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AN EXTENSION OF DISTRIBUTIONAL WAVELET TRANSFORM

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Abstract. We construct a new Boehmian space containing the space $\tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+)$ and define the extended wavelet transform $\mathscr{W}$ of a new Boehmian as a tempered Boehmian. In analogy to the distributional wavelet transform, it is proved that the extended wavelet transform is linear, one-to-one, and continuous with respect to δ-convergence as well as Δ-convergence.

1. Introduction. Let $\mathbb{N}, \mathbb{N}_0, \mathbb{R}, \mathbb{R}_+$ denote the natural numbers, non-negative integers, real numbers and positive real numbers respectively. For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ and $\mathbf{b} \in \mathbb{R}^n$, let $|\beta| = \sum_{i=1}^n \beta_i$ and $\|\mathbf{b}\|$ be the Euclidean norm of $\mathbf{b}$. Let $\tilde{\mathscr{S}}(\mathbb{R}^n \times \mathbb{R}_+) = \{g \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+) : Q_{l,m,k,p}(g) < \infty\}$, where

$$Q_{l,m,k,p}(g) = \sup_{|\beta| \le p} \sup_{(\mathbf{b},a) \in \mathbb{R}^n \times \mathbb{R}_+} (1 + \|\mathbf{b}\|^2)^m \left| a^l \left(\frac{\partial}{\partial a}\right)^k D_{\mathbf{b}}^\beta g \right|$$

for $l, m, k, p \in \mathbb{N}_0$ with $l + m \le k + p$. We denote by $\mathscr{D}(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n)$ the space of infinitely differentiable functions with compact support and the space of rapidly decreasing functions respectively. We endow $\mathscr{S}'(\mathbb{R}^n)$ and $\tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+)$ with the weak* topology.

For a given wavelet $\psi \in \mathbb{R}^n$, the *wavelet transform* $W : \mathscr{S}(\mathbb{R}^n) \to \tilde{\mathscr{S}}(\mathbb{R}^n \times \mathbb{R}_+)$ is defined by

$$(1) \qquad (Wg)(\mathbf{b}, a) = \int_{\mathbb{R}^n} g(\mathbf{t})\, \overline{\psi\left(\frac{\mathbf{t} - \mathbf{b}}{a}\right)}\, \frac{d\mathbf{t}}{a^n} \qquad (\mathbf{b}, a) \in \mathbb{R}^n \times \mathbb{R}_+,$$

and the *distributional wavelet transform* $W' : \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+) \to \mathscr{S}'(\mathbb{R}^n)$ is defined by

$$(2) \qquad (W'G)(f) = G(Wf), \qquad f \in \mathscr{S}(\mathbb{R}^n).$$

For more details, we refer to [14].

On the other hand, Boehmians were introduced by J. Mikusiński and P. Mikusiński [5] with two notions of convergence [6]. Thereafter various

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Boehmian spaces have been constructed to extend various integral transforms. See [3, 4, 7, 11, 12, 15–18].

In [15], a wavelet transform on periodic Boehmians is discussed which extends the wavelet transform on periodic distributions [2]. According to [1], the wavelet transform is the same as the windowed Fourier transform with respect to the window function g, defined by

$$G[f](\nu,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)\bar{g}(\tau-t)e^{i\nu t}\,d\tau, \tag{3}$$

and extends to the space of integrable Boehmians. This transform is different from the usual wavelet transform (see equations (1) and (3)). In the present work, we extend the wavelet transform on $\tilde{\mathscr{S}}'(\mathbb{R}^n\times\mathbb{R}_+)$ [14] to a suitable Boehmian space. Thus the present work and the above-mentioned two works on wavelet transform [15, 1] are different.

2. Auxiliary results

Definition 2.1. Suppose $G\in\tilde{\mathscr{S}}'(\mathbb{R}^n\times\mathbb{R}_+)$ and $\phi\in\mathscr{D}(\mathbb{R}^n)$. We define

$$(G\otimes\phi)(g)=G(g\times\check{\phi}),\qquad g\in\tilde{\mathscr{S}}(\mathbb{R}^n\times\mathbb{R}_+),$$

where

$$(g\times\phi)(\mathbf{b},a)=\int_{\mathbb{R}^n} g(\mathbf{b}-\mathbf{x},a)\phi(\mathbf{x})\,d\mathbf{x},\qquad (\mathbf{b},a)\in\mathbb{R}^n\times\mathbb{R}_+,$$

and $\check{\phi}(\mathbf{x})=\phi(-\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^n$.

Lemma 2.2. *If $g\in\tilde{\mathscr{S}}(\mathbb{R}^n\times\mathbb{R}_+)$ and $\phi\in\mathscr{D}(\mathbb{R}^n)$ then $Q_{l,m,k,p}(g\times\phi)\le CQ_{l,m,k,p}(g)$ for some $C>0$, and hence $g\times\phi\in\tilde{\mathscr{S}}(\mathbb{R}^n\times\mathbb{R}_+)$.*

