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SOME REMARKS ABOUT STRONG PROXIMALITY OF COMPACT FLOWS

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Abstract. This note aims at providing some information about the concept of a strongly proximal compact transformation semigroup. In the affine case, a unified approach to some known results is given. It is also pointed out that a compact flow $(X,\mathcal{S})$ is strongly proximal if (and only if) it is proximal and every point of X has an $\mathcal{S}$-strongly proximal neighborhood in X. An essential ingredient, in the affine as well as in the nonaffine case, turns out to be the existence of a unique minimal subset.

1. Introduction. A compact *flow* (or *transformation semigroup*) $(X,\mathcal{S})$ consists here of a nonempty compact (Hausdorff) space X, a semigroup $\mathcal{S}$, and a mapping $(s,x)\mapsto sx$ of $\mathcal{S}\times X$ into X (called the *action* of $\mathcal{S}$ on X) that satisfies the following conditions:

(i) for every $s\in\mathcal{S}$, the mapping $x\mapsto sx$ of X into itself is continuous,
(ii) $(st)x=s(tx)$ for every $s,t\in\mathcal{S}$ and $x\in X$.

Let $(X,\mathcal{S})$ be a compact flow, and let $(\mathcal{M}^1(X),\mathcal{S})$ be the compact affine flow of all regular Borel probability measures on X which is induced by $(X,\mathcal{S})$; then the compact flow $(X,\mathcal{S})$ is said to be *strongly proximal* if $(\mathcal{M}^1(X),\mathcal{S})$ is proximal [4]. (The origin of this notion lies in Furstenberg's theory of boundaries of Lie groups.) The aim of this note is to provide some new information about this concept.

It has been known for a long time that irreducible compact affine transformation groups are strongly proximal [4]; on the other hand, the main result in [10] asserts that if the compact flow $(X,\mathcal{S})$ is metrizable and strongly proximal, then $(\mathcal{M}^1(X),\mathcal{S})$ is strongly proximal; in Section 3, it is shown (without the hypothesis of metrizability) that these two facts are special cases of the same general property (Theorem 3.4).

The concept of $\mathcal{S}$-strong proximality extends in a natural way to any (closed) subset of X; in Section 4, it is shown that if the compact flow $(X,\mathcal{S})$ is proximal and if every point of X has an $\mathcal{S}$-strongly proximal neighbor-

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hood, then $(X,\mathcal{S})$ is strongly proximal. Strongly proximal minimal compact transformation groups were characterized in [3] (cf. also [5]) by Glasner without any reference to measures; a characterization of the same sort is given in Section 5 for closed $\mathcal{S}$-strongly proximal subsets of X.

2. Preliminaries. In this section, we recall some of the basic notions of topological dynamics used in this paper. More detailed information can be found in [1], [3] and [13].

Let us consider a compact flow $(X,\mathcal{S})$. Let X^X be the semigroup of all (not necessarily continuous) mappings of X into itself. (For every $p,q\in X^X$, the mapping $pq\in X^X$ is given by $(pq)x=p(qx)$, qx being the value of q at $x\in X$.) Let us assume that X^X is provided with the product topology; by Tikhonov's theorem, X^X is compact. For every $q\in X^X$ and every continuous $r\in X^X$, the mappings $p\mapsto pq$ and $p\mapsto rp$ of X^X into itself are continuous. The closure in X^X of the semigroup of all mappings $x\mapsto sx$ of X into itself $(s\in\mathcal{S})$ is a subsemigroup of X^X. This semigroup is denoted by $E(X,\mathcal{S})$ and is called the *enveloping semigroup* of $(X,\mathcal{S})$.

A subset A of X is said to be $\mathcal{S}$-*stable* (or *stable in* $(X,\mathcal{S})$) if $sA\subset A$ for every $s\in\mathcal{S}$. Let Y be a nonempty compact $\mathcal{S}$-stable subset of X. The compact *subflow* $(Y,\mathcal{S})$ of $(X,\mathcal{S})$ consists of Y, $\mathcal{S}$ and the mapping $(s,x)\mapsto sx$ of $\mathcal{S}\times Y$ into Y; its enveloping semigroup $E(Y,\mathcal{S})$ coincides with the set of all mappings of Y into itself induced by those of $E(X,\mathcal{S})$ (note that $pY\subset Y$ for every $p\in E(X,\mathcal{S})$).

For each $x\in X$, let $\mathcal{S}x=\{sx\mid s\in\mathcal{S}\}$ be the orbit of x; then the closure $\overline{\mathcal{S}x}$ of $\mathcal{S}x$ in X is $\mathcal{S}$-stable, and the equality $\overline{\mathcal{S}x}=\{px\mid p\in E(X,\mathcal{S})\}$ holds. An $\mathcal{S}$-*minimal* subset of X is a nonempty subset M of X such that $\overline{\mathcal{S}x}=M$ for each $x\in M$. Obviously, an $\mathcal{S}$-minimal subset of X can equivalently be defined as a closed, nonempty and $\mathcal{S}$-stable subset of X which has no proper subsets with these properties. By Zorn's lemma, every closed nonempty and $\mathcal{S}$-stable subset of X contains an $\mathcal{S}$-minimal subset of X. The compact flow $(X,\mathcal{S})$ is said to be *minimal* if X is an $\mathcal{S}$-minimal subset of X.

