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PRIME NUMBERS WITH BEATTY SEQUENCES

BY

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Abstract. A study of certain Hamiltonian systems has led Y. Long to conjecture the existence of infinitely many primes which are not of the form $p = 2\lfloor \alpha n \rfloor + 1$, where $1 < \alpha < 2$ is a fixed irrational number. An argument of P. Ribenboim coupled with classical results about the distribution of fractional parts of irrational multiples of primes in an arithmetic progression immediately implies that this conjecture holds in a much more precise asymptotic form. Motivated by this observation, we give an asymptotic formula for the number of primes $p = q\lfloor \alpha n + \beta \rfloor + a$ with $n \le N$, where α, β are real numbers such that α is positive and irrational of finite type (which is true for almost all α) and a, q are integers with $0 \le a < q \le N^\kappa$ and $\gcd(a, q) = 1$, where $\kappa > 0$ depends only on α. We also prove a similar result for primes $p = \lfloor \alpha n + \beta \rfloor$ such that $p \equiv a \pmod q$.

1. Introduction. For two fixed real numbers α and β, the corresponding *non-homogeneous Beatty sequence* is the sequence of integers defined by

$$\mathcal{B}_{\alpha,\beta} = (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty}.$$

Beatty sequences appear in a variety of apparently unrelated mathematical settings, and because of their versatility, many arithmetic properties of these sequences have been explored in the literature, such as

- structural properties (see [14, 20, 21, 30, 38]);
- additive properties (see [5, 24]);
- the distribution of arithmetic functions (see [1, 2, 8, 11, 12, 16, 29]);
- the distribution of quadratic non-residues and primitive roots (see [4, 6, 7, 9, 15, 31, 32, 33]).

In 2000, while investigating the Maslov-type index theory for Hamiltonian systems, Long [28] made the following conjecture:

Conjecture. *For every irrational number* $1 < \alpha < 2$, *there are infinitely many prime numbers not of the form* $p = 2\lfloor \alpha n \rfloor + 1$ *for some* $n \in \mathbb{N}$.

Jia [18] has given a lower bound for the number of such primes p in the interval $(x/2, x]$. However, we remark that an even stronger one can be obtained using well established techniques. In fact, with a simple modification

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of an argument given by Ribenboim [34, Chapter 4.V], one can show that the number of such primes $p \le x$ is asymptotic to $(1-\alpha^{-1})\pi(x)$ as $x \to \infty$; see also [10, 26]. Moreover, Ribenboim's method can be applied to the more general problem of estimating

$$\mathcal{N}_{\alpha,\beta;q,a}(x) = \#\{n \le x : p = q\lfloor \alpha n + \beta \rfloor + a \text{ is prime}\},$$

where α, β are fixed real numbers such that α is positive and irrational, and a, q are integers with $0 \le a < q$ and $\gcd(a,q) = 1$. In fact, if a and q are fixed, one easily derives the asymptotic formula

$$\mathcal{N}_{\alpha,\beta;q,a}(x) = (1+o(1))\,\frac{q}{\varphi(q)}\,\pi(x) \qquad (x \to \infty),$$

where the function implied by $o(\cdot)$ depends on α, β and q, and $\varphi(\cdot)$ is the *Euler function*. We remark that some additive problems of Goldbach type involving such primes have also been successfully studied; see [5, 24, 27].

Unfortunately, it seems that these results and techniques are not well known among non-experts (as we have mentioned, [18] gives a proof of a weaker result), so in part our motivation comes from a desire to make these techniques better known and to examine their potential and limitations.

More precisely, we consider here the problem of finding uniform estimates for $\mathcal{N}_{\alpha,\beta;q,a}(x)$ if q is allowed to grow with x. We also consider the same problem for the counting function

$$\mathcal{M}_{\alpha,\beta;q,a}(x) = \#\{n \le x \ : \ p = \lfloor \alpha n + \beta \rfloor \text{ is prime, and } p \equiv a \pmod q\}.$$

In particular, in the case that α is of *finite type* (which is true for *almost all* α in the sense of Lebesgue measure), our main results yield (by partial summation) non-trivial results for both $\mathcal{N}_{\alpha,\beta;q,a}(x)$ and $\mathcal{M}_{\alpha,\beta;q,a}(x)$ even when q grows as a certain power of x.