Proof. In preparation for the main part of the proof, let us first show that $D^\gamma_{\mathbf{b}}(g\times\phi)=(D^\gamma_{\mathbf{b}}g)\times\phi$, where $\gamma=(\gamma_1,\dots,\gamma_i)$ with $\gamma_k=1$ for some $k\in\{1,\dots,n\}$ and $\gamma_i=0$ for $i\ne k$. Let $h\in\mathbb{R}$, $\mathbf{u}_k$ be the unit vector along the x_k-axis and $A>0$ be such that $\operatorname{supp}\phi\subset\{\mathbf{x}\in\mathbb{R}^n:\|\mathbf{x}\|\le A\}$. Using the mean value theorem we get

$$\begin{aligned}\left|\frac{(g\times\phi)(\mathbf{b}+h\mathbf{u}_k,a)-(g\times\phi)(\mathbf{b},a)}{h}-(D^\gamma_{\mathbf{b}}g)(\mathbf{b},a)\right|&\\ \le\int_{\mathbb{R}^n}\left|\frac{g(\mathbf{b}+h\mathbf{u}_k,a)-g(\mathbf{b},a)}{h}-(D^\gamma_{\mathbf{b}}g)(\mathbf{b},a)\right||\phi(\mathbf{x})|\,d\mathbf{x}&\\ \le\int_{\mathbb{R}^n}[|(D^\gamma_{\mathbf{b}}g)(\mathbf{b}+h'\mathbf{u}_k,a)|+|(D^\gamma_{\mathbf{b}}g)(\mathbf{b},a)|]|\phi(\mathbf{x})|\,d\mathbf{x}&\end{aligned}$$

where h' lies between 0 and h. Since the integrand is dominated by the integrable function $2Q_{0,0,0,1}(g)|\phi|$, we can apply the dominated convergence theorem to get

$$\lim_{h\to 0}\int_{\mathbb{R}^n} |(D^\gamma_{\mathbf{b}} g)(\mathbf{b}+h'\mathbf{u}_k,a)-(D^\gamma_{\mathbf{b}} g)(\mathbf{b},a)|\,|\phi(\mathbf{x})|\,d\mathbf{x}$$
$$\le \int_{\mathbb{R}^n} \lim_{h\to 0} |(D^\gamma_{\mathbf{b}} g)(\mathbf{b}+h'\mathbf{u}_k,a)-(D^\gamma_{\mathbf{b}} g)(\mathbf{b},a)|\,|\phi(\mathbf{x})|\,d\mathbf{x}$$
$$= 0 \quad (\text{since } D^\gamma_{\mathbf{b}} g \text{ is continuous}).$$

Similarly we can show that $(\partial/\partial a)(g\times\phi) = ((\partial/\partial a)g)\times\phi$. This proves that $g\times\phi$ is infinitely differentiable and $(\partial/\partial a)^k D^\alpha_{\mathbf{b}}(g\times\phi) = ((\partial/\partial a)^k D^\alpha_{\mathbf{b}} g)\times\phi$ for all $k\in\mathbb{N}$ and $\alpha\in\mathbb{N}_0$. Next for $l,m,k,p\in\mathbb{N}_0$, $\beta\in\mathbb{N}_0^n$ such that $|\beta|\le p$ and $(\mathbf{b},a)\in\mathbb{R}^n\times\mathbb{R}_+$,

$$(1+\|\mathbf{b}\|)^m \left| a^l \left(\frac{\partial}{\partial a}\right)^k D^\beta_{\mathbf{b}}(g\times\phi)(\mathbf{b},a)\right|$$
$$= (1+\|\mathbf{b}\|)^m \left| a^l \int_{\mathbb{R}^n} \left(\left(\frac{\partial}{\partial a}\right)^k D^\beta_{\mathbf{b}} g\right)(\mathbf{b}-\mathbf{x},a)\phi(\mathbf{x})\,d\mathbf{x}\right|$$
$$\le \int_{\|\mathbf{x}\|\le A} (1+(\|\mathbf{b}-\mathbf{x}\|+\|\mathbf{x}\|)^2)^m \left| a^l \left(\left(\frac{\partial}{\partial a}\right)^k D^\beta_{\mathbf{b}} g\right)(\mathbf{b}-\mathbf{x},a)\phi(\mathbf{x})\right| d\mathbf{x}$$
$$\le \int_{\|\mathbf{x}\|\le A} (1+(\|\mathbf{b}-\mathbf{x}\|+A)^2)^m \left| a^l \left(\left(\frac{\partial}{\partial a}\right)^k D^\beta_{\mathbf{b}} g\right)(\mathbf{b}-\mathbf{x},a)\phi(\mathbf{x})\right| d\mathbf{x}$$
$$\le C_1 Q_{l,m,k,p}(g)\int_{\mathbb{R}^n} |\phi(\mathbf{x})|\,d\mathbf{x} \quad \text{for some } C_1>0.$$

By taking $C = C_1\int_{\mathbb{R}^n}|\phi(\mathbf{x})|\,d\mathbf{x}$, we conclude the proof. ∎

LEMMA 2.3. *If $G\in\tilde{\mathscr{S}}'(\mathbb{R}^n\times\mathbb{R}_+)$ and $\phi\in\mathscr{D}(\mathbb{R}^n)$ then $G\otimes\phi\in\tilde{\mathscr{S}}'(\mathbb{R}^n\times\mathbb{R}_+)$.*

Proof. Lemma 2.2 shows that if $g\in\tilde{\mathscr{S}}(\mathbb{R}^n\times\mathbb{R}_+)$, then $G(g\times\check{\phi})$ is meaningful. Moreover, $G\in\tilde{\mathscr{S}}'(\mathbb{R}^n\times\mathbb{R}_+)$ implies that there exist $K>0$ and $l,m,k,p\in\mathbb{N}_0$ such that $l+m\le k+p$ and

$$|G(g\times\check{\phi})| \le KQ_{l,m,k,p}(g\times\check{\phi}) \le KCQ_{l,m,k,p}(g), \quad \forall g\in\tilde{\mathscr{S}}(\mathbb{R}^n\times\mathbb{R}_+).$$

