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ON CERTAIN POROUS SETS IN THE ORLICZ SPACE OF A LOCALLY COMPACT GROUP

BY

IBRAHIM AKBARBAGLU and SAEID MAGHSOUDI (Zanjan)

Abstract. Let G be a locally compact group with a fixed left Haar measure. Given Young functions φ and ψ, we consider the Orlicz spaces $L^\varphi(G)$ and $L^\psi(G)$ on a non-unimodular group G, and, among other things, we prove that under mild conditions on φ and ψ, the set $\{(f,g) \in L^\varphi(G) \times L^\psi(G) : f * g \text{ is well defined on } G\}$ is σ-c-lower porous in $L^\varphi(G) \times L^\psi(G)$. This answers a question raised by Głąb and Strobin in 2010 in a more general setting of Orlicz spaces. We also prove a similar result for non-compact locally compact groups.

1. Preliminaries and background. Throughout this paper, G denotes a locally compact group with a fixed left Haar measure λ. Also, let $L^0(G)$ denote the set of all equivalence classes of λ-measurable complex-valued functions on G. For measurable functions f and g on G, the *convolution*

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\,d\lambda(y)$$

is defined at each point $x \in G$ for which the function $y \mapsto f(y)g(y^{-1}x)$ is Haar integrable. If $f * g(x)$ is defined for λ-almost all $x \in G$, then we say that the convolution of functions $f * g$ is defined on G.

For each $x \in G$ the formula $\lambda_x(A) = \lambda(Ax)$ defines a left invariant regular Borel measure λ_x on G. Thus, the uniqueness of left Haar measure implies that for each $x \in G$ there is a positive number, say $\Delta(x)$, such that $\lambda_x = \Delta(x)\lambda$. The function $\Delta : G \to (0,\infty)$ defined in this way is called the *modular function* of G. It is clear that Δ is a continuous homomorphism on G. Moreover, for every measurable subset A of G,

$$\lambda(A^{-1}) = \int_A \Delta(x^{-1})\,d\lambda(x);$$

for more details see [F] or [HR]. The group G is called *unimodular* whenever $\Delta = 1$. In this case, left Haar measure and right Haar measure coincide.

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For $1 \le p \le \infty$, classical Lebesgue spaces on G with respect to the Haar measure λ will be denoted by $L^p(G)$ with the norm $\|\cdot\|_p$ as defined in [HR].

We deal with a problem which has its origin in the 1960's. Żelazko [Z1] proved that if G is locally compact abelian group and $p \in (1,\infty)$, then $L^p(G)$ is closed under convolution if and only if G is compact. Independently Rajagopalan [R] and Żelazko [Z2] asked if that statement holds for an arbitrary locally compact group G—this was known as the L^p*-conjecture* or *Żelazko conjecture*. Several authors found the affirmative answer in special cases. The L^p-conjecture was finally settled by Saeki [S] in 1990; in the same paper one can also find a brief history of the struggle with this hypothesis.

Rickert [RI] proved in 1967 that if G is a non-compact locally compact group, $1/p + 1/q < 1$, then there are $f \in L^p(G)$ and $g \in L^q(G)$ such that $f * g$ does not exist. This result was reproved 40 years later in [ANR]. Note that this observation implies the L^p-conjecture for $p > 2$. Recently, Głąb and Strobin [GS1] proved that the set of pairs $(f,g) \in L^p(G) \times L^q(G)$ such that $f*g$ exists is small, namely σ-porous, if $1/p+1/q < 1$. They asked for which pairs (p,q) the convolution $f * g$ exists for every $(f,g) \in L^p(G) \times L^q(G)$; moreover, if the convolution does not exist for some pair, what is the size of those pairs $(f,g) \in L^p(G) \times L^q(G)$ such that $f * g$ exists. The solution to that problem is the following:

- If $1/p + 1/q \ge 1$ and G is unimodular, then $f * g$ exists for $(f,g) \in L^p(G) \times L^q(G)$ (follows from Young's inequality).
- If $p \in (1,\infty)$, $q \in [1,\infty)$ and G is not unimodular, then there is a pair $(f,g) \in L^p(G) \times L^q(G)$ such that $f*g$ does not exist and the set of all pairs for which the convolution exists is σ-porous (proved recently by the authors [AM]).
- If $p = q = 1$, then $f * g$ exists for $(f,g) \in L^p(G) \times L^q(G)$, where G is any locally compact group.
- If $p = q \in (0,1)$ and G is discrete, then $f * g$ exists for $(f,g) \in L^p(G) \times L^q(G)$.

The aim of this paper is to draw a similar picture for a generalization of L^p spaces, namely Orlicz spaces.

Orlicz spaces are genuine generalizations of the usual L^p-spaces. They have been thoroughly investigated. We refer to two excellent books [KR] and [RR] for more details. Also [M] and [Z] provide some useful information on the subject. Moreover, there have been generalizations and studies of Orlicz spaces in several directions; see for example [K].