2. Notation. The symbol $\|x\|$ is used to denote the distance from the real number x to the nearest integer; that is,

$$\|x\| = \min_{n \in \mathbb{Z}} |x - n| \qquad (x \in \mathbb{R}).$$

As usual, we denote by $\lfloor x \rfloor$, $\lceil x \rceil$, and $\{x\}$ the greatest integer $\le x$, the least integer $\ge x$, and the fractional part of x, respectively.

We also put $\mathbf{e}(x) = e^{2\pi i x}$ for all real numbers x and use $\Lambda(\cdot)$ to denote the *von Mangoldt function*:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n \text{ is a power of a prime } p, \\ 0 & \text{otherwise.} \end{cases}$$

Throughout the paper, the implied constants in the symbols O, $\ll$ and $\gg$ may depend on the parameters α and β but are absolute unless indicated otherwise. We recall that the notations $A \ll B$, $B \gg A$ and $A = O(B)$ are all equivalent to the statement that $|A| \le c|B|$ for some constant $c > 0$.

3. Preliminaries. Recall that the *discrepancy* $D(M)$ of a sequence of (not necessarily distinct) real numbers $a_1, \ldots, a_M \in [0,1)$ is defined by

$$D(M) = \sup_{\mathcal{I} \subseteq [0,1)} \left| \frac{V(\mathcal{I}, M)}{M} - |\mathcal{I}| \right|, \tag{1}$$

where the supremum is taken over all subintervals $\mathcal{I} = (c,d)$ of the interval $[0,1)$, $V(\mathcal{I}, M)$ is the number of positive integers $m \le M$ such that $a_m \in \mathcal{I}$, and $|\mathcal{I}| = d - c$ is the length of $\mathcal{I}$.

For an irrational number γ, we define its *type* τ by the relation

$$\tau = \sup\Big\{ \varrho \in \mathbb{R} : \liminf_{\substack{n \to \infty \\ (n \in \mathbb{N})}} n^{\varrho} \, \|\gamma n\| = 0 \Big\}.$$

Using *Dirichlet's approximation theorem*, it is easily seen that $\tau \ge 1$ for every irrational number γ. The celebrated theorems of Khinchin [19] and of Roth [35, 36] assert that $\tau = 1$ for almost all real (in the sense of the Lebesgue measure) and all irrational algebraic numbers γ, respectively; see also [13, 37].

For every irrational number γ, it is well known that the sequence of fractional parts $\{\gamma\}, \{2\gamma\}, \{3\gamma\}, \ldots$ is *uniformly distributed modulo* 1 (for instance, see [23, Chapter 1, Example 2.1]). If γ is of finite type, this statement can be made more precise. Let $D_{\gamma,\delta}(M)$ denote the discrepancy of the sequence of fractional parts $(\{\gamma m + \delta\})_{m=1}^{M}$. By [23, Chapter 2, Theorem 3.2] we have:

LEMMA 3.1. *Let γ be a fixed irrational number of finite type $\tau < \infty$. Then for all $\delta \in \mathbb{R}$ the following bound holds*:

$$D_{\gamma,\delta}(M) \le M^{-1/\tau + o(1)} \qquad (M \to \infty),$$

where the function implied by $o(\cdot)$ depends only on γ.

The following elementary result characterizes the set of numbers that occur in a Beatty sequence $\mathcal{B}_{\alpha,\beta}$ in the case that $\alpha > 1$:

LEMMA 3.2. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$. Then an integer m has the form $m = \lfloor \alpha n + \beta \rfloor$ for some integer n if and only if*

$$0 < \{\alpha^{-1}(m - \beta + 1)\} \le \alpha^{-1}.$$

The value of n is determined uniquely by m.