Thus $G\otimes\phi\in\tilde{\mathscr{S}}'(\mathbb{R}^n\times\mathbb{R}_+)$. ∎

LEMMA 2.4. *If $G_1,G_2\in\tilde{\mathscr{S}}'(\mathbb{R}^n\times\mathbb{R}_+)$, $\phi\in\mathscr{D}(\mathbb{R}^n)$ and $\alpha\in\mathbb{C}$, then*

(1) $(G_1+G_2)\otimes\phi = G_1\otimes\phi + G_2\otimes\phi$.
(2) $(\alpha G_1)\otimes\phi = \alpha(G_1\otimes\phi)$.

This follows from the linearity of the integral.

Theorem 2.5 (Convolution theorems). *If* $G \in \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+)$, $f \in \mathscr{S}(\mathbb{R}^n)$ *and* $\phi \in \mathscr{D}(\mathbb{R}^n)$ *then*

(1) $W(f * \phi) = (Wf) \times \phi$,
(2) $W'(G \otimes \phi) = (W'G) * \phi$.

Proof. (1) For $(\mathbf{b}, a) \in \mathbb{R}^n \times \mathbb{R}_+$,

$$W(f * \phi)(\mathbf{b}, a) = \int_{\mathbb{R}^n} (f * \phi)(\mathbf{t})\, \overline{\psi\left(\frac{\mathbf{t} - b}{a}\right)}\, \frac{dt}{a^n} \tag{4}$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{t} - \mathbf{x})\phi(\mathbf{x})\, d\mathbf{x}\, \overline{\psi\left(\frac{\mathbf{t} - \mathbf{b}}{a}\right)}\, \frac{d\mathbf{t}}{a^n}.$$

Since the integrand $(\mathbf{t}, \mathbf{x}) \mapsto f(\mathbf{t} - \mathbf{x})\phi(\mathbf{x})\, \overline{\psi\left(\frac{\mathbf{t}-\mathbf{b}}{a}\right)}$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$, it is measurable. As $\psi, f \in \mathscr{S}(\mathbb{R}^n)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$, we get $\sup_{\mathbf{x} \in \mathbb{R}^n} |\psi(\mathbf{x})| = K < \infty$ and $f, \phi \in \mathscr{L}^1(\mathbb{R}^n)$. Therefore

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \left| f(\mathbf{t} - \mathbf{x})\phi(\mathbf{x})\, \overline{\psi\left(\frac{\mathbf{t} - \mathbf{b}}{a}\right)} \right| d\left(\mathbf{x} \times \frac{d\mathbf{t}}{a^n}\right)$$

$$\leq K \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(\mathbf{t} - \mathbf{x})\phi(\mathbf{x})|\, d\mathbf{x}\, \frac{d\mathbf{t}}{a^n}$$

$$= K \int_{\mathbb{R}^n} |\phi(\mathbf{x})|\, d\mathbf{x} \int_{\mathbb{R}^n} |f(\mathbf{t} - \mathbf{x})|\, \frac{d\mathbf{t}}{a^n} < \infty,$$

and hence the integrand is an integrable function on $\mathbb{R}^n \times \mathbb{R}^n$. Thus we can apply Fubini's theorem to the integral on the right hand side of (4) and the integral is equal to

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{t} - \mathbf{x})\, \overline{\psi\left(\frac{\mathbf{t} - \mathbf{b}}{a}\right)}\, \frac{d\mathbf{t}}{a^n}\, \phi(\mathbf{x})\, d\mathbf{x}$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{s})\, \overline{\psi\left(\frac{\mathbf{s} + \mathbf{x} - \mathbf{b}}{a}\right)}\, \frac{d\mathbf{s}}{a^n}\, \phi(\mathbf{x})\, d\mathbf{x}$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{s})\, \overline{\psi\left(\frac{\mathbf{s} - (\mathbf{b} - \mathbf{x})}{a}\right)}\, \frac{d\mathbf{s}}{a^n}\, \phi(\mathbf{x})\, d\mathbf{x}$$

$$= \int_{\mathbb{R}^n} (Wf)(\mathbf{b} - \mathbf{x}, a)\phi(x)\, d\mathbf{x} = (Wf \times \phi)(\mathbf{b}, a).$$

(2) For $f \in \mathscr{S}(\mathbb{R}^n)$,

$$W'(G \otimes \phi)(f) = (G \otimes \phi)(Wf) = G((Wf) \times \check{\phi}) = G(W(f * \check{\phi}))$$

$$= (W'G)(f * \check{\phi}) = ((W'G) * \phi)(f). \ \blacksquare$$

3. Boehmian spaces. We shall first recall the construction of a Boehmian space from [10]. Let Γ be a topological vector space, $(S, *)$ be a commutative semigroup, $\star : \Gamma \times S \to \Gamma$ satisfying

- $(f+g) \star s = (f \star s) + (g \star s),\ f, g \in \Gamma,\ s \in S,$
- $(\alpha f) \star s = \alpha(f \star s),\ f \in \Gamma,\ s \in S,\ \alpha \in \mathbb{C},$
- $f \star (s * t) = (f \star s) \star t,\ f \in \Gamma,\ s, t \in S,$
- $f_i \star s \to f \star s$ in Γ as $n \to \infty$ if $f_i \to f$ in Γ as $n \to \infty$ and $s \in S$;

and let Δ be a collection of sequences from S with the following properties:

- if $(s_i), (t_i) \in \Delta$ then $(s_i * t_i) \in \Delta$,
- $f \star s_i \to f$ in Γ as $n \to \infty$ if $f \in \Gamma$ and $(s_i) \in \Delta$.