A function $\varphi : \mathbb{R} \to [0,\infty]$ is called a *Young function* if φ is convex, even, and left continuous with $\varphi(0) = 0$; we also assume that φ is neither identically zero nor identically infinity on $\mathbb{R}$. A Young function φ is called *finite* if $\varphi(x) < \infty$ for all $x \in \mathbb{R}$. The Young function $\widetilde{\varphi}$ *complementary* to φ

is defined by

$$\widetilde{\varphi}(y) = \sup\{x|y| - \varphi(x) : x \geq 0\} \in [0, \infty]$$

for $y \in \mathbb{R}$; then $(\varphi, \widetilde{\varphi})$ is called a complementary pair of Young functions. We also need an inverse of a Young function. For a Young function φ and $y \in [0, \infty)$ let

$$\varphi^{-1}(y) = \sup\{x \geq 0 : \varphi(x) \leq y\}.$$

We say a Young function φ is Δ_2-*regular* whenever there exist $k > 0$ and $x_0 \geq 0$ such that $\varphi(2x) \leq k\varphi(x)$ for all $x \geq x_0$, with $x_0 > 0$ if G is compact and $x_0 = 0$ otherwise.

A Young function φ is said to be an N-*function* (or a nice Young function) whenever

$$\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty \quad \text{and} \quad \lim_{x\to 0} \frac{\varphi(x)}{x} = 0.$$

As an elementary example of an N-function we can consider $\varphi(x) = |x|^p/p$ for $p > 1$; then $\widetilde{\varphi}(x) = |x|^q/q$, where $1/p + 1/q = 1$.

It is a consequence of [RR, Corollary 1.3.2] and [KR, p. 11] that φ is finite-valued and vanishes only at the origin. Moreover, if φ is an N-function, then so is $\widetilde{\varphi}$ and by [RR, Proposition 2.1.1(ii)] we have

$$x < \varphi^{-1}(x)\widetilde{\varphi}^{-1}(x) \leq 2x \tag{1.1}$$

for all $x > 0$.

Let φ be a Young function. For each $f \in L^0(G)$ we define

$$\rho_\varphi(f) = \int_G \varphi(|f(x)|)\, d\lambda(x).$$

Given a Young function φ, the *Orlicz space* $L^\varphi(G)$ is defined by

$$L^\varphi(G) = \{f \in L^0(G) : \rho_\varphi(af) < \infty \text{ for some } a > 0\}.$$

Similarly, let

$$M^\varphi(G) = \{f \in L^0(G) : \rho_\varphi(af) < \infty \text{ for all } a > 0\}.$$

Then $L^\varphi(G)$ is a Banach space under the norm $N_\varphi(\cdot)$ (called *the Luxemburg–Nakano norm*) defined for $f \in L^\varphi(G)$ by

$$N_\varphi(f) = \inf\{k > 0 : \rho_\varphi(f/k) \leq 1\}.$$

It is well known that

$$N_\varphi(f) \leq 1 \quad \text{if and only if} \quad \rho_\varphi(f) \leq 1, \tag{1.2}$$

and

$$N_\varphi(\chi_F) = \left[\varphi^{-1}\left(\frac{1}{\lambda(F)}\right)\right]^{-1} \tag{1.3}$$

(see [RR, Corollary 3.4.7]). Here χ_A denotes the characteristic function of a subset A.

Another well-known norm on $L^\varphi(G)$, called the *Orlicz norm*, is defined by

$$\|f\|_\varphi = \sup\Big\{\int_G |fg|\,d\lambda : N_{\widetilde{\varphi}}(g) \le 1\Big\}.$$

It follows from [RR, Proposition 3.3.4] that the Orlicz norm is equivalent to the Luxemburg–Nakano norm; in fact we have

$$N_\varphi(f) \le \|f\|_\varphi \le 2N_\varphi(f) \tag{1.4}$$

for all $f \in L^\varphi(G)$.

Let us recall the notion of porosity from [GS1]; for more details see also [ZA]. Let X be a metric space. The open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x,r)$. For a given number $0 < c \le 1$, a subset M of X is called *c-lower porous* if

$$\liminf_{R\to 0+} \frac{\gamma(x,M,R)}{R} \ge \frac{c}{2}$$

for all $x \in M$, where

$$\gamma(x,M,R) = \sup\{r \ge 0 : \exists z \in X,\ B(z,r) \subseteq B(x,R)\setminus M\}.$$

It is clear that M is c-lower porous if and only if

$$\forall x \in M, \forall \alpha \in (0,c/2), \exists r_0 > 0, \forall r \in (0,r_0), \exists z \in X, B(z,\alpha r) \subseteq B(x,r)\setminus M.$$

A set is called *σ-c-lower porous* if it is a countable union of c-lower porous sets with the same constant $c > 0$. It is easy to see that a σ-c-lower porous set is meager, and the notion of σ-porosity is stronger than that of meagerness.

Our purpose in this work is to consider the porosity of the set of all pairs $(f,g) \in M^\varphi(G) \times M^\psi(G)$ (or $(f,g) \in L^\varphi(G) \times L^\psi(G)$) for which $f * g$ is λ-a.e. finite on G for given Young functions φ and ψ under certain mild conditions. Finally, let us remark that in [RA], a sufficient condition on a Young function φ has been given for $L^\varphi(G)$ to be an algebra under convolution multiplication; see also [GS2, H] for a similar study with respect to the pointwise product.

2. Main results. We commence with our first main result which answers a question raised in [GS1] and generalizes the main theorem of [AM] to a more general setting of Orlicz spaces. For the proof we use some ideas from [GS1].