Proof. It is easy to see that an integer m has the form $m = \lfloor \alpha n + \beta \rfloor$ for some integer n if and only if the inequalities

$$\frac{m - \beta}{\alpha} \le n < \frac{m - \beta + 1}{\alpha}$$

hold, and since $\alpha > 1$ the value of n is determined uniquely. ∎

We also need the following statement, which is a simplified and weakened version of a theorem of Balog and Perelli [3] (see also [25]):

LEMMA 3.3. *For an arbitrary real number* ϑ *and coprime integers* a, q *with* $0 \le a < q$, *if* $|\vartheta - b/d| \le 1/L$ *and* $\gcd(b, d) = 1$, *then the bound*

$$\sum_{\substack{n \le L \\ n \equiv a \pmod q}} \Lambda(n)\, \mathbf{e}(\vartheta n) \ll \left(\frac{L}{d^{1/2}} + d^{1/2} L^{1/2} + L^{4/5} \right) (\log L)^3$$

holds, *where the implied constant is absolute.*

Finally, we use the *Siegel–Walfisz theorem* (see, for example, the book [17] by Huxley), which asserts that for any fixed constant $B > 0$ and uniformly for integers $L \ge 3$ and $0 \le a < q \le (\log L)^B$ with $\gcd(a, q) = 1$, one has

$$\sum_{\substack{n \le L \\ n \equiv a \pmod q}} \Lambda(n) = \frac{L}{\varphi(q)} + O(L \exp(-C_B \sqrt{\log L})), \tag{2}$$

where $C_B > 0$ is an absolute constant that depends only on B.

4. Bounds on exponential sums. The following result may be well known but does not seem to be recorded in the literature. Thus, we present it here with a complete proof.

THEOREM 4.1. *Let* γ *be a fixed irrational number of finite type* $\tau < \infty$. *Then*, *for every real number* $0 < \varepsilon < 1/(8\tau)$, *there is a number* $\eta > 0$ *such that the bound*

$$\Big| \sum_{m \le M} \Lambda(qm + a)\, \mathbf{e}(\gamma k m) \Big| \le M^{1-\eta}$$

holds for all integers $1 \le k \le M^{\varepsilon}$ *and* $0 \le a < q \le M^{\varepsilon/4}$ *with* $\gcd(a, q) = 1$ *provided that* M *is sufficiently large.*

Proof. Fix a constant ϱ such that

$$1 \le \tau < \varrho < \frac{1}{8\varepsilon}. \tag{3}$$

Since γ is of type τ, for some constant $c > 0$ we have

$$\|\gamma d\| > c d^{-\varrho} \qquad (d \ge 1). \tag{4}$$

Let k, a, q be integers with the properties stated in the proposition, and write

$$\sum_{m \le M} \Lambda(qm + a)\, \mathbf{e}(\gamma k m) = \mathbf{e}(-\vartheta a) \sum_{\substack{n \le L \\ n \equiv a \pmod q}} \Lambda(n)\, \mathbf{e}(\vartheta n), \tag{5}$$

where $\vartheta = \gamma k / q$ and $L = qM + a$. Let b/d be the convergent in the continued fraction expansion of ϑ which has the largest denominator d not exceeding

$L^{1-\varepsilon}$; then

$$\left|\frac{\gamma k}{q} - \frac{b}{d}\right| \le \frac{1}{dL^{1-\varepsilon}}. \tag{6}$$

Multiplying by qd, we deduce from (4) that

$$\frac{q}{L^{1-\varepsilon}} \ge |\gamma kd - bq| \ge \|\gamma kd\| > c(kd)^{-\varrho}.$$

Thus, since $k \le L^{\varepsilon}$ and $q \le L^{\varepsilon/4} \le L^{\varepsilon}$, we see that under the condition (3) the bound

$$d \ge CL^{(1-2\varepsilon)/\varrho-\varepsilon} \ge CL^{1/(4\varrho)} \tag{7}$$

holds, where $C = c^{1/\varrho}$ and L is sufficiently large.

