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EUCLIDEAN COMPONENTS FOR A CLASS OF SELF-INJECTIVE ALGEBRAS

BY

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Abstract. We determine the length of composition series of projective modules of G-transitive algebras with an Auslander–Reiten component of Euclidean tree class. We thereby correct and generalize a result of Farnsteiner [Math. Nachr. 202 (1999)]. Furthermore we show that modules with certain length of composition series are periodic. We apply these results to G-transitive blocks of the universal enveloping algebras of restricted p-Lie algebras and prove that G-transitive principal blocks only allow components with Euclidean tree class if $p = 2$. Finally, we deduce conditions for a smash product of a local basic algebra Γ with a commutative semisimple group algebra to have components with Euclidean tree class, depending on the components of the Auslander–Reiten quiver of Γ.

1. Introduction. The stable Auslander–Reiten quiver of a finite-dimensional algebra can be viewed as part of a presentation of the stable module category, and it is an important invariant which has many applications. It is a locally finite graph, where the vertices correspond to the isomorphism classes of indecomposable modules. Each connected component is isomorphic to $\mathbb{Z}[T]/\Gamma$ where T is a tree, and Γ is an admissible group of automorphism. (See [Ben, 4.15.6] for details.)

For many self-injective algebras it is known that the possibilities for T are restricted: it can only be Dynkin, or Euclidean, or one of a few infinite trees (see [W], [E], [ES]). In this paper we study algebras with Euclidean components. Recently the study of self-injective algebras with Euclidean Auslander–Reiten components has attracted much attention, for example all self-injective algebras of Euclidean type have this property (see the survey article [Sk, Section 4]).

These have also been studied in the context of reduced enveloping algebras by Farnsteiner, and we discovered that [F2, 4.6] is not correct. In Theorem [F2, 4.6] Farnsteiner proves a necessary condition for certain blocks of universal enveloping algebras $u(L, \chi)$ of a restricted p-Lie algebra L with $p > 2$ to have an Auslander–Reiten component with Euclidean tree class.

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Unfortunately one crucial step in the proof is wrong. As all other results in the case of $\tilde{D}_n$-tree class depend on this step, we need a different proof and theorem.

In the first section we give a proof for the more general setup of G-transitive blocks of Frobenius algebras in any characteristic.

In the second section we apply the main result of the first section to G-transitive blocks of $u(L, \chi)$. We show that some of the results of Farnsteiner's paper remain true while others need additional assumptions. We can show that a G-transitive principal block of $u(L, \chi)$ does not have Auslander–Reiten components of Euclidean tree class if $p > 2$.

In the last section we determine conditions for the smash product of a basic local algebra Γ and a semisimple commutative group algebra to have Auslander–Reiten components of Euclidean tree class depending on the tree class of components of Γ.

Let B be an indecomposable Frobenius algebra. We introduce for a group $G \subset \mathrm{Aut}(B)$ the G-transitive algebra B. This means that G acts transitively on the set of simple modules in B. We denote by $l(P)$ the length of an indecomposable projective module P. We show in 3.3, 3.11, 3.7 that the following holds for G-transitive algebras that have Auslander–Reiten components of Euclidean tree class:

(1) All non-periodic Auslander–Reiten components are isomorphic to either $\mathbb{Z}[\tilde{A}_{1,2}]$ or $\mathbb{Z}[\tilde{D}_n]$ for n odd and $n > 5$.
(2) In the first case $l(P) = 4$ and all indecomposable modules of length $0 \bmod l(P)$ and $2 \bmod l(P)$ are periodic.
(3) In the second case $l(P) = 8$ and all indecomposable modules of length $4 \bmod l(P)$ are periodic.

In [F2] Farnsteiner introduces $G(L)$, the group of group-like elements of $u^*(L)$, and shows that it can be embedded into the automorphism group of $u(L, \chi)$. He proves for $G(L)$-transitive blocks over a field of characteristic $p > 2$ (see [F2, 4.6]) that:

(1) All non-periodic Auslander–Reiten components are isomorphic to either $\mathbb{Z}[\tilde{A}_{1,2}]$ or $\mathbb{Z}[\tilde{D}_5]$. All indecomposable modules of length 2 are periodic.
(2) In the first case $l(P) = 4$.
(3) In the second case $l(P) = 8$.

For his results in the case of tree class $\tilde{D}_n$ he first shows that $n = 5$. As this step is wrong, a different proof was needed for the more general setup. In the new proof we can show that $n > 5$, which contradicts Farnsteiner's result.