We equip the space $L^\varphi(G) \times L^\psi(G)$ with the complete norm

$$N_{\varphi,\psi}(f,g) = \max\{N_\varphi(f), N_\psi(g)\} \quad (f \in L^\varphi(G),\ g \in L^\psi(G)).$$

Let us remark that $M^\varphi(G)$ is non-trivial if φ is finite.

THEOREM 2.1. *Let G be a non-unimodular locally compact group, φ be an N-function, and ψ be a finite Young function. For any compact symmetric* *neighborhood V of the identity element of G the set*

$$E_V = \{(f, g) \in M^{\varphi}(G) \times M^{\psi}(G) : \exists x \in V, |f * g(x)| < \infty\}$$

is σ-c-lower porous in $M^{\varphi}(G) \times M^{\psi}(G)$ for some $c > 0$.

Proof. Since G is not unimodular, there exists $b \in G$ with $\Delta(b) > q^4$, where $q = \sup\{\Delta(x) : x \in V\}$. Hence for distinct natural numbers m and n we have

$$Vb^n \cap Vb^m = \emptyset \quad \text{and} \quad b^{-n}V^2 \cap b^{-m}V^2 = \emptyset.$$

Now for a natural number n, put

$$E_n = \Big\{(f, g) \in M^{\varphi}(G) \times M^{\psi}(G) : \exists x \in V, \int_G |f(y)|\,|g(y^{-1}x)|\,d\lambda(y) \le n\Big\}.$$

So, $E_V = \bigcup_{n\in\mathbb{N}} E_n$. Therefore we only need to show that for each $n \in \mathbb{N}$, E_n is c-lower porous for some constant $c > 0$. To this end, let $c \in (0, 1)$ be such that

$$\frac{2c}{1-c}\left(1 + q\frac{\lambda(V^2)}{\lambda(V)}\right) = 1.$$

Then for $0 < \alpha < c$,

$$\frac{2\alpha}{1-\alpha}\left(1 + q\frac{\lambda(V^2)}{\lambda(V)}\right) < 1.$$

By continuity of the map $x \mapsto (2\alpha/x)(1+q\lambda(V^2)/\lambda(V))$ on $(0, 1)$, there exist $0 < \beta < 1 - \alpha$ and $d < 1$ such that

$$\frac{2\alpha}{\beta(1-d)}\left(1 + q\frac{\lambda(V^2)}{\lambda(V)}\right) < 1.$$

Put $C = \bigcup_{n\in\mathbb{N}} Vb^n$ and $D = \bigcup_{n\in\mathbb{N}} b^{-n}V^2$. Fix $n \in \mathbb{N}$ and $(f, g) \in E_n$. Since $f \in L^{\varphi}(G)$ and $g \in L^{\psi}(G)$, we infer that

$$N_{\varphi}(f\chi_C) < \infty \quad \text{and} \quad N_{\psi}(g\chi_D) < \infty.$$

Hence, given $r > 0$, there exists $n_0 \in \mathbb{N}$ such that if we set $C_0 = \bigcup_{k=n_0}^{\infty} Vb^k$ and $D_0 = \bigcup_{k=n_0}^{\infty} b^{-k}V^2$, then

$$N_{\varphi}(f\chi_{C_0}) + N_{\psi}(g\chi_{D_0}) < (1 - \alpha - \beta)r.$$

Now, we choose a natural number $n_1 \ge n_0$ such that

$$d^2\beta^2r^2p\left[\widetilde{\varphi}^{-1}\left(\frac{1}{\lambda(V)q^{4n_1-1}}\right)\right]^{-1}\psi^{-1}\left(\frac{1}{\lambda(V^2)}\right) > n, \tag{2.1}$$

where $\widetilde{\varphi}$ is the complementary N-function to φ and $p > 0$ is the quantity

$$1 - \frac{2\alpha}{\beta(1-d)}\left(1 + q\frac{\lambda(V^2)}{\lambda(V)}\right).$$

Set $A = Vb^{n_1}$ and $B = b^{-n_1}V^2$ and note that

$$\lambda(A^{-1}) = \lambda(V), \quad \lambda(B) = \lambda(V^2).$$

Next let s, t be real numbers defined by

$$sN_\varphi\left(\varphi^{-1}\left(\frac{1}{\lambda(V)\Delta}\right)\chi_A\right) = \beta r \quad \text{and} \quad tN_\psi(\chi_B) = \beta r. \tag{2.2}$$

Define functions $\widetilde{f}$ and $\widetilde{g}$ on G by setting

$$\widetilde{f}(y) = s\varphi^{-1}\left(\frac{1}{\lambda(V)\Delta(y)}\right)\chi_A(y) + f(y)\chi_{G\setminus A}(y) \quad \text{and} \quad \widetilde{g} = t\chi_B + g\chi_{G\setminus B}.$$

Since $A \subseteq C_0$, we have

$$\begin{aligned} N_\varphi(\widetilde{f} - f) &\le N_\varphi\left(s\varphi^{-1}\left(\frac{1}{\lambda(V)\Delta}\right)\chi_A\right) + N_\varphi(f\chi_A) \\ &\le \beta r + (1 - \alpha - \beta)r = r - \alpha r. \end{aligned}$$

In a similar way, $N_\psi(\widetilde{g} - g) \le r - \alpha r$. It follows that $B((\widetilde{f}, \widetilde{g}), \alpha r) \subseteq B((f, g), r)$.