Inserting (7) into (6) and using (3) again, we conclude that

$$\left|\frac{\gamma k}{q} - \frac{b}{d}\right| \le \frac{1}{CL^{1+1/(4\varrho)-\varepsilon}} \le \frac{1}{L}$$

if L is sufficiently large. We are therefore in a position to apply Lemma 3.3; from (3), (7), and the fact that $d \le L^{1-\varepsilon}$, it follows that the bound

$$\sum_{\substack{n\le L\\ n\equiv a \,(\mathrm{mod}\, q)}} \Lambda(n)\,\mathbf{e}(\vartheta n) \ll (L^{1-1/(8\varrho)} + L^{1-\varepsilon/2})(\log L)^3 \le L^{1-\varepsilon/3}$$

holds for all sufficiently large L. Since $L \ll qM \le M^{1+\varepsilon/4}$, the result now follows from simple calculations after inserting this estimate into (5). ∎

Using similar arguments, we have:

Theorem 4.2. *Let γ be a fixed irrational number of finite type $\tau < \infty$. Then, for every real number $0 < \varepsilon < 1/(8\tau)$, there is a number $\eta > 0$ such that the bound*

$$\Bigg|\sum_{\substack{m\le M\\ m\equiv a \,(\mathrm{mod}\, q)}} \Lambda(m)\,\mathbf{e}(\gamma km)\Bigg| \le M^{1-\eta}$$

holds for all integers $1 \le k \le M^{\varepsilon}$ and $0 \le a < q \le M^{\varepsilon/4}$ with $\gcd(a,q) = 1$ provided that M is sufficiently large.

5. Main results

Theorem 5.1. *Let α and β be a fixed real numbers with α positive, irrational, and of finite type. Then there is a constant $\kappa > 0$ such that for all integers $0 \le a < q \le N^{\kappa}$ with $\gcd(a,q) = 1$, we have*

$$\sum_{n\le N} \Lambda(q\lfloor \alpha n + \beta\rfloor + a) = \alpha^{-1} \sum_{m\le \lfloor \alpha N+\beta\rfloor} \Lambda(qm+a) + O(N^{1-\kappa})$$

where the implied constant depends only on α and β.

Proof. Suppose first that $\alpha > 1$. It is obvious that if α is of finite type, then so is α^{-1}. We choose

$$0 < \varepsilon < \frac{1}{16\tau},$$

where $1 \le \tau < \infty$ is the type of α^{-1}.

Put $\gamma = \alpha^{-1}$, $\delta = \alpha^{-1}(1-\beta)$, and $M = \lfloor \alpha N + \beta \rfloor$. By Lemma 3.2, it follows that

$$(8) \qquad S_{\alpha,\beta;q,a}(N) = \sum_{n \le N} \Lambda(q\lfloor \alpha n + \beta \rfloor + a) = \sum_{\substack{m \le M \\ 0 < \{\gamma m + \delta\} \le \gamma}} \Lambda(qm + a) + O(1)$$
$$= \sum_{m \le M} \Lambda(qm + a)\, \psi(\gamma m + \delta) + O(1),$$

where $\psi(x)$ is the periodic function with period one for which

$$\psi(x) = \begin{cases} 1 & \text{if } 0 < x \le \gamma, \\ 0 & \text{if } \gamma < x \le 1. \end{cases}$$

By a classical result of Vinogradov (see [40, Chapter I, Lemma 12]) it is known that for any Δ such that

$$0 < \Delta < \frac{1}{8} \quad \text{and} \quad \Delta \le \frac{1}{2}\min\{\gamma, 1-\gamma\},$$

there is a real-valued function $\psi_\Delta(x)$ with the following properties:

(i) $\psi_\Delta(x)$ is periodic with period one;
(ii) $0 \le \psi_\Delta(x) \le 1$ for all $x \in \mathbb{R}$;
(iii) $\psi_\Delta(x) = \psi(x)$ if $\Delta \le x \le \gamma - \Delta$ or if $\gamma + \Delta \le x \le 1 - \Delta$;
(iv) $\psi_\Delta(x)$ can be represented as a Fourier series

$$\psi_\Delta(x) = \gamma + \sum_{k=1}^{\infty} (g_k \mathbf{e}(kx) + h_k \mathbf{e}(-kx)),$$

where the coefficients satisfy the uniform bound

$$(9) \qquad \max\{|g_k|, |h_k|\} \ll \min\{k^{-1}, k^{-2}\Delta^{-1}\} \qquad (k \ge 1).$$