The compact flow $(X,\mathcal{S})$ is said to be *proximal* if for every $(x,y)\in X\times X$ there is a net (s_α) in $\mathcal{S}$ and $z\in X$ such that $\lim s_\alpha x=z$ and $\lim s_\alpha y=z$; it is equivalent to say that for every $(x,y)\in X\times X$, there is $p\in E(X,\mathcal{S})$ such that $px=py$; it is also equivalent to say that for every $(x,y)\in X\times X$ and every entourage V (in the unique uniformity) of X, there is $s\in\mathcal{S}$ such that $(sx,sy)\in V$. It is well known (and easy to prove) that if $(X,\mathcal{S})$ is proximal, then $(X,\mathcal{S})$ has a unique minimal subset.

A compact flow $(K,\mathcal{S})$ is said to be *affine* if K is a convex subspace of a locally convex Hausdorff topological vector space and, for each $s\in\mathcal{S}$, the mapping $x\mapsto sx$ of K into itself is affine. If $(K,\mathcal{S})$ is a compact affine flow, then each member of $E(K,\mathcal{S})$ is an affine mapping. Let $(X,\mathcal{S})$ be a compact flow, $\mathcal{C}(X)$ the Banach space of all continuous real-valued functions on X with the supremum norm, and $\mathcal{C}^*(X)$ the topological dual of $\mathcal{C}(X)$ equipped with the weak* topology. Let $\mathcal{M}^1(X)$ be the compact convex subspace of $\mathcal{C}^*(X)$ formed by all regular Borel probability measures on X; then the compact affine flow $(\mathcal{M}^1(X), \mathcal{S})$ consists of $\mathcal{M}^1(X)$, $\mathcal{S}$ and the mapping $(s,\mu) \mapsto s\mu$ of $\mathcal{S}\times\mathcal{M}^1(X)$ into $\mathcal{M}^1(X)$, $s\mu$ being defined by $(s\mu)(f) = \mu(sf)$, and $(sf)(x) = f(sx)$ for every $f \in \mathcal{C}(X)$ and every $x \in X$. For every $x \in X$ and $A \subset X$, let $\delta_x \in \mathcal{M}^1(X)$ be the point mass at x and $\delta_A = \{\delta_x \mid x \in A\}$; then $x \mapsto \delta_x$ is a homeomorphism of X onto the set δ_X of extreme points of $\mathcal{M}^1(X)$, and $\delta_{sx} = s\delta_x$ for $(s,x) \in \mathcal{S} \times X$ (therefore, we may regard $(X, \mathcal{S})$ as a compact subflow of $(\mathcal{M}^1(X), \mathcal{S})$ via $x \mapsto \delta_x$).

The standard notions of functional analysis used here can be found for instance in [2]. As in [2], if A is a subset of a locally convex Hausdorff topological vector space, $\operatorname{co} A$ (respectively $\overline{\operatorname{co}}\, A$) denotes the convex (respectively closed convex) hull of A. If A is convex, the set of all extreme points of A is denoted by $\operatorname{ext} A$.

3. Strong proximality of affine flows. First, let us recall the following basic concept (introduced by Glasner in [4]).

Definition 3.1. A compact flow $(X, \mathcal{S})$ is said to be *strongly proximal* if the compact affine flow $(\mathcal{M}^1(X), \mathcal{S})$ is proximal.

Characterization 3.2 below of strong proximality is proved in [4]. (In [4], $\mathcal{S}$ is assumed to be a group; however, the proof works for any semigroup.)

Lemma 3.2. *Let $(X, \mathcal{S})$ be a compact flow. Then $(X, \mathcal{S})$ is strongly proximal if and only if for every $\mu \in \mathcal{M}^1(X)$ the closure $\overline{\mathcal{S}\mu}$ of $\mathcal{S}\mu$ in $\mathcal{M}^1(X)$ meets δ_X (that is, there is a net (s_α) in $\mathcal{S}$ and $x \in X$ such that* $\lim s_\alpha\mu = \delta_x$*).*

In connection with Section 4, the following equivalences should be pointed out.

Lemma 3.3. *Let $(X, \mathcal{S})$ be a compact flow and $M = \{x \in X \mid \forall \mu \in \mathcal{M}^1(X),\ \delta_x \in \overline{\mathcal{S}\mu}\}$. Then the following conditions are equivalent.*

(1) *$(X, \mathcal{S})$ is strongly proximal.*
(2) *M is the unique minimal subset of* $(X, \mathcal{S})$.
(3) *M is nonempty.*

Proof. (1)⇒(2): Since $(\mathcal{M}^1(X), \mathcal{S})$ is proximal, it has a unique minimal subset M'. Since every nonempty closed and $\mathcal{S}$-stable subset of $\mathcal{M}^1(X)$ contains a minimal subset of $(\mathcal{M}^1(X), \mathcal{S})$, M' is contained in X and in $\overline{\mathcal{S}\mu}$ ($\mu \in \mathcal{M}^1(X)$); in other words, $M' \subset M$. To conclude that $M = M'$, it suffices to remark that if $x \in M'$, then $M' = \overline{\mathcal{S}x} \supset M$.

(2)⇒(3): Obvious.

(3)⇒(1): If $M \neq \emptyset$, then $\overline{\mathcal{S}\mu} \cap X \neq \emptyset$ for every $\mu \in \mathcal{M}^1(X)$, and by Lemma 3.2, $(X, \mathcal{S})$ is strongly proximal. ∎

Recall that if $(K, \mathcal{S})$ is a compact affine flow, X a compact $\mathcal{S}$-stable subset of K such that $\overline{\mathrm{co}}\, X = K$, and $\beta : \mathcal{M}^1(X) \to K$ the barycenter mapping, then β is surjective, affine and continuous [9]. Moreover, $\beta(s\mu) = s\beta(\mu)$ for all $s \in \mathcal{S}$ and $\mu \in \mathcal{M}(X)$, as is pointed out for instance in [3].