A pair of sequences $((f_i), (s_i))$ is said to be a *quotient* if

$$f_i \star s_j = f_j \star s_i, \quad \forall i, j \in \mathbb{N}. \tag{5}$$

We denote by $\frac{f_i}{s_i}$ a quotient. An equivalence relation $\sim$ on the collection of all quotients is defined by

$$\frac{f_i}{s_i} \sim \frac{g_i}{t_i} \quad \text{if} \quad f_i \star t_j = g_j \star s_i, \quad \forall i, j \in \mathbb{N}. \tag{6}$$

The collection of all equivalence classes is denoted by $\mathscr{B} = \mathscr{B}(\Gamma, (S, *), \star, \Delta)$ and each equivalence class is called a *Boehmian*. On the space of Boehmians we define addition, scalar multiplication and multiplication by $s \in S$ as follows:

$$\left[\frac{f_i}{s_i}\right] + \left[\frac{g_i}{t_i}\right] = \left[\frac{(f_i \star t_i) + (g_i \star s_i)}{s_i * t_i}\right],$$

$$\alpha\left[\frac{f_i}{s_i}\right] = \left[\frac{\alpha f_i}{s_i}\right],$$

$$\left[\frac{f_i}{s_i}\right] \star s = \left[\frac{f_i \star s}{s_i}\right].$$

Every member f of Γ is identified with the Boehmian $\left[\frac{f \star s_i}{s_i}\right]$ for any $(s_i) \in \Delta$. The space of Boehmians is also equipped with two notions of convergence, namely δ-convergence and Δ-convergence. The following two definitions and the lemma can be found in [6].

Definition 3.1 (δ-convergence). Let (X_i) be a sequence in $\mathscr{B}$ and $X \in \mathscr{B}$. We say that $X_i \xrightarrow{\delta} X$ as $i \to \infty$ if there exists $(s_j) \in \Delta$ such that $X_i \star s_j, X \star s_j \in \Gamma$ for all $i, j \in \mathbb{N}$ and for each $j \in \mathbb{N}$, $X_i \star s_j \to X \star s_j$ as $i \to \infty$ in Γ.

Lemma 3.2. *$X_i \xrightarrow{\delta} X$ as $i \to \infty$ if and only if there exist $f_{i,j}, f_j \in \Gamma$, $i, j \in \mathbb{N}$, and $(s_j) \in \Delta$ such that $X_i = \left[\frac{f_{i,j}}{s_j}\right]$, $X = \left[\frac{f_j}{s_j}\right]$ and for each $j \in \mathbb{N}$, $f_{i,j} \to f_j$ as $i \to \infty$ in Γ.*

Definition 3.3 (Δ-convergence). We say that $X_i \stackrel{\Delta}{\to} X$ as $i \to \infty$ if there exists $(s_i) \in \Delta$ with $(X_i - X) \star s_i \in \Gamma$ for all $i \in \mathbb{N}$ and $(X_i - X) \star s_i \to 0$ as $i \to \infty$ in Γ.

For more details about the two notions of convergence on $\mathscr{B}$, we refer to [6].

The space of tempered Boehmians was introduced by P. Mikusiński in [8, 9] and slightly modified in [16]. Now we recall the definition of tempered Boehmians from [16]. The space of tempered Boehmians is defined as $\mathscr{B}_2 = \mathscr{B}(\mathscr{S}'(\mathbb{R}^n), (\mathscr{D}(\mathbb{R}^n), *), *, \Delta_0)$, where $*$ is the convolution between $\mathscr{S}'(\mathbb{R}^n)$ and $\mathscr{S}(\mathbb{R}^n)$ defined by $(u * f)(g) = u(g * \check{f})$ for $g \in \mathscr{S}'(\mathbb{R}^n)$, and Δ_0 is the family of sequences (ϕ_i) from $\mathscr{D}(\mathbb{R}^n)$ satisfying

- $\int_{\mathbb{R}^n} \phi_i(\mathbf{x})\, d\mathbf{x} = 1$ for all $i \in \mathbb{N}$,
- there exists $M > 0$ such that $\int_{\mathbb{R}^n} |\phi_i(\mathbf{x})|\, d\mathbf{x} \le M$ for all $i \in \mathbb{N}$,
- $s(\phi_i) \to 0$ as $i \to \infty$, where $s(\phi_i) = \sup\{\|\mathbf{x}\| : \phi_i(\mathbf{x}) \neq 0\}$.

Now we prove the following lemmas to construct the Boehmian space $\mathscr{B}_1 = \mathscr{B}(\tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+), (\mathscr{D}(\mathbb{R}^n), *), \otimes, \Delta_0)$.

Lemma 3.4. *If $G \in \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+)$, $g \in \tilde{\mathscr{S}}(\mathbb{R}^n \times \mathbb{R}_+)$ and $\phi_1, \phi_2 \in \mathscr{D}(\mathbb{R}^n)$ then*

(1) $g \times (\phi_1 * \phi_2) = (g \times \phi_1) \times \phi_2$,
(2) $G \otimes (\phi_1 * \phi_2) = (G \otimes \phi_1) \otimes \phi_2$.