Additionally we show that the number of non-periodic components is equal to the number of isomorphism types of simple modules in B.

In the second section we can verify Farnsteiner's statement that supersolvable algebras in characteristic $p > 2$ do not have Auslander–Reiten components of Euclidean tree class. An example in [E, 2.3] shows that this statement is wrong for $p = 2$. We also show in 4.2 that if B is a G-transitive block of $u(L, \chi)$ that has an Auslander–Reiten component of Euclidean tree class, then $p = 2$ or the dimension of a simple module in B is divisible by 2.

In the last section we introduce the smash product of a basic local algebra Γ and a semisimple group algebra kG, where $G \subset \mathrm{Aut}(\Gamma)$ is an abelian group. We show that the smash products are special cases of G-transitive algebras and describe how to construct the Gabriel quiver of the smash product from the Gabriel quiver of Γ.

For a finite-dimensional algebra A, we introduce $G(A)$, the free abelian group with basis the isomorphism types of indecomposable modules. We have a bilinear form $(-,-)_A$ induced by $\dim_k \mathrm{Hom}_A(-,-)$ defined on $G(A)$. We develop some properties of this bilinear form in 5.6, 5.7.

We use these general results to show in 5.15 that Auslander–Reiten sequences of the smash product restricted to Γ are sums of Auslander–Reiten sequences in Γ that are twists of each other. Using this result, we prove that the smash product has an Auslander–Reiten component of tree class $\tilde{D}_n$ if and only if Γ has an Auslander–Reiten component of tree class $\tilde{D}_n$. If the smash product has an Auslander–Reiten component of tree class $\tilde{A}_{1,2}$, then so does Γ. The converse is not true, as we show by providing a counterexample. We can in this case restrict the component to $Z[\tilde{A}_{12}]$ and $\mathbb{Z}[\tilde{A}_n]$ for certain $n \in \mathbb{N}$ (see 5.17).

2. Euclidean components for G-transitive algebras. For general background on Auslander–Reiten theory we refer to [ASS] or [Ben]. Let F be a field, let B be an indecomposable Frobenius algebra over F with Nakayama automorphism ν and let P be a projective indecomposable B-module. We denote by τ the Auslander–Reiten translation of B. As B is a Frobenius algebra, we have $\tau \cong \Omega^2\nu$ by [Ben, 4.12.9]. Furthermore, we denote by B-mod the category of finite-dimensional left B-modules.

Let $\alpha \in \mathrm{Aut}(B)$ and $M, N \in B$-mod. Then we denote by M_α the module which is isomorphic to M as an abelian group and the action of B on M_α is given by $b.m := \alpha(b)m$ for all $b \in B$ and $m \in M$. We denote by $l(M)$ the length of the composition series of M, by $c(M)$ its complexity and by $\mathrm{Irr}(M, N)$ the space of irreducible maps from M to N.

Let G be a subgroup of $\mathrm{Aut}(B)$. We call B a *G-transitive block* if for any two simple left B-modules V and W there is an element $g \in G$ such that $V_g \cong W$. We denote by $T_s(B)$ the stable Auslander–Reiten quiver of B.

From now on we assume that B is G-transitive.

We first prove a result on the length of the modules appearing at the end of an Auslander–Reiten sequence.

LEMMA 2.1. *Let M be an indecomposable, non-projective B-module. Then $l(\tau(M)) = l(M) + nl(P)$ for some $n \in \mathbb{Z}$.*

Proof. As B is G-transitive all projective indecomposable modules of B have the same length. Therefore the length of projective covers of left B-modules are multiples of $l(P)$. It follows that for all $i \in \mathbb{N}$ there is an $n \in \mathbb{Z}$ such that $l(\Omega^i(M)) = l(M) + nl(P)$. Therefore $l(\tau(M)) = l(\Omega^2(M_{\nu^{-1}})) = l(M_{\nu^{-1}}) + nl(P) = l(M) + nl(P)$ for some $n \in \mathbb{Z}$. ∎

The next lemma proves a condition which ensures that all indecomposable length 2 modules of a block have complexity one.