Statements 3.5 and 3.6 below are closely related. The first one, which dates back to 1975, is ascribed by Glasner [4] to Furstenberg; the second one has recently been stated by Raja [10] under the assumption that X is metrizable. In fact, they are two special cases of the following theorem.

THEOREM 3.4. *Let $(K, \mathcal{S})$ be a compact affine flow. Suppose that K has a unique minimal subset M under $\mathcal{S}$ and that M contains an extreme point of K. Then $(K, \mathcal{S})$ is strongly proximal.*

Proof. Let $z \in M \cap \operatorname{ext} K$. Let $\mu \in \mathcal{M}^1(K)$, and let us show that $\overline{\mathcal{S}\mu}$ contains the point mass δ_z; it will follow from Lemma 3.2 that $(K, \mathcal{S})$ is strongly proximal. Let $\beta : \mathcal{M}^1(K) \to K$ be the barycenter mapping. The subset $\overline{\mathcal{S}\beta(\mu)}$ of K being nonempty closed and $\mathcal{S}$-stable, and M being the unique minimal subset of K, M is contained in $\overline{\mathcal{S}\beta(\mu)}$ and there exists a net (s_α) in $\mathcal{S}$ such that $(s_\alpha \beta(\mu))$ converges in K to z. Because $\mathcal{M}^1(K)$ is compact, we may suppose that $(s_\alpha \mu)$ converges in $\mathcal{M}^1(K)$, and then

$$\beta(\lim s_\alpha \mu) = \lim \beta(s_\alpha \mu) = \lim s_\alpha \beta(\mu) = z.$$

Since $z \in \operatorname{ext} K$, $\beta^{-1}(\{z\})$ is the singleton $\{\delta_z\}$ through a basic result by Bauer [9]; consequently, $\lim s_\alpha \mu = \delta_z$, which proves the theorem. ∎

Let us recall that a compact affine flow is called *irreducible* if it contains no nonempty proper closed convex stable set.

COROLLARY 3.5. *Every irreducible compact affine flow is strongly proximal.*

Proof. Let $(K, \mathcal{S})$ be an irreducible compact affine flow, and let z be an extreme point of K. Let M be a minimal subset of $(K, \mathcal{S})$. Since $(K, \mathcal{S})$ is irreducible, $\overline{\mathrm{co}}\, M = K$, and by Milman's theorem, $z \in M$; as a consequence, M is the unique minimal subset of $(K, \mathcal{S})$. By Theorem 3.4, $(K, \mathcal{S})$ is strongly proximal. ∎

COROLLARY 3.6. *Let $(X, \mathcal{S})$ be a strongly proximal compact flow. Then the compact affine flow $(\mathcal{M}^1(X), \mathcal{S})$ is strongly proximal.*

Proof. Let M be the unique minimal subset of $(X, \mathcal{S})$. The set δ_X of all extreme points of $\mathcal{M}^1(X)$ being nonempty closed and $\mathcal{S}$-stable, it contains the unique minimal subset δ_M of $(\mathcal{M}^1(X), \mathcal{S})$, and by Theorem 3.4, $(\mathcal{M}^1(X), \mathcal{S})$ is strongly proximal. ∎

Statement 3.8 below is both a consequence and a generalization of Corollary 3.6. In order to obtain it, let us recall the following property from [4] (with a simple proof).

LEMMA 3.7. *Let $(X,\mathcal{S})$ and $(Y,\mathcal{S})$ be two compact flows and let $\phi : (X,\mathcal{S}) \to (Y,\mathcal{S})$ be a homomorphism* (*i.e. ϕ is a continuous mapping of X into Y, and $\phi(sx) = s\phi(x)$ for all $s \in \mathcal{S}$ and $x \in X$*). *Suppose that $(X,\mathcal{S})$ is strongly proximal and ϕ is surjective. Then $(Y,\mathcal{S})$ is strongly proximal.*

Proof. Let $\nu \in \mathcal{M}^1(Y)$. Since $\phi(\mathcal{M}^1(X)) = \phi(\overline{\mathrm{co}}\, X) = \overline{\mathrm{co}}\, \phi(X) = \mathcal{M}^1(Y)$, there is $\mu \in \mathcal{M}^1(X)$ such that $\phi(\mu) = \nu$. By Lemma 3.2, there is $x \in X$ such that $\delta_x \in \overline{\mathcal{S}\mu}$, and since $\delta_{\phi(x)} = \phi(\delta_x) \in \phi(\overline{\mathcal{S}\mu}) = \overline{\mathcal{S}(\phi(\mu))} = \overline{\mathcal{S}\nu}$, we can see that $(Y,\mathcal{S})$ is strongly proximal. ∎

COROLLARY 3.8. *Let $(K,\mathcal{S})$ be a compact affine flow and let X be a compact $\mathcal{S}$-stable subset of K such that $\overline{\mathrm{co}}\, X = K$. If $(X,\mathcal{S})$ is strongly proximal, then $(K,\mathcal{S})$ is strongly proximal.*

Proof. $(\mathcal{M}^1(X),\mathcal{S})$ is strongly proximal by Corollary 3.6; therefore, since the barycenter mapping $\beta : (\mathcal{M}^1(X),\mathcal{S}) \to (K,\mathcal{S})$ is a surjective (affine) homomorphism, $(K,\mathcal{S})$ is also strongly proximal by Lemma 3.7. ∎

If in Theorem 3.4 it is not assumed that M contains an extreme point of K, then $(K,\mathcal{S})$ is not necessarily strongly proximal. Let us verify that $(K,\mathcal{S})$ remains proximal, which is probably known. We begin with a lemma.