Therefore, from (8) we deduce that

$$(10) \qquad S_{\alpha,\beta;q,a}(N) = \sum_{m \le M} \Lambda(qm + a)\psi_\Delta(\gamma m + \delta) + O(1 + V(\mathcal{I}, M)\log N),$$

where $V(\mathcal{I}, M)$ denotes the number of positive integers $m \le M$ such that

$$\{\gamma m + \delta\} \in \mathcal{I} = [0, \Delta) \cup (\gamma - \Delta, \gamma + \Delta) \cup (1 - \Delta, 1).$$

Since $|\mathcal{I}| \ll \Delta$, it follows from the definition (1) and Lemma 3.1, that

$$(11) \qquad V(\mathcal{I}, M) \ll \Delta N + N^{1-\varepsilon},$$

where the implied constant depends only on α.

To estimate the sum in (10), we use the Fourier expansion for $\psi_\Delta(\gamma m+\delta)$ and change the order of summation, obtaining

$$\begin{aligned}(12)\qquad &\sum_{m\le M} \Lambda(qm+a)\,\psi_\Delta(\gamma m+\delta)\\ &= \gamma\sum_{m\le M}\Lambda(qm+a) + \sum_{k=1}^{\infty} g_k\,\mathbf{e}(\delta k)\sum_{m\le M}\Lambda(qm+a)\,\mathbf{e}(\gamma k m)\\ &\quad + \sum_{k=1}^{\infty} h_k\,\mathbf{e}(-\delta k)\sum_{m\le M}\Lambda(qm+a)\,\mathbf{e}(-\gamma k m).\end{aligned}$$

By Theorem 4.1 and the bound (9), we see that for $0\le a<q\le M^{\varepsilon/4}$, we have

$$(13)\qquad \sum_{k\le M^\varepsilon} g_k\,\mathbf{e}(\delta k)\sum_{m\le M}\Lambda(qm+a)\,\mathbf{e}(\gamma k m) \ll M^{1-\eta}\sum_{k\le M^\varepsilon} k^{-1} \ll M^{1-\eta/2}$$

for some $\eta>0$ that depends only on α. Similarly,

$$(14)\qquad \sum_{k\le M^\varepsilon} h_k\,\mathbf{e}(-\delta k)\sum_{m\le M}\Lambda(qm+a)\,\mathbf{e}(-\gamma k m) \ll M^{1-\eta/2}.$$

On the other hand, using the trivial bound

$$\Big|\sum_{m\le M}\Lambda(qm+a)\,\mathbf{e}(\gamma k m)\Big| \le \sum_{n\le N}\Lambda(n)\ll N,$$

we have

$$\begin{aligned}(15)\qquad &\sum_{k>M^\varepsilon} g_k\,\mathbf{e}(\delta k)\sum_{m\le M}\Lambda(qm+a)\,\mathbf{e}(\gamma k m)\\ &\qquad\ll N\sum_{k>M^\varepsilon} k^{-2}\Delta^{-1} \ll N^{1-\varepsilon}\Delta^{-1},\end{aligned}$$

and

$$(16)\qquad \sum_{k>M^\varepsilon} h_k\,\mathbf{e}(-\delta k)\sum_{m\le M}\Lambda(qm+a)\,\mathbf{e}(-\gamma k m) \ll N^{1-\varepsilon}\Delta^{-1}.$$

Inserting the bounds (13)–(16) into (12), we obtain

$$\begin{aligned}(17)\qquad &\sum_{m\le M}\Lambda(qm+a)\,\psi_\Delta(\gamma m+\delta)\\ &\qquad= \gamma\sum_{m\le M}\Lambda(qm+a) + O(M^{1-\eta/2}+N^{1-\varepsilon}\Delta^{-1}),\end{aligned}$$

where the constant implied by $O(\cdot)$ depends only on α and β.

Substituting (11) and (17) in (10) and choosing $\Delta = N^{-\varepsilon/4}$ leads to

$$S_{\alpha,\beta;q,a}(N) = \gamma \sum_{m \le M} \Lambda(qm + a) + O(N^{1-\kappa}), \tag{18}$$

for some κ which depends only on α. This concludes the proof in the case that $\alpha > 1$.

