Gurariy where it was proved that the set of all differentiable functions from $C[0,1]$ is not spaceable. It is well known that the set of all differentiable functions in $C[0,1]$ is dense but meager. We will prove that even dense $\mathcal{G}_\delta$-sets in Banach spaces may not be spaceable.

THEOREM 4.2. *Let Y be an infinite-dimensional closed subspace of ℓ_1. Then there is $y \in Y$ such that $E(y)$ contains an interval.*

Proof. Let $\varepsilon_n \searrow 0$. Let x_1 be any non-zero element of Y with $\|x_1\|_1 = 1 + \varepsilon_1$. Since $x_1 \in \ell_1$, there is n_1 with $\sum_{n=n_1+1}^{\infty} |x_1(n)| \le \varepsilon_1$. Let E_1 consist of all finite sums $\sum_{n=1}^{n_1} \delta_n x_1(n)$ where $\delta_i \in \{0,1\}$. Then E_1 is a finite set with $\min E_1 = \sum_{n=1}^{n_1} x_1^-(n)$, $\max E_1 = \sum_{n=1}^{n_1} x_1^+(n)$ and $1 \le \max E_1 - \min E_1 \le 1 + \varepsilon_1$.

Let $Y_1 = Y \cap \{x \in \ell_1 : x(n) = 0$ for every $n \le n_1\}$. As $\{x \in \ell_1 : x(n) = 0$ for every $n \le n_1\}$ has a finite codimension, Y_1 is infinite-dimensional. Let x_2 be any non-zero element of Y_1 with $\|x_2\|_1 = 1 + \varepsilon_2$. Since $x_2 \in \ell_1$, there is $n_2 > n_1$ with $\sum_{n=n_2+1}^{\infty} |x_i(n)| \le \varepsilon_2$, $i = 1, 2$. Let E_2 consist of all finite sums $\sum_{n=n_1+1}^{n_2} \delta_n x_2(n)$, where $\delta_i \in \{0,1\}$. Then E_2 is a finite set with $\min E_2 = \sum_{n=n_1+1}^{n_2} x_2^-(n)$, $\max E_2 = \sum_{n=n_1+1}^{n_2} x_2^+(n)$ and $1 \le \max E_2 - \min E_2 \le 1 + \varepsilon_2$.

Proceeding inductively, we define natural numbers $n_1 < n_2 < \cdots$ and infinite-dimensional closed spaces $Y \supset Y_1 \supset Y_2 \supset \cdots$ such that $Y_k = \{x \in Y : x(n) = 0$ for every $n \le n_k\}$, non-zero elements $x_k \in Y_{k-1}$ with $\|x_k\|_1 = 1 + \varepsilon_k$

and $\sum_{n=n_k+1}^{\infty} |x_i(n)| \le \varepsilon_k$, $i = 1, \dots, k$, and finite sets E_k consisting of all sums $\sum_{n=n_{k-1}+1}^{n_k} \delta_n x_k(n)$ where $\delta_i \in \{0,1\}$. Note that $1 \le \operatorname{diam}(E_k) \le 1 + \varepsilon_k$. Consider $y = \sum_{k=1}^{\infty} x_k/2^k$. We claim that $E(y)$ contains the interval $I := [\min E_1, \max E_1]$.

Note that for any $t \in I$ there is $t_1 \in E_1$ with $|t - t_1| \le (1+\varepsilon_1)/2$. Since $1 \le \operatorname{diam}(E_2) \le 1 + \varepsilon_2$, there is $t_2 \in E_1 + \frac{1}{2}E_2$ with $|t - t_2| \le (1+\varepsilon_2)/2^2$. Hence, there is $\tilde{t} \in E(x_1 + x_2/2)$ with $|t - \tilde{t}| \le (1+\varepsilon_2)/2^2 + \varepsilon_1$. Since $1 \le \operatorname{diam}(E_k) \le 1+\varepsilon_k$, inductively we can find $t_k \in E_1 + \frac{1}{2}E_2 + \dots + \frac{1}{2^{k-1}}E_k$ with $|t - t_k| \le (1+\varepsilon_k)/2^k$. Hence, there is $\tilde{t} \in E(x_1 + x_2/2 + \dots + x_k/2^{k-1})$ with

$$|t-\tilde{t}| \le (1+\varepsilon_k)/2^k + \varepsilon_{k-1} + \varepsilon_{k-1}/2 + \dots + \varepsilon_{k-1}/2^{k-1} \le (1+\varepsilon_k)/2^k + 2\varepsilon_{k-1}.$$

Since $E(y)$ is closed and contains $E(x_1 + x_2/2 + \dots + x_k/2^{k-1})$, it follows that $t \in E(y)$ and consequently $I \subset E(y)$. ∎

Immediately we get the following.

COROLLARY 4.3. *The set $\mathcal{C}$ is not spaceable.*

We end the paper with some open questions on the set $\mathcal{MC}$.

PROBLEM 4.4.

(i) *Is $\mathcal{MC}$ $\mathfrak{c}$-algebrable?*
(ii) *Is $\mathcal{MC}$ an $\mathcal{F}_\sigma$-subset of ℓ_1?*
(iii) *Is $\mathcal{MC}$ spaceable?*

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