LEMMA 3.9. *Let $(X,\mathcal{S})$ be a compact flow and let $(Y,\mathcal{S})$ be a compact subflow of $(X,\mathcal{S})$. Then the following conditions are equivalent.*

(1) *$(X,\mathcal{S})$ is proximal.*
(2) *$(Y,\mathcal{S})$ is proximal and $\overline{\mathcal{S}x} \cap Y \neq \emptyset$ for every $x \in X$.*

Proof. (1)⇒(2): First, since $(X,\mathcal{S})$ is proximal, the subflow $(Y,\mathcal{S})$ of $(X,\mathcal{S})$ is proximal. Next, the sets Y and $\overline{\mathcal{S}x}$ being nonempty, closed in X and $\mathcal{S}$-stable, they contain the unique minimal subset of X and consequently $\overline{\mathcal{S}x} \cap Y \neq \emptyset$ $(x \in X)$.

(2)⇒(1): Let $(x,y) \in X \times X$. By hypothesis, there is $p \in E(X,\mathcal{S})$ such that $p(x) \in Y$. In the same way, there is $q \in E(X,\mathcal{S})$ such that $q(py) \in Y$. Since $q(px)$ and $q(py)$ belong to Y, there is $r \in E(X,\mathcal{S})$ such that $r(q(px)) = r(q(py))$. Since $rqp \in E(X,\mathcal{S})$, the pair (x,y) is proximal. ∎

PROPOSITION 3.10. *Let $(K,\mathcal{S})$ be a compact affine flow. Then $(K,\mathcal{S})$ is proximal if and only if K has a unique minimal subset under $\mathcal{S}$.*

Proof. If $(K,\mathcal{S})$ is proximal, then $(K,\mathcal{S})$ has a unique minimal subset. Let us suppose that $(K,\mathcal{S})$ has a unique minimal subset M, and let us show that $(M,\mathcal{S})$ is proximal; since $\overline{\mathcal{S}x} \cap M = M \neq \emptyset$ for every $x \in K$, this will imply, by Lemma 3.9, that $(K,\mathcal{S})$ is proximal. Let $(x,y) \in M \times M$; the subset

$\overline{\mathrm{co}}\,\mathcal{S}(\frac{1}{2}x + \frac{1}{2}y)$ of K being nonempty closed and $\mathcal{S}$-stable, and M being the unique minimal subset of K, M is contained in $\overline{\mathrm{co}}\,\mathcal{S}(\frac{1}{2}x+\frac{1}{2}y)$; consequently, $x, y \in \overline{\mathrm{co}}\,\mathcal{S}(\frac{1}{2}x+\frac{1}{2}y)$. Let z be an extreme point of $\overline{\mathrm{co}}\,\mathcal{S}(\frac{1}{2}x+\frac{1}{2}y)$; by Milman's theorem, $z \in \overline{\mathcal{S}}(\frac{1}{2}x + \frac{1}{2}y)$. Let (s_α) be a net in $\mathcal{S}$ such that $(s_\alpha(\frac{1}{2}x + \frac{1}{2}y))$ converges to z. By compactness, we may assume that the nets $(s_\alpha x)$ and $(s_\alpha y)$ converge, so that

$$z = \lim s_\alpha \left(\frac{1}{2}\,x + \frac{1}{2}\,y\right) = \frac{1}{2}\lim s_\alpha x + \frac{1}{2}\lim s_\alpha y,$$

and since z is an extreme point of $\overline{\mathrm{co}}\,\mathcal{S}(\frac{1}{2}x + \frac{1}{2}y)$, $\lim s_\alpha x = \lim s_\alpha y = z$. Hence the flow $(M, \mathcal{S})$ is proximal. ∎

4. Local versus global strong proximality. The main result of this section is Theorem 4.5. It reduces questions of strong proximality to proximality and a local strong proximality condition. Obviously, in order to describe such a phenomenon, a "local" concept of strong proximality is needed. Bearing in mind that if $(X, \mathcal{S})$ is a compact flow, then the $\mathcal{S}$-proximality of a set $A \subset X$ means that $\overline{\mathcal{S}\mu} \cap \delta_X \neq \emptyset$ for every $\mu \in \mathcal{M}^1(X)$ supported by a finite subset of A [1], and that the $\mathcal{S}$-strong proximality of $(X, \mathcal{S})$ means that $\overline{\mathcal{S}\mu} \cap \delta_X \neq \emptyset$ for every $\mu \in \mathcal{M}^1(X)$, we naturally lay down the following definition.

Definition 4.1. Let $(X, \mathcal{S})$ be a compact flow.

(1) We shall say that a closed subset A of X is *$\mathcal{S}$-strongly proximal* if $\overline{\mathcal{S}\mu} \cap \delta_X \neq \emptyset$ for every $\mu \in \mathcal{M}^1_A(X)$, with $\mathcal{M}^1_A(X) = \{\mu \in \mathcal{M}^1(X) \mid \mu(A) = 1\}$.

(2) For convenience's sake, we shall say that a subset A of X is *$\mathcal{S}$-strongly proximal* if every compact subset of A is $\mathcal{S}$-strongly proximal. Obviously, if A is $\mathcal{S}$-strongly proximal, then so is any subset of A.

Lemma 4.2 (which is closely related to the above Lemma 3.3) usefully completes Definition 4.1(1). It will be involved in the proof of Theorem 4.5.