Proof. (1) Let $(\mathbf{b}, a) \in \mathbb{R}^n \times \mathbb{R}_+$. Then

$$(g \times (\phi_1 * \phi_2))(\mathbf{b}, a) = \int_{\mathbb{R}^n} g(\mathbf{b} - \mathbf{x}, a)(\phi_1 * \phi_2)(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^n} g(\mathbf{b} - \mathbf{x}, a) \int_{\mathbb{R}^n} \phi_1(\mathbf{x} - \mathbf{t})\phi_2(\mathbf{t})\, d\mathbf{t}\, d\mathbf{x}.$$

Since g is bounded and $\phi_1, \phi_2 \in \mathscr{D}(\mathbb{R}^n)$, as in the proof of Theorem 2.5 we find that $g(\mathbf{b} - \mathbf{x}, a)\phi_1(\mathbf{x} - \mathbf{t})\phi_2(\mathbf{t})$ is integrable on the product space and hence we can apply Fubini's theorem. Thus the last integral is equal to

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(\mathbf{b} - \mathbf{x}, a)\phi_1(\mathbf{x} - \mathbf{t})\phi_2(\mathbf{t})\, d\mathbf{x}\, d\mathbf{t} \\
&\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(\mathbf{b} - (\mathbf{y} + \mathbf{t}), a)\phi_1(\mathbf{y})\, d\mathbf{y}\, \phi_2(\mathbf{t})\, d\mathbf{t} \\
&\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g((\mathbf{b} - \mathbf{t}) - \mathbf{y}), a)\phi_1(\mathbf{y})\, d\mathbf{y}\, \phi_2(\mathbf{t})\, d\mathbf{t} \\
&\quad = \int_{\mathbb{R}^n} (g \times \phi_1)(\mathbf{b} - \mathbf{t}, a)\phi_2(\mathbf{t})\, d\mathbf{t} = ((g \times \phi_1) \times \phi_2)(\mathbf{b}, a).
\end{aligned}$$

(2) Let $g \in \tilde{\mathscr{S}}(\mathbb{R}^n \times \mathbb{R}_+)$. Then

$$\begin{aligned}(G \otimes (\phi_1 * \phi_2))(g) &= G(g \times (\phi_1 * \phi_2)^\vee) \\ &= G(g \times (\check{\phi}_1 * \check{\phi}_2)) \\ &= G(g \times (\check{\phi}_2 * \check{\phi}_1)) \qquad \text{(since } * \text{ is commutative)} \\ &= G((g \times \check{\phi}_2) \times \check{\phi}_1) \qquad \text{(using (1) of this lemma)} \\ &= (G \otimes \phi_1)(g \times \check{\phi}_2) \\ &= ((G \otimes \phi_1) \otimes \phi_2)(g). \ \blacksquare\end{aligned}$$

LEMMA 3.5. *If $G_i \to G$ as $i \to \infty$ in $\tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+)$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$ then $G_i \otimes \phi \to G$ as $i \to \infty$ in $\tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+)$.*

Proof. We note first that $g \times \check{\phi} \in \tilde{\mathscr{S}}(\mathbb{R}^n \times \mathbb{R}_+)$ for all $g \in \tilde{\mathscr{S}}(\mathbb{R}^n \times \mathbb{R}_+)$, by Lemma 2.2. If $G_i \to G$ as $i \to \infty$ in $\tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+)$ then for each $g \in \tilde{\mathscr{S}}(\mathbb{R}^n \times \mathbb{R}_+)$, $(G_i - G)(g) \to 0$ as $i \to \infty$. Therefore

$$\begin{aligned}((G_i \otimes \phi) - (G \otimes \phi))(g) &= ((G_i - G) \otimes \phi)(g) \\ &= (G_i - G)(g \times \check{\phi}) \to 0 \quad \text{ as } i \to \infty. \ \blacksquare\end{aligned}$$

LEMMA 3.6. *If $g \in \tilde{\mathscr{S}}(\mathbb{R}^n \times \mathbb{R}_+)$ and $(\phi_i) \in \Delta_0$ then $g \times \phi_i \to g$ as $i \to \infty$ in $\tilde{\mathscr{S}}(\mathbb{R}^n \times \mathbb{R}_+)$.*

Proof. Fix $l, m, k, p \in \mathbb{N}_0$ such that $l + m \le k + p$ and $\beta \in \mathbb{N}_0^n$ with $|\beta| \le p$. The mean-value theorem applied to $(\partial/\partial a)^k D_{\mathbf{b}}^\beta g(\cdot, a)$ gives the estimate

$$(7) \quad \left|\left(\frac{\partial}{\partial a}\right)^k D_{\mathbf{b}}^\beta g(\mathbf{b} - \mathbf{x}, a) - \left(\frac{\partial}{\partial a}\right)^k D_{\mathbf{b}}^\beta g(\mathbf{b}, a)\right| \le \|x\| \cdot \left\|\nabla\left(\frac{\partial}{\partial a}\right)^k D_{\mathbf{b}}^\beta g(\mathbf{t}, a)\right\|,$$

where $\mathbf{t} = (1-h)(\mathbf{b}-\mathbf{x}) + h\mathbf{b} = \mathbf{b} + (h-1)\mathbf{x}$ $(0 < h < 1)$. If $\beta = (\beta_1, \ldots, \beta_n)$ then put