Thus it remains only to prove that $B((\widetilde{f}, \widetilde{g}), \alpha r) \cap E_n = \emptyset$. Let $(h, k) \in B((\widetilde{f}, \widetilde{g}), \alpha r)$. We will show that $(h, k) \notin E_n$. To prove this, let

$$\begin{aligned} A_1 &= \left\{x \in A : |h(x)| < sd\varphi^{-1}\left(\frac{1}{\lambda(V)\Delta(x)}\right)\right\}, \\ B_1 &= \{x \in B : |k(x)| < td\}. \end{aligned}$$

Then

$$\begin{aligned} \alpha r &> N_\varphi(h - \widetilde{f}) \ge N_\varphi((h - \widetilde{f})\chi_{A_1}) \ge s(1 - d)N_\varphi\left(\varphi^{-1}\left(\frac{1}{\lambda(V)\Delta}\right)\chi_{A_1}\right) \\ &\overset{(1.4)}{\ge} \frac{s(1-d)}{2}\int_{A_1} \varphi^{-1}\left(\frac{1}{\lambda(V)\Delta(y)}\right)\widetilde{\varphi}^{-1}\left(\frac{1}{\lambda(V)\Delta(y)}\right) d\lambda(y) \\ &\overset{(1.1)}{>} \frac{s(1-d)}{2}\int_{A_1} \frac{1}{\lambda(V)\Delta(y)}\, d\lambda(y). \end{aligned}$$

Hence

$$\begin{aligned} \lambda(A_1^{-1}) \le \frac{2\alpha r}{s(1-d)}\lambda(V) &= N_\varphi\left(\varphi^{-1}\left(\frac{1}{\lambda(V)\Delta}\right)\chi_A\right)\frac{2\alpha}{\beta(1-d)}\lambda(V) \\ &\overset{(1.2)}{\le} \frac{2\alpha}{\beta(1-d)}\lambda(V). \end{aligned}$$

In the same way, we get

$$N_\psi(\chi_{B_1}) \le \frac{2\alpha r}{t(1-d)} = N_\psi(\chi_B)\frac{2\alpha}{\beta(1-d)},$$

and, by (1.3), we have

$$\lambda(B_1) \le \lambda(B)\frac{2\alpha}{\beta(1-d)} = \lambda(V^2)\frac{2\alpha}{\beta(1-d)}.$$

The above inequalities also show that the sets $A_2 := A\setminus A_1$ and $B_2 := B\setminus B_1$ are of positive measure and so non-empty. Now let $z \in V$ be an arbitrary element, and define $E = (A_2^{-1}z) \cap B_2$ and $F = zE^{-1}$. It can be easily seen that $A^{-1}z \subseteq B$, and thus $A_2^{-1}z \subseteq B$. Hence

$$\begin{aligned}\lambda(F^{-1}) &= \lambda(Ez^{-1}) = \lambda(A_2^{-1}) - \lambda(A_2^{-1} \setminus (B_2z^{-1}))\\ &\geq \lambda(A_2^{-1}) - \lambda((B \setminus B_2)z^{-1}) = \lambda(A_2^{-1}) - \lambda(B_1z^{-1})\\ &= \lambda(V) - \lambda(A_1^{-1}) - \Delta(z^{-1})\lambda(B_1) \geq \lambda(V)p.\end{aligned}$$

Also, $F \subseteq A_2$, $E \subseteq B_2$ and $F^{-1}z = E$. Furthermore, if $y \in A$ then $y = xb^{n_1}$ for some $x \in V$. Therefore, noting that $\Delta(b) > q^4$, we find that $\Delta(y^{-1}) < q^{1-4n_1}$. Finally, we conclude

$$\begin{aligned}\int_F |h(y)|\,|k(y^{-1}z)|\,d\lambda(y) &\geq d^2st \int_F \varphi^{-1}\left(\frac{1}{\lambda(V)\Delta(y)}\right) d\lambda(y)\\ &\overset{(1.1)}{\geq} d^2st \int_F \frac{1}{\lambda(V)\Delta(y)\widetilde{\varphi}^{-1}\left(\frac{1}{\lambda(V)\Delta(y)}\right)}\,d\lambda(y)\\ &\overset{(1.3),(2.2)}{\geq} d^2\beta^2r^2p\left[\widetilde{\varphi}^{-1}\left(\frac{1}{\lambda(V)q^{4n_1-1}}\right)\right]^{-1}\psi^{-1}\left(\frac{1}{\lambda(V^2)}\right)\\ &\overset{(2.1)}{>} n,\end{aligned}$$

which proves the theorem. ∎

The fact that a σ-c-lower porous set is of first category together with Theorem 2.1 yields the following corollary.

Corollary 2.2. *Let G be a non-unimodular locally compact group, φ be an N-function, and ψ be a finite Young function. Then the set*

$$E = \{(f, g) \in M^\varphi(G) \times M^\psi(G) : f * g \text{ is not finite } \lambda\text{-a.e.}\}$$

is residual in $M^\varphi(G)\times M^\psi(G)$. In particular, if φ and ψ are also Δ_2-regular, then the set

$$E' = \{(f, g) \in L^\varphi(G) \times L^\psi(G) : f * g \text{ is not finite } \lambda\text{-a.e.}\}$$

is residual in $L^\varphi(G) \times L^\psi(G)$.

We state the following simple observation for later use.