LEMMA 2.2. *Suppose P has length 4. Then every indecomposable B-module of length 2 has complexity one.*

Proof. Let M be an indecomposable B-module of length 2. The module M has a simple top and therefore an indecomposable projective cover. Then $\Omega(M)$ is also an indecomposable length 2 module and has therefore an indecomposable projective cover. It follows that $l(\Omega^n(M)) = 2$ for all $n \in \mathbb{N}$. Therefore the complexity of M is one. ∎

In general not every module of a Frobenius algebra with complexity one is periodic. In particular, the periodicity need not hold for an algebra with Auslander–Reiten component of Euclidean tree class, as is shown in the next

EXAMPLE 2.3 ([SM]). Let $A_q := F\langle x, y\rangle/(x^2, y^2, xy - qyx)$ where $q \neq 0$ and where q is not a root of unity. Let $M_\gamma = \mathrm{Span}\{v, xv\}$ be the two-dimensional module with $yv = \gamma xv$ for $\gamma \in F^*$. The projective cover of M_γ is given by $\pi : A_q \to M_\gamma$ with $\pi(1) = v$. Then $\Omega(M_\gamma) = \mathrm{Span}\{xy, y - \gamma x\} \cong M_{\gamma q}$. As q is not a root of unity we have $\Omega^k(M_\gamma) \ncong M_\gamma$ for all $k \in \mathbb{N}$. Therefore M_γ is not periodic but has complexity one. Furthermore, the Auslander–Reiten component containing the simple module is isomorphic to $\mathbb{Z}[\tilde{A}_{12}]$.

In the case of Auslander–Reiten components of Euclidean tree class, we know that the simple modules are non-periodic.

LEMMA 2.4. *Suppose that $T_s(B)$ has a component θ of Euclidean tree class. Then all simple modules are non-periodic and lie in an Auslander–Reiten component isomorphic to θ.*

Proof. As θ has Euclidean tree class it is attached to a projective indecomposable module P by [W, 2.4] and [ASS, IV, 5.5]. If $l(P) < 4$, then $l(P)$ is uniserial, which is a contradiction to the Euclidean tree class by

[Ben, 4.16.2]. As P is attached to θ the indecomposable module $P/\operatorname{soc} P$ lies in θ. The map induced by Ω restricted to θ is an isomorphism. Therefore $\Omega(P/\operatorname{soc} P) = \operatorname{soc} P$ is contained in a component isomorphic to θ. So $\operatorname{soc}(P)$ is not periodic. As B is G-transitive, all simple modules are non-periodic and lie in components isomorphic to θ. ■

We define certain stable graph automorphisms for G-transitive blocks with an Auslander–Reiten component θ of Euclidean tree class. Note that by [W, 2.4] and [ASS, IV, 5.5], there is at least one projective indecomposable module P attached to θ. Those maps have been defined in the proof of [F2, 4.6] similarly.

Definition 2.5. Suppose that the stable Auslander–Reiten quiver $T_s(B)$ has a component θ of Euclidean tree class. Define $\phi_g : T_s(B) \to T_s(B)$ by $M \mapsto \Omega(M_g)$. Then ϕ_g is a stable graph isomorphism. For any $g \in G$ define the map $A_g : T_s(B) \to T_s(B)$ by $M \mapsto M_g$. We denote by θ_g the component of $T_s(B)$ which is the image of $A_g(\theta)$.

For the rest of the section we fix, for any component θ of Euclidean tree class, a projective indecomposable module P that is attached to θ and an element $g \in G$ such that $\phi := \phi_g|_\theta$ is an automorphism of θ. Furthermore, let $S := \phi(P/\operatorname{soc} P)$.

Also $\phi_g|_\theta$ is an automorphism of θ if and only if $P_{g^{-1}}$ is attached to $\Omega(\theta)$. We can see this as follows. If $P_{g^{-1}}$ is attached to $\Omega(\theta)$, then A_g induces an isomorphism from $\Omega(\theta)$ to θ and $\phi_g|_\theta$ is an automorphism.

Suppose $\phi_g|_\theta$ is an automorphism. Then $A_{g^{-1}}$ induces an isomorphism from θ to $\Omega(\theta)$. As P is attached to θ, $P_{g^{-1}}$ is attached to $\Omega(\theta)$.

Note that S is a simple module that belongs to θ.

First we need to show that the twisting action of ν commutes with the twisting action of any automorphism of B.