Lemma 4.2. *Let $(X, \mathcal{S})$ be a compact flow and let A be a closed subset of X. Set $N = \{x \in X \mid \forall \mu \in \mathcal{M}^1_A(X),\ \delta_x \in \overline{\mathcal{S}\mu}\}$. Then:*

(1) *N is a closed and $\mathcal{S}$-stable subset of X.*

(2) *A is $\mathcal{S}$-strongly proximal if and only if N is nonempty.*

Proof. (1) $N = \bigcap\{\overline{\mathcal{S}\mu} \cap X \mid \mu \in \mathcal{M}^1_A(X)\}$, and $\overline{\mathcal{S}\mu} \cap X$ is a closed and $\mathcal{S}$-stable subset of X for all $\mu \in \mathcal{M}^1_A(X)$; therefore N is a closed and $\mathcal{S}$-stable subset of X.

(2) Since $N \subset \overline{\mathcal{S}\mu} \cap X$ $(\mu \in \mathcal{M}^1_A(X))$, if N is nonempty then A is $\mathcal{S}$-strongly proximal. Conversely, let us suppose that A is $\mathcal{S}$-strongly proximal. If $\mu_1, \ldots, \mu_n \in \mathcal{M}^1_A(X)$, then $n^{-1}\sum_{i=1}^n \mu_i \in \mathcal{M}^1_A(X)$ and consequently

$\overline{\mathcal{S}(n^{-1}\sum_{i=1}^{n}\mu_i)}\cap X\neq\emptyset$. Let (s_α) be a net in $\mathcal{S}$ such that $(s_\alpha(n^{-1}\sum_{i=1}^{n}\mu_i))$ converges in $\mathcal{M}^1(X)$ to a point $x\in X$. By compactness, we may assume that the nets $(s_\alpha\mu_i)$ $(i=1,\ldots,n)$ converge in $\mathcal{M}^1(X)$, so that

$$x=\lim s_\alpha\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)=\frac{1}{n}\sum_{i=1}^{n}\lim s_\alpha\mu_i,$$

and since x is an extreme point of $\mathcal{M}^1(X)$, $\lim s_\alpha\mu_i=x$ $(i=1,\ldots,n)$. It follows that the collection $\{\overline{\mathcal{S}\mu}\cap X\mid\mu\in\mathcal{M}^1_A(X)\}$ of closed subsets of X has the finite intersection property; X being compact, its intersection N is nonempty. ∎

The proof of Theorem 4.5 essentially rests on the following two lemmas. (The second of these will be used again in the proof of Lemma 5.3.)

LEMMA 4.3. *Let $(X,\mathcal{S})$ be a proximal compact flow. Let A be an $\mathcal{S}$-strongly proximal closed subset of X and $x\in X$. Then $A\cup\{x\}$ is $\mathcal{S}$-strongly proximal.*

Proof. Let $\mu\in\mathcal{M}^1(X)$ be such that $\mu(A\cup\{x\})=1$. Let $\alpha\in[0,1]$ and $\nu\in\mathcal{M}^1_A(X)$ be such that $\mu=\alpha\nu+(1-\alpha)\delta_x$. As A is $\mathcal{S}$-strongly proximal, there is $p\in E(\mathcal{M}^1(X),\mathcal{S})$ such that $p\nu\in\delta_X$, and as $(X,\mathcal{S})$ is proximal, there is $q\in E(\mathcal{M}^1(X),\mathcal{S})$ such that $q(p\nu)=q(p\delta_x)$. Let $r=qp$; then $r\in E(\mathcal{M}^1(X),\mathcal{S})$ and $r\nu=r\delta_x$. The mapping r being affine, $r\mu=\alpha r\nu+(1-\alpha)r\delta_x=r\delta_x$. The conditions $r\mu\in\overline{\mathcal{S}\mu}$, $r\delta_x\in\delta_X$ and $r\mu=r\delta_x$ imply $\overline{\mathcal{S}\mu}\cap\delta_X\neq\emptyset$; consequently, $A\cup\{x\}$ is $\mathcal{S}$-strongly proximal. ∎

LEMMA 4.4. *Let $(X,\mathcal{S})$ be a compact flow and let $\mu\in\mathcal{M}^1(X)$. Let A be an $\mathcal{S}$-strongly proximal closed subset of X and let $N=\{x\in X\mid\forall\nu\in\mathcal{M}^1_A(X),\ \delta_x\in\overline{\mathcal{S}\nu}\}$. Then $\mu(A)\le\sup\{\nu(\{x\})\mid\nu\in\overline{\mathcal{S}\mu}\}$ for every $x\in N$.*

Proof. Let $x\in N$. Let us suppose that $\mu(A)>0$ and define $\mu'\in\mathcal{M}^1(X)$ by $\mu'(B)=\mu(A\cap B)/\mu(A)$ for every $B\in\mathcal{B}(X)$. Since $\mu'(A)=1$, we have $\delta_x\in\overline{\mathcal{S}\mu'}$. Let (s_α) be a net in $\mathcal{S}$ such that $\lim s_\alpha\mu'=\delta_x$, and let θ be a cluster point of the net $(s_\alpha\mu)$ in $\overline{\mathcal{S}\mu}$. For every open subset U of X containing x and every $\varepsilon>0$, there is α_0 such that $(s_\alpha\mu')(U)>1-\varepsilon$ for $\alpha\ge\alpha_0$; that implies $(s_\alpha\mu)(U)>(1-\varepsilon)\mu(A)$ for $\alpha\ge\alpha_0$, and since the mapping $\nu\mapsto\nu(\overline{U})$ of $\overline{\mathcal{S}\mu}$ into $\mathbb{R}$ is upper semicontinuous [12], $\theta(\overline{U})\ge(1-\varepsilon)\mu(A)$. The regularity of θ implies that $\theta(\{x\})\ge(1-\varepsilon)\mu(A)$, and as this holds for every $\varepsilon>0$, we obtain $\mu(A)\le\theta(\{x\})\le\sup\{\nu(\{x\})\mid\nu\in\overline{\mathcal{S}\mu}\}$. ∎