Lemma 2.3. *Let G be a non-compact locally compact group. Then for any compact subsets A and B of G, there exists $x \in G$ such that*

$$xB \cap A = \emptyset = Bx^{-1} \cap A.$$

Our second main result is a generalization of the main theorem in [GS1] from classical Lebesgue spaces to Orlicz spaces.

THEOREM 2.4. *Let G be a non-compact locally compact group, φ be an N-function, and ψ be a finite-valued Young function with*

$$\limsup_{x\to\infty} x\varphi^{-1}\left(\frac{1}{x}\right)\psi^{-1}\left(\frac{1}{x}\right)=\infty.$$

Then for every compact symmetric neighborhood V of the identity element of G, the set

$$E_V=\{(f,g)\in M^{\varphi}(G)\times M^{\psi}(G):\exists x\in V,\ |f*g(x)|<\infty\}.$$

is σ-c-lower porous for some $c>0$.

Proof. Since G is not compact, by Lemma 2.3 there exists a sequence $(a_n)_n$ in G such that, for distinct natural numbers m,n,

$$a_mV^2\cap a_nV^2=\emptyset,\qquad Va_m^{-1}\cap Va_n^{-1}=\emptyset,\qquad \Delta(a_n)\le 1.$$

Now for every natural number n, put

$$E_n=\Big\{(f,g)\in M^{\varphi}(G)\times M^{\psi}(G):\exists x\in V,\ \int_G |f(y)|\,|g(y^{-1}x)|\,d\lambda(y)\le n\Big\}.$$

So, $E_V=\bigcup_{n\in\mathbb{N}}E_n$. We will show that for each $n\in\mathbb{N}$, E_n is c-lower porous for some constant $c>0$, which will complete the proof.

To prove this, let $c\in(0,1)$ be such that

$$\frac{2c}{1-c}\left(1+q\frac{\lambda(V^2)}{\lambda(V)}\right)=1,$$

where $q=\sup\{\Delta(x):x\in V\}$. Fix $0<\alpha<c$; then

$$\frac{2\alpha}{1-\alpha}\left(1+q\frac{\lambda(V^2)}{\lambda(V)}\right)<1.$$

By continuity of the map $x\mapsto(2\alpha/x)(1+q\lambda(V^2)/\lambda(V))$ on $(0,1)$, there exist $0<\beta<1-\alpha$ and $d<1$ such that

$$\frac{\alpha}{\beta(1-d)}\left(1+q\frac{\lambda(V^2)}{\lambda(V)}\right)<1.$$

Put $C=\bigcup_{n\in\mathbb{N}}Va_n^{-1}$ and $D=\bigcup_{n\in\mathbb{N}}a_nV^2$. Fix $n\in\mathbb{N}$ and $(f,g)\in E_n$. Now from $f\in M^{\varphi}(G)$ and $g\in M^{\psi}(G)$, we infer that

$$N_{\varphi}(f\chi_C)<\infty\quad\text{and}\quad N_{\psi}(g\chi_D)<\infty.$$

Hence, given $r>0$, there exists $n_0\in\mathbb{N}$ such that for $C_0=\bigcup_{k=n_0}^{\infty}Va_k^{-1}$ and $D_0=\bigcup_{k=n_0}^{\infty}a_kV^2$,

$$N_{\varphi}(f\chi_{C_0})+N_{\psi}(g\chi_{D_0})<(1-\alpha-\beta)r.$$

Now, by our assumption on φ^{-1} and ψ^{-1} we can find $n_1>n_0$ such that

$$\theta\lambda(V)\varphi^{-1}\left(\frac{q}{\theta\lambda(V)}\right)\psi^{-1}\left(\frac{q}{\theta\lambda(V)}\right)\ge n\left[d^2\beta^2r^2pq^{-2}\frac{\lambda(V)}{2\lambda(V^2)}\right]^{-1}, \tag{2.3}$$

where $\theta=n_1-n_0+1$ and $p>0$ is the quantity

$$1-\frac{2\alpha}{\beta(1-d)}\left(1+q\frac{\lambda(V^2)}{\lambda(V)}\right).$$

Setting $A=\bigcup_{k=n_0}^{n_1}Va_k^{-1}$ and $B=\bigcup_{k=n_0}^{n_1}a_kV^2$, and noting that V is a symmetric neighborhood and A, B are unions of disjoint subsets, we find that

$$\lambda(A^{-1})=(n_1-n_0+1)\lambda(V),\quad \lambda(B)=(n_1-n_0+1)\lambda(V^2).$$

Next let s, t be real numbers defined by

$$sN_\varphi\left(\varphi^{-1}\left(\frac{1}{\lambda(A^{-1})\Delta}\right)\chi_A\right)=\beta r\quad\text{and}\quad tN_\psi(\chi_B)=\beta r.$$

Define functions $\widetilde{f}$ and $\widetilde{g}$ on G by setting

$$\widetilde{f}(y)=s\varphi^{-1}\left(\frac{1}{\lambda(A^{-1})\Delta(y)}\right)\chi_A(y)+f(y)\chi_{G\setminus A}(y)\quad\text{and}\quad \widetilde{g}=t\chi_B+g\chi_{G\setminus B}.$$

By an argument similar to the proof of Theorem 2.1 we have

$$N_\varphi(\widetilde{f}-f)\le r-\alpha r\quad\text{and}\quad N_\psi(\widetilde{g}-g)\le r-\alpha r.$$

It follows that $B((\widetilde{f},\widetilde{g}),\alpha r)\subseteq B((f,g),r)$.