$$\beta_j^{(i)} = \begin{cases} \beta_j + 1 & \text{if } j = i \\ \beta_j & \text{if } j \ne i \end{cases} \qquad \text{for } i, j \in \{1, \ldots, n\},$$

and $\beta^{(i)} = (\beta_1^{(i)}, \ldots, \beta_n^{(i)}) \in \mathbb{N}_0^n$, $\forall i = 1, \ldots, n$. Then we also have

$$(8) \quad \left\|\nabla\left(\frac{\partial}{\partial a}\right)^k D_{\mathbf{b}}^\beta g(\mathbf{t}, a)\right\| \le C_2 \sum_{i=1}^n \left|\left(\frac{\partial}{\partial a}\right)^k D_{\mathbf{b}}^{\beta^{(i)}} g(\mathbf{t}, a)\right|$$

for some $C_2 > 0$. Therefore for every $(\mathbf{b}, a) \in \mathbb{R}^n \times \mathbb{R}_+$,

$$|a|^l(1+\|\mathbf{b}\|^2)^m\left|\left(\frac{\partial}{\partial a}\right)^k D_{\mathbf{b}}^{\beta}(g\times\phi_i - g)(\mathbf{b},a)\right|$$

$$= |a|^l(1+\|\mathbf{b}\|^2)^m\left|\int\limits_{\mathbb{R}^n}\left[\left(\frac{\partial}{\partial a}\right)^k D_{\mathbf{b}}^{\beta} g(\mathbf{b}-\mathbf{x},a) - \left(\frac{\partial}{\partial a}\right)^k D_{\mathbf{b}}^{\beta} g(\mathbf{b},a)\,\phi_i(\mathbf{x})\right]d\mathbf{x}\right|$$

$$\text{(by using } \textstyle\int_{\mathbb{R}^n}\phi_i(\mathbf{x})\,d\mathbf{x} = 1,\ \forall i\in\mathbb{N})$$

$$\le \int\limits_{\mathbb{R}^n}|a|^l(1+\|\mathbf{b}\|^2)^m\left|\left(\frac{\partial}{\partial a}\right)^k D_{\mathbf{b}}^{\beta} g(\mathbf{b}-\mathbf{x},a) - \left(\frac{\partial}{\partial a}\right)^k D_{\mathbf{b}}^{\beta} g(\mathbf{b},a)\right| |\phi_i(\mathbf{x})|\,d\mathbf{x}$$

$$\le C_2\int\limits_{\mathbb{R}^n}|a|^l(1+(\|\mathbf{t}\|+\|\mathbf{x}\|)^2)^m\|\mathbf{x}\|\sum_{i=1}^{n}\left|\left(\frac{\partial}{\partial a}\right)^k D_{\mathbf{b}}^{\beta^{(i)}} g(\mathbf{t},a)\right| |\phi_i(\mathbf{x})|\,d\mathbf{x}$$

$$\text{(since } \|\mathbf{b}\|\le\|\mathbf{t}\|+\|(1-h)\mathbf{x}\|\le\|\mathbf{t}\|+\|\mathbf{x}\|)$$

$$\le C_2\int\limits_{\mathbb{R}^n}|a|^l(1+(\|\mathbf{t}\|+s(\phi_i))^2)^m\|\mathbf{x}\|\sum_{i=1}^{n}\left|\left(\frac{\partial}{\partial a}\right)^k D_{\mathbf{b}}^{\beta^{(i)}} g(\mathbf{t},a)\right| |\phi_i(\mathbf{x})|\,d\mathbf{x}$$

$$\le C_2C_3MQ_{m,l,k,p+1}(g)s(\phi_i)$$

for some $C_3 > 0$ and $M > 0$ such that $\int_{\mathbb{R}^n}|\phi_i(\mathbf{x})|\,d\mathbf{x}\le M$ for all $i\in\mathbb{N}$. Since $s(\phi_i)\to 0$ as $i\to\infty$, we obtain

$$Q_{m,l,k,p}(g\times\phi_i - g)\to 0 \quad \text{as } i\to\infty,$$

which proves the lemma. ∎

Lemma 3.7. *If $G\in\tilde{\mathscr{S}}'(\mathbb{R}^n\times\mathbb{R}_+)$ and $(\phi_i)\in\Delta_0$ then $G\otimes\phi_i\to G$ as $i\to\infty$ in $\tilde{\mathscr{S}}'(\mathbb{R}^n\times\mathbb{R}_+)$.*

Proof. Suppose $g\in\tilde{\mathscr{S}}(\mathbb{R}^n\times\mathbb{R}_+)$. Then Lemma 2.2 gives $g\times\check{\phi}_i\in\tilde{\mathscr{S}}(\mathbb{R}^n\times\mathbb{R}_+)$ for all $i\in\mathbb{N}$. It is obvious that $(\check{\phi}_i)\in\Delta_0$. Therefore Lemma 3.6 leads to $g\times\check{\phi}_i\to g$ as $i\to\infty$ in $\tilde{\mathscr{S}}(\mathbb{R}^n\times\mathbb{R}_+)$. Hence

$$\lim_{i\to\infty}(G\otimes\phi_i)(g) = \lim_{i\to\infty}G(g\times\check{\phi}_i) = G(\lim_{i\to\infty}g\times\check{\phi}_i) = G(g). \ ∎$$