THEOREM 4.5. *A compact flow $(X,\mathcal{S})$ is strongly proximal if and only if it is proximal and every point of X has an $\mathcal{S}$-strongly proximal neighborhood in X.*

Proof. Of course, if $(X,\mathcal{S})$ is strongly proximal, then $(X,\mathcal{S})$ is proximal and every point of X has an $\mathcal{S}$-strongly proximal neighborhood. Let us prove the converse. For every $y \in X$, let us choose an $\mathcal{S}$-strongly proximal open neighborhood U_y of y in X. Let M be the unique minimal subset of $(X, \mathcal{S})$ and let $x \in M$. Let $\mu \in \mathcal{M}^1(X)$. The subset $\{x\}$ of X being closed, the mapping $\nu' \mapsto \nu'(\{x\})$ of $\overline{\mathcal{S}\mu}$ into $\mathbb{R}$ is upper semicontinuous; consequently, since $\overline{\mathcal{S}\mu}$ is a compact space, there exists $\nu \in \overline{\mathcal{S}\mu}$ such that $\nu(\{x\}) = \sup\{\nu'(\{x\}) \mid \nu' \in \overline{\mathcal{S}\mu}\}$.

Let us show that $\delta_x = \nu$, which will imply $\delta_x \in \overline{\mathcal{S}\mu}$ and prove the theorem. Let us suppose that this is not the case; then $\nu(X \setminus \{x\}) > 0$ and since the open covering $\{U_y \mid y \in X\}$ of the compact X has a finite subcover, we can choose $z \in X$ such that $\nu(U_z \setminus \{x\}) > 0$. Let K be a compact subset of $U_z \setminus \{x\}$ such that $\nu(K) > 0$; then K is $\mathcal{S}$-strongly proximal and by Lemma 4.3, so is $K \cup \{x\}$. By Lemmas 4.2 and 4.4, $\nu(K \cup \{x\}) \leq \sup\{\nu'(\{x\}) \mid \nu' \in \overline{\mathcal{S}\nu}\}$; moreover, $\sup\{\nu'(\{x\}) \mid \nu' \in \overline{\mathcal{S}\nu}\} \leq \nu(\{x\})$ because $\overline{\mathcal{S}\nu} \subset \overline{\mathcal{S}\mu}$; therefore $\nu(K \cup \{x\}) \leq \nu(\{x\})$, which contradicts $\nu(K \cup \{x\}) = \nu(K) + \nu(\{x\}) > \nu(\{x\})$. ■

REMARK 4.6. (1) Let $(X, \mathcal{S})$ be a compact flow with a unique minimal subset M. If there exists in M a point x which has an $\mathcal{S}$-strongly proximal neighborhood V in X, then every point of X has an $\mathcal{S}$-strongly proximal neighborhood in X. Indeed, let $y \in X$ and choose $s \in \mathcal{S}$ such that sy belongs to the interior of V in X; then $s^{-1}(V)$ is an $\mathcal{S}$-strongly proximal neighborhood of y in X. (Let K be a compact subset of $s^{-1}(V)$ and let $\mu \in \mathcal{M}^1_K(X)$. The mapping s being continuous, $s(K)$ is a compact subset of V. Since $(s\mu)(s(K)) = 1$ and $s(K)$ is $\mathcal{S}$-strongly proximal, $\delta_y \in \overline{\mathcal{S}(s\mu)}$ for some $y \in X$; consequently, as $\overline{\mathcal{S}(s\mu)} = \overline{(\mathcal{S}s)\mu} \subset \overline{\mathcal{S}\mu}$, K is $\mathcal{S}$-strongly proximal.)

(2) Let $(X, \mathcal{S})$ be a compact flow. A subset A of X is said to be $\mathcal{S}$-*contractible* to a point $x \in X$ if for every neighborhood V of x in X there is $s \in \mathcal{S}$ such that $s(A) \subset V$. If A is $\mathcal{S}$-contractible to at least one point of X, then A is said to be $\mathcal{S}$-contractible. It has been proved by Margulis [8] that if $(X, \mathcal{S})$ is proximal and if every point of X has an $\mathcal{S}$-contractible neighborhood in X, then $(X, \mathcal{S})$ is strongly proximal. (In [8], $(X, \mathcal{S})$ is assumed to be a compact metrizable transformation group, but with minor changes, the proof works for any semigroup $\mathcal{S}$ and any compact X.) Lemma 5.1 below shows, in an obvious way, that any $\mathcal{S}$-contractible subset of X is $\mathcal{S}$-strongly proximal; consequently, Margulis's result follows from Theorem 4.5.

5. Strong proximality without measures. In view of Theorem 4.5, looking for a criterion for strong proximality of subsets of a given compact flow $(X, \mathcal{S})$ naturally comes to mind. The above Definition 4.1 involves the members of $\mathcal{M}^1(X)$; in this section, a criterion for $\mathcal{S}$-strong proximality of closed subsets of X is given without any reference to measures (Proposition

5.2). The special case of Proposition 5.2 when $(X,\mathcal{S})$ is a minimal compact transformation group and $A_0 = A = X$ is due to Glasner [3] (cf. also [5]).