We now prove that $B((\widetilde{f},\widetilde{g}),\alpha r)\cap E_n=\emptyset$. For any fixed $(h,k)\in B((\widetilde{f},\widetilde{g}),\alpha r)$, if we set

$$A_1=\left\{x\in A:|h(x)|<sd\varphi^{-1}\left(\frac{1}{\lambda(A^{-1})\Delta(x)}\right)\chi_A(x)\right\},$$
$$B_1=\{x\in B:|k(x)|<td\},$$

then the subsets $A_2:=A\setminus A_1$ and $B_2:=B\setminus B_1$ are of positive measure and so non-empty. Now let $z\in V$ be an arbitrary element, and define $E=(A_2^{-1}z)\cap B_2$ and $F=zE^{-1}$. It follows that $\lambda(F^{-1})\ge\lambda(A^{-1})p$. Also, $F\subseteq A_2$, $E\subseteq B_2$ and $F^{-1}z=E$. Furthermore, if $y\in A$ then $y=xa_n^{-1}$ for some $x\in V$ and some $n_0\le n\le n_1$. Therefore, noting that $\Delta(a_n)\le 1$, we find that $\Delta(y^{-1})\le q$. Finally, we conclude that

$$\begin{aligned}
\int_F|h(y)|\,|k(y^{-1}z)|\,d\lambda(y)&\ge d^2st\int_F\varphi^{-1}\left(\frac{1}{\lambda(A^{-1})\Delta(y)}\right)d\lambda(y)\\
&\overset{(1.1)}{\ge} d^2st\int_F\frac{1}{\lambda(A^{-1})\Delta(y)\widetilde{\varphi}^{-1}\left(\frac{1}{\lambda(A^{-1})\Delta(y)}\right)}\,d\lambda(y)\\
&\ge d^2\beta^2r^2p\left[\widetilde{\varphi}^{-1}\left(\frac{q}{\lambda(A^{-1})}\right)\right]^{-1}\psi^{-1}\left(\frac{1}{\lambda(B)}\right)\\
&\ge\frac12 d^2\beta^2r^2pq^{-2}\lambda(A^{-1})\varphi^{-1}\left(\frac{q}{\lambda(A^{-1})}\right)\psi^{-1}\left(\frac{q}{\lambda(B)}\right)\\
&\ge\frac12 d^2\beta^2r^2pq^{-2}\frac{\lambda(V)}{\lambda(V^2)}\lambda(A^{-1})\varphi^{-1}\left(\frac{q}{\lambda(A^{-1})}\right)\psi^{-1}\left(\frac{q}{\lambda(A^{-1})}\right)\overset{(2.3)}{>}n,
\end{aligned}$$

as required. ∎

Corollary 2.5. *Let G be a locally compact group and let φ be an N-function with* $\limsup_{x\to 0} \varphi(x)/x^2 = 0$. *Then $f*g$ exists for all $f,g \in L^\varphi(G)$ if and only if G is compact. In particular, for $2 < p < \infty$, $f*g$ exists for all $f,g \in L^p(G)$ if and only if G is compact.*

Proof. By [RR, Proposition 3.1.7], we see that if G is compact, then $L^\varphi(G) \subseteq L^1(G)$ for every N-function φ, thus the "if" part is immediate.

For the converse, first observe that, by our hypothesis on φ, for a given $0 < \epsilon < 1$, there exists $0 < \delta < 1$ such that $\varphi(\sqrt{x}) < \epsilon x$ for all $x \in (0,\delta)$. This implies that $\sqrt{x} < \varphi^{-1}(\epsilon x)$. Now for each $y \in (0,\epsilon\delta)$, we have $\sqrt{\epsilon^{-1}y} < \varphi^{-1}(y)$, hence $\varphi^{-1}(y)^2/y > 1/\epsilon$, which is equivalent to $\limsup_{x\to\infty} x\varphi^{-1}(1/x)^2 = \infty$. This together with Theorem 2.4 proves the assertion. ∎

The following direct consequence of Theorem 2.4 is the main theorem of [GS1].

Corollary 2.6. *Let G be a non-compact locally compact group. If $p,q > 1$ are such that $1/p + 1/q < 1$, then, for every compact subset $K \subset G$, the set $E_K = \{(f,g) \in L^p(G) \times L^q(G) : \exists x \in K,\ f*g(x)$ is well defined$\}$ is σ-c-lower porous for some $c > 0$.*

In view of Theorem 2.4, the anonymous referee of the paper asked a natural question: Suppose that G is unimodular and

$$\limsup_{x\to\infty} x\varphi^{-1}(1/x)\psi^{-1}(1/x) < \infty.$$

Is it true that $f*g$ exists for every $(f,g) \in L^\varphi(G)\times L^\psi(G)$? We are unable to answer this question; but let us recall an analogue of Young's inequality for Orlicz spaces from [RR], which gives an affirmative answer in the following special case.