Remark 3.8. If $\Lambda\in\mathscr{S}'(\mathbb{R}^n)$ and $f\in\mathscr{S}(\mathbb{R}^n)$ then we know that $\Lambda*f$ is a function defined by $(\Lambda*f)(x)=\Lambda(\tau_x\check{f})$ for $x\in\mathbb{R}^n$. But this technique is not applicable for $G\otimes\phi$ when $G\in\tilde{\mathscr{S}}'(\mathbb{R}^n\times\mathbb{R}_+)$ and $\phi\in\mathscr{D}(\mathbb{R}^n)$, because G acts on functions of $(\mathbf{b},a)\in\mathbb{R}^n\times\mathbb{R}^+$ and ϕ is a function of $\mathbf{x}\in\mathbb{R}^n$. Therefore we could not decide whether $G\otimes\phi$ is a function or not. However, the conclusion of Lemma 2.3 is sufficient for our purpose. For this reason, we construct the Boehmian space $\mathscr{B}_1$ by using the distribution space $\tilde{\mathscr{S}}'(\mathbb{R}^n\times\mathbb{R}_+)$ and we use the definition of tempered Boehmian space $\mathscr{B}(\mathscr{S}'(\mathbb{R}^n),(\mathscr{D}(\mathbb{R}^n),*),*,\Delta_0)$ of [16] instead of $\mathscr{B}(\mathscr{I}(\mathbb{R}^n),(\mathscr{D}(\mathbb{R}^n),*),*,\Delta_0)$ of [8].

4. Generalized wavelet transform. In this section, we are going to define the extended wavelet transform and discuss its properties.

DEFINITION 4.1. We define the *extended wavelet transform* $\mathscr{W}$ from $\mathscr{B}_1$ into $\mathscr{B}_2$ by

$$\mathscr{W}\left(\left[\frac{G_i}{\phi_i}\right]\right) = \left[\frac{W'G_i}{\phi_i}\right] \quad \text{for } \left[\frac{G_i}{\phi_i}\right] \in \mathscr{B}_1.$$

LEMMA 4.2. *The extended wavelet transform $\mathscr{W} : \mathscr{B}_1 \to \mathscr{B}_2$ is well defined.*

Proof. First we prove that if $\left[\frac{G_i}{\phi_i}\right] \in \mathscr{B}_1$ then $\left[\frac{W'G_i}{\phi_i}\right] \in \mathscr{B}_2$. If $\left[\frac{G_i}{\phi_i}\right] \in \mathscr{B}_1$, then we observe that $G_i \in \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+)$ for all $i \in \mathbb{N}$, $(\phi_i) \in \Delta_0$ and

$$G_i \otimes \phi_j = G_j \otimes \phi_i, \quad \forall i, j \in \mathbb{N}. \tag{9}$$

Then $W'G_i \in \mathscr{S}'(\mathbb{R}^n)$ for all $i \in \mathbb{N}$. Applying the wavelet transform $W' : \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+) \to \mathscr{S}'(\mathbb{R}^n)$ on both sides of (9) and using Theorem 2.5(2), we can write

$$(W'G_i) * \phi_j = (W'G_j) * \phi_i, \quad \forall i, j \in \mathbb{N}. \tag{10}$$

Therefore $\frac{W'G_i}{\phi_i}$ is a quotient; as a consequence, $\left[\frac{W'G_i}{\phi_i}\right] \in \mathscr{B}_2$. Moreover, $\left[\frac{G_i}{\phi_i}\right] = \left[\frac{H_i}{\phi_i}\right]$ in $\mathscr{B}_2$ implies that

$$G_i \otimes \phi_j = H_j \otimes \phi_i, \quad \forall i, j \in \mathbb{N}. \tag{11}$$

and hence

$$(W'G_i) * \phi_j = (W'H_j) * \phi_i, \quad \forall i, j \in \mathbb{N}. \tag{12}$$

Therefore $\mathscr{W}\left[\frac{G_i}{\phi_i}\right] = \mathscr{W}\left[\frac{H_i}{\phi_i}\right]$ in $\mathscr{B}_2$, which completes the proof. ∎

LEMMA 4.3 (Consistency). *The extended wavelet transform $\mathscr{W} : \mathscr{B}_1 \to \mathscr{B}_2$ is consistent with the wavelet transform $W' : \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+) \to \mathscr{S}'(\mathbb{R}^n)$.*

Proof. If $G \in \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+)$, then the Boehmian representing G in $\mathscr{B}_1$ is $\left[\frac{G\otimes\phi_i}{\phi_i}\right]$ for any $(\phi_i) \in \Delta_0$. Now

$$\mathscr{W}\left(\left[\frac{G \otimes \phi_i}{\phi_i}\right]\right) = \left[\frac{W'(G \otimes \phi_i)}{\phi_i}\right] = \left[\frac{(W'G) * \phi_i}{\phi_i}\right],$$

which is the identification of $W'(G)$ in $\mathscr{B}_2$. ∎

THEOREM 4.4. *The extended wavelet transform $\mathscr{W} : \mathscr{B}_1 \to \mathscr{B}_2$ is linear.*

The proof of this theorem is straightforward.