LEMMA 5.1. *Let $(X,\mathcal{S})$ be a compact flow. Let A be a closed subset of X, let A_0 be a dense subset of A and let $x \in X$. Then the following conditions are equivalent.*

(1) $\delta_x \in \overline{\mathcal{S}\mu}$ *for all* $\mu \in \mathcal{M}^1_A(X)$.
(2) *For every neighborhood V of δ_x in $\mathcal{M}^1(X)$, there exists a finite subset $\mathcal{F}$ of $\mathcal{S}$ such that* $\{\mu \in \mathcal{M}^1_A(X) \mid \exists s \in \mathcal{F}, s\mu \in V\} = \mathcal{M}^1_A(X)$.
(3) *For every neighborhood V of δ_x in $\mathcal{M}^1(X)$, there exists a finite subset $\mathcal{F}$ of $\mathcal{S}$ such that $\{\mu \in \mathcal{M}^1_A(X) \mid \exists s \in \mathcal{F},\ s\mu \in V\}$ is dense in $\mathcal{M}^1_A(X)$.*
(4) *For every neighborhood V of x in X and every $\varepsilon > 0$, there is a finite subset $\mathcal{F}$ of $\mathcal{S}$ such that if $(x_1,\dots,x_n)$ is any finite sequence of points in A_0, then for some $s \in \mathcal{F}$ all sx_i are in V except for at most $[n\varepsilon]$ indices $i \in \{1,\dots,n\}$.*

Proof. (1)⇒(2): Let $\mathcal{V}(\delta_x)$ denote the set of all neighborhoods of δ_x in $\mathcal{M}^1(X)$. Let $V \in \mathcal{V}(\delta_x)$ and let U be its interior in $\mathcal{M}^1(X)$. For every $s \in \mathcal{S}$, the subset Ω_s of $\mathcal{M}^1_A(X)$ defined by $\Omega_s = \{\mu \in \mathcal{M}^1_A(X) \mid s\mu \in U\}$ is open [12], and by (1), $\bigcup_{s\in\mathcal{S}} \Omega_s = \mathcal{M}^1_A(X)$. Since the space $\mathcal{M}^1_A(X)$ is compact, there exists a finite subset $\mathcal{F}$ of $\mathcal{S}$ such that $\bigcup_{s\in\mathcal{F}} \Omega_s = \mathcal{M}^1_A(X)$, and since $U \subset V$, $\{\mu \in \mathcal{M}^1_A(X) \mid \exists s \in \mathcal{F},\ s\mu \in V\}$ is equal to $\mathcal{M}^1_A(X)$.

(2)⇒(3) is obvious.

(3)⇒(1): Let $V \in \mathcal{V}(\delta_x)$. Let us choose a finite subset $\mathcal{F}$ of $\mathcal{S}$ such that the set $B = \{\mu \in \mathcal{M}^1_A(X) \mid \exists s \in \mathcal{F},\ s\mu \in V\}$ is dense in $\mathcal{M}^1_A(X)$. The inclusion $B \subset \bigcup_{s\in\mathcal{F}} s^{-1}(V)$ implies

$$\mathcal{M}^1_A(X) = \overline{B} \subset \overline{\bigcup_{s\in\mathcal{F}} s^{-1}(V)} = \bigcup_{s\in\mathcal{F}} \overline{s^{-1}(V)}.$$

If $\mu \in \mathcal{M}^1_A(X)$, then there is $s \in \mathcal{F}$ such that $\mu \in \overline{s^{-1}(V)}$, and since s is continuous,

$$s\mu \in s(\overline{s^{-1}(V)}) \subset \overline{s(s^{-1}(V))} \subset \overline{V};$$

consequently, $\overline{\mathcal{F}\mu} \cap \overline{V} \neq \emptyset$. It follows that for each fixed μ in $\mathcal{M}^1_A(X)$, the collection $\{\overline{\mathcal{S}\mu} \cap \overline{V} \mid V \in \mathcal{V}(\delta_x)\}$ of closed subsets of $\mathcal{M}^1(X)$ has the finite intersection property. The space $\mathcal{M}^1(X)$ being compact, the intersection of this collection is nonempty, and since $\bigcap_{V\in\mathcal{V}(\delta_x)} \overline{V} = \{\delta_x\}$, we conclude that $\delta_x \in \overline{\mathcal{S}\mu}$.

(3)⇔(4): For every open subset G of X and for every $\varepsilon > 0$, let $\Omega_{G,\varepsilon} = \{\mu \in \mathcal{M}^1(X) \mid \mu(G) > 1-\varepsilon\}$. It is well known that the collection of all those $\Omega_{G,\varepsilon}$ which contain δ_x is a fundamental system of neighborhoods of x in $\mathcal{M}^1(X)$ [12]. Consequently, the equivalence between (3) and (4) is established if we observe that $\{n^{-1}\sum_{i=1}^{n}\delta_{x_i} \mid x_1,\ldots,x_n \in A_0, n\geq 1\}$ is dense in $\mathcal{M}^1_A(X)$, and that for every $x_1,\ldots,x_n\in A_0$ and every $s\in\mathcal{S}$, $s(n^{-1}\sum_{i=1}^{n}\delta_{x_i})$ belongs to $\Omega_{G,\varepsilon}$ if and only if all sx_i are in G except for at most $[n\varepsilon]$ indices $i\in\{1,\ldots,n\}$. ∎

Proposition 5.2. *Let $(X,\mathcal{S})$ be a compact flow which has a unique minimal subset M, and let $x\in M$. Let A be a closed subset of X, and let A_0 be a dense subset of A. Then the following conditions are equivalent.*