Theorem 2.7. *Let G be a unimodular locally compact group, φ_i, $i = 1,2$, be N-functions, and φ_3 be an arbitrary Young function satisfying*

$$\varphi_1^{-1}(x)\varphi_2^{-1}(x) \le x\varphi_3^{-1}(x)$$

for $x \ge 0$. Then for any $f_i \in L^{\varphi_i}(G)$, $i = 1,2$, the convolution $f_1 * f_2$ belongs to $L^{\varphi_3}(G)$ and moreover

$$N_{\varphi_3}(f_1 * f_2) \le 2N_{\varphi_1}(f_1)N_{\varphi_2}(f_2).$$

From this theorem it follows that if G is unimodular, φ,ψ are N-functions, and the function $x \mapsto x\varphi^{-1}(1/x)\psi^{-1}(1/x)$ is bounded on $(0,\infty)$, then $f*g \in L^\infty(G)$ for all $f \in L^\varphi(G)$ and $g \in L^\psi(G)$. To see this it is sufficient to put $\varphi_3(x) = 0$ for $|x| \le 1$ and $\varphi_3(x) = \infty$ otherwise.

Theorem 2.8. *Let G be a non-compact locally compact group and let φ be an N-function. Then for any compact symmetric neighborhood V of the* *identity, the set*

$$E_V = \{(f,g) \in M^\varphi(G) \times L^\infty(G) : \exists x \in V,\ |f * g(x)| < \infty\}$$

is σ-c-lower porous in $M^\varphi(G) \times L^\infty(G)$ for some $c > 0$.

Proof. Since G is not compact, by Lemma 2.3 there exists a sequence (a_n) in G such that for any distinct $m,n \in \mathbb{N}$,

$$a_m V^2 \cap a_n V^2 = \emptyset, \quad V a_m^{-1} \cap V a_n^{-1} = \emptyset, \quad \Delta(a_n) \le 1.$$

Now for every natural number n put

$$E_n = \Big\{(f,g) \in M^\varphi(G) \times L^\infty(G) : \exists x \in V,\ \int_G |f(y)|\,|g(y^{-1}x)|\,d\lambda(y) \le n\Big\}.$$

Thus, $E_V = \bigcup_{n=1}^\infty E_n$. Hence it is enough to show that for any natural number n, E_n is σ-c-lower porous for some $c > 0$. To prove this, let $c = 1/3$; then $2c/(1-c) = 1$. Fix $0 < \alpha < 1/3$, so $2\alpha/(1-\alpha) < 1$. By continuity of the map $x \mapsto 2\alpha/x$ on $(0,1)$ there are $0 < \beta < 1-\alpha$ and $d < 1$ such that $2\alpha/\beta(1-d) < 1$.

Set $A = \bigcup_{n=1}^\infty V a_n^{-1}$ and $B = \bigcup_{n=1}^\infty a_n V^2$. Fix a natural number n and some $r > 0$. Let $(f,g) \in E_n$. Since $f \in M^\varphi(G)$, we have $N_\varphi(f\chi_A) < \infty$. It follows that there exists an $n_0 \in \mathbb{N}$ such that for $A_0 := \bigcup_{k=n_0}^\infty V a_k^{-1}$, we obtain

$$N_\varphi(f\chi_{A_0}) < 1 - \alpha - \beta.$$

Now, let $n_1 > n_0$ be such that

$$\bigg[\widetilde{\varphi}^{-1}\bigg(\frac{q}{(n_1 - n_0 + 1)\lambda(V)}\bigg)\bigg]^{-1} > n(r^2(1-2\alpha)dp\beta)^{-1},$$

where $q = \sup_{x\in V} \Delta(x)$, $p = 1 - 2\alpha/\beta(1-d)$ and $\widetilde{\varphi}$ is the complementary N-function to φ.

Put $C = \bigcup_{k=n_0}^{n_1} V a_k^{-1}$ and $D = \bigcup_{k=n_0}^{n_1} a_k V^2$. Define

$$\widetilde{f}(y) = M\varphi^{-1}\bigg(\frac{1}{\lambda(C^{-1})\Delta(y)}\bigg)\chi_C(y) + f(y)\chi_{G\setminus C}(y)$$

and

$$\widetilde{g}(x) = \begin{cases} g(x) + r(1-\alpha) & \text{if } \operatorname{Re}(g(x)) \ge 0, \\ g(x) - r(1-\alpha) & \text{if } \operatorname{Re}(g(x)) < 0, \end{cases}$$

where M is a constant with $MN_\varphi(f\chi_C) = \beta r$. Then it is clear that

$$N_\varphi(\widetilde{f} - f) < r(1-\alpha) \quad \text{and} \quad \|\widetilde{g} - g\|_\infty = r(1-\alpha).$$

Hence $B((\widetilde{f},\widetilde{g}),\alpha r) \subset B((f,g),r)$. Therefore, it remains only to show that $B((\widetilde{f},\widetilde{g}),\alpha r) \cap E_n = \emptyset$.