THEOREM 4.5. *The extended wavelet transform $\mathscr{W} : \mathscr{B}_1 \to \mathscr{B}_2$ is one-to-one.*

Proof. Let $\left[\frac{G_i}{\phi_i}\right], \left[\frac{H_i}{\delta_i}\right] \in \mathscr{B}_1$ be such that $\mathscr{W}\left(\left[\frac{G_i}{\phi_i}\right]\right) = \mathscr{W}\left(\left[\frac{H_i}{\delta_i}\right]\right)$ in $\mathscr{B}_2$ or $\left[\frac{W'G_i}{\phi_i}\right] = \left[\frac{W'H_i}{\delta_i}\right]$. Then

$$(W'G_i) * \delta_j = (W'H_j) * \phi_i, \quad \forall i, j \in \mathbb{N}. \tag{13}$$

By Theorem 2.5(2), it follows that

$$W'(G_i \otimes \delta_j) = W'(H_j \otimes \phi_i), \quad \forall i, j \in \mathbb{N}. \tag{14}$$

Since the wavelet transform $W' : \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+) \to \mathscr{S}'(\mathbb{R}^n)$ is one-to-one, we get

$$G_i \otimes \delta_j = H_j \otimes \phi_i, \quad \forall i, j \in \mathbb{N}. \tag{15}$$

Thus $\left[\frac{G_i}{\phi_i}\right] = \left[\frac{H_i}{\delta_i}\right]$ in $\mathscr{B}_1$. ∎

THEOREM 4.6. *The extended wavelet transform $\mathscr{W} : \mathscr{B}_1 \to \mathscr{B}_2$ is continuous with respect to δ-convergence.*

Proof. Let $X_i \xrightarrow{\delta} X$ as $i \to \infty$ in $\mathscr{B}_1$. Then by Lemma 3.2, there exist $G_{i,j} \in \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+)$ for $i, j \in \mathbb{N}$ and $(\phi_j) \in \Delta_0$ such that $X_i = \left[\frac{G_{i,j}}{\phi_j}\right]$, $X = \left[\frac{G_j}{\phi_j}\right]$ and for each $j \in \mathbb{N}$,

$$G_{i,j} \to G_j \quad \text{in } \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+) \text{ as } i \to \infty. \tag{16}$$

Since the wavelet transform $W' : \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+) \to \mathscr{S}'(\mathbb{R}^n)$ is continuous, we get

$$W'G_{i,j} \to W'G_j \quad \text{in } \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+) \text{ as } i \to \infty. \tag{17}$$

Since $\mathscr{W}X_i = \left[\frac{W'G_{i,j}}{\phi_j}\right]$ and $\mathscr{W}X = \left[\frac{W'G_j}{\phi_j}\right]$, again by using Lemma 3.2 we conclude that $\mathscr{W}X_i \xrightarrow{\delta} \mathscr{W}X$ in $\mathscr{B}_2$ as $i \to \infty$. Thus $\mathscr{W}$ is continuous with respect to δ-convergence. ∎

LEMMA 4.7. *If $X \in \mathscr{B}_1$ and $\phi \in \mathscr{D}(\mathbb{R}^n)$ then $\mathscr{W}(X \otimes \phi) = (\mathscr{W}X) * \phi$.*

Proof. Let $X = \left[\frac{G_i}{\phi_i}\right]$. Then

$$\mathscr{W}(X \otimes \phi) = \mathscr{W}\left(\left[\frac{G_i \otimes \phi}{\phi_i}\right]\right) = \left[\frac{W'(G_i \otimes \phi)}{\phi_i}\right] = \left[\frac{(W'G_i) * \phi}{\phi_i}\right]$$
$$= \left[\frac{W'G_i}{\phi_i}\right] * \phi = (\mathscr{W}X) * \phi. \ \blacksquare$$

THEOREM 4.8. *The extended wavelet transform $\mathscr{W} : \mathscr{B}_1 \to \mathscr{B}_2$ is continuous with respect to Δ-convergence.*

Proof. Let $X_i \xrightarrow{\Delta} X$ in $\mathscr{B}_1$ as $i \to \infty$. This means that there exist sequences (F_i) from $\tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+)$ and $(\phi_i) \in \Delta_0$ such that $X_i \otimes \phi_i - X \otimes \phi_i =$

$\left[\frac{F_i \otimes \phi_j}{\phi_j}\right]$ for all $i \in \mathbb{N}$ and

$$F_i \to 0 \quad \text{in } \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+) \text{ as } i \to \infty. \tag{18}$$

By the continuity of the wavelet transform $W' : \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+) \to \mathscr{S}'(\mathbb{R}^n)$, it follows that

$$W'F_i \to 0 \quad \text{in } \mathscr{S}'(\mathbb{R}^n) \text{ as } i \to \infty. \tag{19}$$

Using Lemma 4.7, we get, for all $i \in \mathbb{N}$,

$$\begin{aligned}(\mathscr{W} X_i - \mathscr{W} X) * \phi_i &= (\mathscr{W}(X_i - X)) * \phi_i = \mathscr{W}((X_i - X) \otimes \phi_i) \\ &= \mathscr{W}\left(\left[\frac{F_i * \phi_j}{\phi_j}\right]\right) = \left[\frac{(W'F_i) * \phi_j}{\phi_j}\right].\end{aligned}$$

Consequently, $\mathscr{W} X_i \xrightarrow{\Delta} \mathscr{W} X$ in $\mathscr{B}_2$ as $i \to \infty$. ∎

Finally, we point out that an inversion formula for the wavelet transform $W' : \tilde{\mathscr{S}}'(\mathbb{R}^n \times \mathbb{R}_+) \to \mathscr{S}'(\mathbb{R}^n)$ has not yet been found, and the same problem concerns $\mathscr{W} : \mathscr{B}_1 \to \mathscr{B}_2$.

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