(1) *A is an $\mathcal{S}$-strongly proximal subset of X.*
(2) *$\delta_x\in\overline{\mathcal{S}\mu}$ for every $\mu\in\mathcal{M}^1_A(X)$.*
(3) *For every neighborhood V of x in X and every $\varepsilon>0$, there is a finite subset $\mathcal{F}$ of $\mathcal{S}$ such that if $(x_1,\ldots,x_n)$ is any finite sequence of points in A_0, then for some $s\in\mathcal{F}$ all sx_i are in V except for at most $[n\varepsilon]$ indices $i\in\{1,\ldots,n\}$.*

Proof. Let $N=\{y\in X\mid \forall\mu\in\mathcal{M}^1_A(X),\ \delta_y\in\overline{\mathcal{S}\mu}\}$. By Lemma 4.2, A is $\mathcal{S}$-strongly proximal if and only if $N\neq\emptyset$. The closed and $\mathcal{S}$-stable subset N of X being nonempty if and only if $M\subset N$, and M being contained in N if and only if $x\in N$, conditions (1) and (2) are equivalent. By Lemma 5.1, (2)⇔(3). ∎

To see whether a subset A of a compact flow $(X,\mathcal{S})$ is $\mathcal{S}$-strongly proximal, we have to check whether the compact subsets of A are (by Definition 4.1(2)). Proposition 5.4 below provides us with a description of the $\mathcal{S}$-strong proximality of arbitrary subsets of X in the spirit of the definition given in 4.1(1) for closed subsets.

Lemma 5.3. *Let $(X,\mathcal{S})$ be a compact flow and let $\mu\in\mathcal{M}^1(X)$. Suppose that there exists a collection $\mathcal{K}$ of $\mathcal{S}$-strongly proximal closed subsets of X which is stable under finite unions and satisfies* $\sup\{\mu(A)\mid A\in\mathcal{K}\}=1$. *Then $\overline{\mathcal{S}\mu}\cap\delta_X\neq\emptyset$.*

Proof. For every $A\in\mathcal{K}$, let $N_A=\{x\in X\mid\forall\nu\in\mathcal{M}^1_A(X),\ \delta_x\in\overline{\mathcal{S}\nu}\}$. As $\mathcal{K}$ is stable under finite unions, Lemma 4.2 shows that the collection $\{N_A\mid A\in\mathcal{K}\}$ of closed subsets of X has the finite intersection property; therefore, X being compact, $\bigcap_{A\in\mathcal{K}}N_A$ is nonempty and we can choose a point x in this intersection. As in the proof of Theorem 4.5, there is $\nu\in\overline{\mathcal{S}\mu}$ such that $\nu(\{x\})=\sup\{\nu'(\{x\})\mid\nu'\in\overline{\mathcal{S}\mu}\}$. By Lemma 4.4, $\sup\{\nu'(\{x\})\mid \nu'\in\overline{\mathcal{S}\mu}\}=1$; consequently, $\nu=\delta_x$ and since $\nu\in\overline{\mathcal{S}\mu}$, δ_x belongs to $\overline{\mathcal{S}\mu}\cap\delta_X$. ∎

Proposition 5.4. *Let $(X,\mathcal{S})$ be a compact flow and let A be any subset of X. Then the following conditions are equivalent.*

(1) *A is an $\mathcal{S}$-strongly proximal subset of X.*
(2) *$\overline{\mathcal{S}\mu} \cap \delta_X \neq \emptyset$ for every $\mu \in \mathcal{M}^1(X)$ such that $\mu_*(A) = 1$ ($\mu_*(A)$ being the inner measure of A relative to μ).*

Proof. (1)⇒(2): Let $\mathcal{B}(X)$ be the Borel σ-algebra of X; by definition, $\mu_*(A) = \sup\{\mu(B) \mid A \supset B \in \mathcal{B}(X)\}$ for every $\mu \in \mathcal{M}^1(X)$. Let $\mathcal{K}$ be the collection of all compact subsets of A. Then $\mathcal{K}$ is stable under finite unions, and $\mu_*(A) = \sup\{\mu(K) \mid K \in \mathcal{K}\}$ (by regularity of μ). Consequently, by Lemma 5.3, if $\mu_*(A) = 1$ and if A is $\mathcal{S}$-strongly proximal (i.e. if any $K \in \mathcal{K}$ is $\mathcal{S}$-strongly proximal), then $\overline{\mathcal{S}\mu} \cap \delta_X \neq \emptyset$.

(2)⇒(1): Let K be any compact subset of A and let $\mu \in \mathcal{M}^1(X)$ be such that $\mu(K) = 1$. Since $\mu_*(A) = 1$, $\overline{\mathcal{S}\mu} \cap \delta_X \neq \emptyset$ by hypothesis. Consequently, K is $\mathcal{S}$-strongly proximal, and by definition, so is A. ■

As noted by Margulis [8], if $(M, \mathcal{G})$ is a proximal compact metrizable transformation group and if μ is a discrete Borel probability on M, then $\overline{\mathcal{G}\mu} \cap \delta_M \neq \emptyset$. Our concluding remark is closely related to that of Margulis.

REMARK 5.5. A topological space is said to be *scattered* if each of its nonempty subsets contains an isolated point. It is called *σ-scattered* if it is a countable union of scattered subspaces. If $(X, \mathcal{S})$ is a proximal compact flow and if A is a σ-scattered subspace of X, then A is $\mathcal{S}$-strongly proximal. Indeed, if A is finite, this follows from Lemma 4.3; if A is countable, it follows from the finite case and from Lemma 5.3; if A is compact and scattered, it follows from the countable case and from the fact that every $\mu \in \mathcal{M}^1(A)$ is discrete (i.e., there is a countable subset of A with full measure) [11]; finally, the general case follows from the fact, proved in [7], that every compact subspace of a σ-scattered space is scattered.

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