Take any $(h,k) \in B((\widetilde{f},\widetilde{g}),\alpha r)$ and let

$$C_1 = \left\{ x \in C : |h(x)| \geq dM\varphi^{-1}\left(\frac{1}{\lambda(C^{-1})\Delta(x)}\right)\right\}, \quad C_2 = C \setminus C_1.$$

Then

$$\lambda(C_2^{-1}) < \frac{2\alpha}{\beta(1-d)}\lambda(C^{-1}) \quad \text{and} \quad |k(x)| \geq r(1-2\alpha).$$

Now let $z \in V$ be an arbitrary element. Since

$$\lambda(C_1^{-1}) = \lambda(C^{-1}) - \lambda(C_2^{-1}) \geq p\lambda(C^{-1}),$$

we have

$$\begin{aligned}\int_F |h(y)|\,|k(y^{-1}z)|\,d\lambda(y) &\geq r(1-2\alpha)dM \int_{C_1} \varphi^{-1}\left(\frac{1}{\lambda(C^{-1})\Delta(y)}\right) d\lambda(y) \\ &> r(1-2\alpha)dM \int_{C_1} \frac{d\lambda(y)}{\lambda(C^{-1})\Delta(y)\widetilde{\varphi}^{-1}\big(\frac{1}{\lambda(C^{-1})\Delta(y)}\big)} \\ &\geq r^2(1-2\alpha)dp\beta\left[\widetilde{\varphi}^{-1}\left(\frac{q}{(n_1-n_0+1)\lambda(V)}\right)\right]^{-1} > n,\end{aligned}$$

which completes the proof. ∎

We conclude this work with the following example.

EXAMPLE 2.9. (a) Let φ be a Young function. It is easy to see that if φ is a Young function with $\lim_{x\to 0}\varphi(x)/x \neq 0$, then $L^\varphi(G) \subseteq L^1(G)$, and hence $f * g$ exists for all $f, g \in L^\varphi(G)$. For instance, let $\varphi(x) = e^{|x|} - 1$ for $x \in \mathbb{R}$. Then $L^\varphi(\mathbb{R})$ is, in fact, closed with respect to convolution multiplication. Note that φ does not satisfy the condition $\lim_{x\to 0}\varphi(x)/x^2 = 0$.

(b) Let $\varphi(x) = |x|^\alpha(1 + \ln|x|)$. By Corollary 2.5, if $\alpha > 2$, then for any locally compact group G, the convolution of any two elements of $L^\varphi(G)$ exists if and only if G is compact. For $\alpha = 1$, by part (a), $L^\varphi(G)$ is closed under convolution multiplication for any locally compact group G.

(c) Let G be a locally compact group and let ψ be an N-function, so that $\lim_{x\to 0}\psi(x)/x = 0$. It can be easily verified that the function φ defined by $\varphi(x) := |x|\psi(x)$ for all $x \in [0,\infty)$ is also an N-function, which satisfies the condition of Corollary 2.5. According to this corollary, if the convolution of any two elements of $L^\varphi(G)$ exists, then G must be compact. Various examples of such Young functions can be found in [KR].

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REFERENCES

[ANR] F. Abtahi, R. Nasr-Isfahani and A. Rejali, *On the L^p-conjecture for locally compact groups*, Arch. Math. (Basel) 89 (2007), 237–242.

[AM] I. Akbarbaglu and S. Maghsoudi, *An answer to a question on the convolution of functions*, Arch. Math. (Basel) 98 (2012), 545–553.

[F] G. B. Folland, *A First Course in Abstract Harmonic Analysis*, CRC Press, Boca Raton, 1995.

[GS1] S. Głąb and F. Strobin, *Porosity and the L^p-conjecture*, Arch. Math. (Basel) 95 (2010), 583–592.

[GS2] S. Głąb and F. Strobin, *Dichotomies for L^p spaces*, J. Math. Anal. Appl. 368 (2010), 382–390.

[HR] E. Hewitt and K. Ross, *Abstract Harmonic Analysis I*, Springer, New York, 1970.

[H] H. Hudzik, *Orlicz spaces of essentially bounded functions and Banach–Orlicz algebras*, Arch. Math. (Basel) 44 (1985), 535–538.

[KR] M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, Noordhoff, 1961.

[K] W. Kunze, *Noncommutative Orlicz spaces and generalized Arens algebras*, Math. Nachr. 147 (1990), 123–138.

[M] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Math. 1034, Springer, Berlin, 1983.

[R] M. Rajagopalan, *L_p-conjecture for locally compact groups, I*, Trans. Amer. Math. Soc. 125 (1966), 216–222.

[RA] M. M. Rao, *Convolutions of vector fields, III. Amenability and spectral properties*, in: Real and Stochastic Analysis, Trends Math., Birkhäuser, Boston, 2004, 375–401.

[RR] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Dekker, New York, 1991.

[RI] N. W. Rickert, *Convolution of L^p functions*, Proc. Amer. Math. Soc. 18 (1967), 762–763.

[S] S. Saeki, *The L^p-conjecture and Young's inequality*, Illinois J. Math. 34 (1990), 614–627.

[Z] A. C. Zaanen, *Riesz Spaces, II*, North-Holland Math. Library, 30, North-Holland, Amsterdam, 1983.

[ZA] L. Zajíček, *On σ-porous sets in abstract spaces*, Abstr. Appl. Anal. 5 (2005), 509–534,

[Z1] W. Żelazko, *On the algebras L_p of locally compact groups*, Colloq. Math. 8 (1961), 115–120.

[Z2] W. Żelazko, *A note on L_p-algebras*, Colloq. Math. 10 (1963), 53–56.

Ibrahim Akbarbaglu, Saeid Maghsoudi
Department of Mathematics
University of Zanjan
Zanjan 45195-313, Iran
E-mail: ibrahim.akbarbaglu@znu.ac.ir
s_maghsodi@znu.ac.ir

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