If $\alpha < 1$, we put $t = \lceil \alpha^{-1} \rceil$ and write

$$\sum_{n \le N} \Lambda(q\lfloor \alpha n + \beta \rfloor + a) = \sum_{j=0}^{t-1} \sum_{n \le (N-j)/t} \Lambda(q\lfloor \alpha t n + \alpha j + \beta \rfloor + a).$$

Applying the preceding argument with the irrational number $\alpha t > 1$, we conclude the proof. ∎

In particular, using the Siegel–Walfisz theorem (2) to estimate the sum in (18) for "small" a and q, we obtain:

Corollary 5.2. *Under the conditions of Theorem 5.1, for any constant $B > 0$ and uniformly for all integers $N \ge 3$ and $0 \le a < q \le (\log N)^B$ with $\gcd(a, q) = 1$, we have*

$$\sum_{n \le N} \Lambda(q\lfloor \alpha n + \beta \rfloor + a) = \frac{q}{\varphi(q)}\, N + O(N \exp(-C\sqrt{\log N}\,))$$

for some constant $C > 0$ that depends only on α, β and B.

In the special case that $(a, q) = (0, 1)$ or $(1, 2)$ (the latter case corresponding to primes in the Long conjecture), we can use a well known bound on the error term in the Prime Number Theorem (proved independently by Korobov [22] and Vinogradov [39]) to achieve the following sharper result:

Corollary 5.3. *Suppose that $(a, q) = (0, 1)$ or $(a, q) = (1, 2)$. Then, under the conditions of Theorem* 5.1, *for any constant $B > 0$ and uniformly for all integers $N \ge 3$, we have*

$$\sum_{n \le N} \Lambda(q\lfloor \alpha n + \beta \rfloor + a) = qN + O(N \exp(-c(\log N)^{3/5}(\log\log N)^{-1/5}))$$

for some absolute constant $c > 0$.

Finally, using Theorem 4.2 in place of Theorem 4.1, we obtain the following analogues of Theorem 5.1 and its two corollaries:

Theorem 5.4. *Let α and β be fixed real numbers with α positive, irrational, and of finite type. Then there is a constant $\kappa > 0$ such that for all integers $0 \le a < q \le N^\kappa$ with $\gcd(a, q) = 1$, we have*

$$\sum_{\substack{n \le N \\ \lfloor \alpha n + \beta \rfloor \equiv a \pmod q}} \Lambda(\lfloor \alpha n + \beta \rfloor) = \alpha^{-1} \sum_{\substack{m \le \lfloor \alpha N + \beta \rfloor \\ m \equiv a \pmod q}} \Lambda(m) + O(N^{1-\kappa})$$

where the implied constant depends only on α and β.

COROLLARY 5.5. *Under the conditions of Theorem* 5.4, *for any constant* $B > 0$ *and uniformly for all integers* $N \ge 3$ *and* $0 \le a < q \le (\log N)^B$ *with* $\gcd(a,q) = 1$, *we have*

$$\sum_{\substack{n \le N \\ \lfloor \alpha n + \beta \rfloor \equiv a \pmod q}} \Lambda(\lfloor \alpha n + \beta \rfloor) = \frac{N}{\varphi(q)} + O(N \exp(-C\sqrt{\log N}\,))$$

for some constant $C > 0$ *that depends only on* α, β *and* B.

COROLLARY 5.6. *Suppose that* $(a,q) = (0,1)$ *or* $(a,q) = (1,2)$. *Then, under the conditions of Theorem* 5.4, *for any constant* $B > 0$ *and uniformly for all integers* $N \ge 3$, *we have*

$$\sum_{\substack{n \le N \\ \lfloor \alpha n + \beta \rfloor \equiv a \pmod q}} \Lambda(\lfloor \alpha n + \beta \rfloor) = N + O(N \exp(-c(\log N)^{3/5}(\log \log N)^{-1/5}))$$

for some absolute constant $c > 0$.

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