Lemma 2.6. *Let A be a Frobenius algebra. For all $g \in \operatorname{Aut}(A)$ and A-modules M we have $M_{\nu g} \cong M_{g\nu}$.*

Proof. Let $(-,-)$ and $\{-,-\}$ be two associative non-degenerate bilinear forms, ν and ν_1 the corresponding Nakayama automorphisms, and let $f := (-,1)$ and $f_1 := \{-,1\}$ be the corresponding linear forms. Then $\pi : A \to A^*$, $a \mapsto af$, and $\pi_1 : A \to A^*$, $a \mapsto af_1$, are B-module isomorphisms. Therefore there exist $x, y \in A$ such that $xf = f_1$ and $yf_1 = f$. It follows that $x = y^{-1}$. Set $u := \nu(x)$. Then

$$\{a,b\} = (ab,x) = (a,bx) = (\nu(b)u,a) = \{\nu(b)u, ax^{-1}\} = \{u^{-1}\nu(b)u, a\}$$

for all $a, b \in A$. Let $C_u : A \to A$, $a \mapsto u^{-1}au$ for all $a \in A$. Then $\nu_1 = C_u \circ \nu$.

Let $g \in \mathrm{Aut}(A)$. Then $\{-,-\} := (-,-) \circ (g \times g)$ is an associative non-degenerate bilinear form. It has Nakayama automorphism $g^{-1} \circ \nu \circ g$ as

$$\{a,b\} = (g(a), g(b)) = (\nu(g(b)), g(a)) = \{g^{-1}(\nu(g(b))), a\}$$

for all $a, b \in A$. By the first part there exists an invertible element $u \in A$ such that $g^{-1} \circ \nu \circ g = C_u \circ \nu$. Therefore $M_{g\nu g^{-1}} \cong M_{C_u \circ \nu}$ for all left B-modules M. But $M_\nu \cong M_{C_u \circ \nu}$ via the automorphism $\phi : M_\nu \to M_{C_u \circ \nu}$, $m \mapsto u^{-1}m$, and therefore we conclude $M_{g\nu} \cong M_{\nu g}$. ■

3. Restrictions on tree classes. We say for the rest of the article that the algebra A *satisfies* (C) if

every module with complexity one is Ω-periodic.

We say the algebra *satisfies* (C′) if

every module with complexity one is Ω-periodic and τ-periodic.

Note that if ν has finite order and (C) holds then (C′) is also true. We introduce the following

Assumption 3.1. For B, we assume that all elements in G have finite order. Furthermore, we assume that B satisfies (C′) and that $T_s(B)$ has a component θ of Euclidean tree class. Let P be a projective indecomposable module attached to θ.

We have the following condition for the existence of a non-periodic indecomposable module of length 3.

Lemma 3.2. *Suppose B satisfies* (C). *If all indecomposable modules of length* 2 *have complexity one*, *then all indecomposable modules of length* 3 *are non-periodic. Also there is no uniserial module of length* 3.

Proof. Every indecomposable module of length 3 has a simple top or a simple socle. Let M be an indecomposable module of length 3 with simple top. Such an element exists, because a factor module of P of length 3 has simple top and is therefore indecomposable. Then there exists an exact sequence $0 \to S \to M \to L \to 0$ such that L is an indecomposable module of length 2 and S is a simple module. Then $c(S) \le \max\{c(M), c(L)\}$. By 2.4, S is non-periodic and therefore $c(S) \ge 2$. As $c(L) = 1$, we have $c(M) \ge 2$ and therefore M has to be non-periodic. If M has a simple socle we can find an indecomposable module N of length 2 such that $0 \to N \to L \to S \to 0$ is an exact sequence. By the same argument as in the previous case, M has to be non-periodic.

We assume that there is a uniserial module M of length 3 with composition series S_1, S_2, S_3. Let $L = \mathrm{rad}(M)$ and $N = M/S_3$. Then an exact sequence is given by $0 \to L \to S_2 \oplus M \to N \to 0$. As L and N are indecomposable modules of length 2 they are periodic. Therefore $c(M) \leq 1$ and M is therefore periodic. This contradicts the first part. ∎

The proof of the next theorem goes along the lines of the proof of [F2, 4.6]. As the setup here is more general and the author uses properties of the universal enveloping algebra of restricted p-Lie algebras, we give a proof for our setup.

THEOREM 3.3. *Let B be as in* 3.1. *Then the following statements hold*:

(1) *θ is isomorphic to $\mathbb{Z}[\tilde{A}_{12}]$ or $\mathbb{Z}[\tilde{D}_n]$ with n odd.*
(2) *The group G acts transitively on the non-periodic components.*
(3) *If $\theta \cong \mathbb{Z}[\tilde{A}_{12}]$, then all projective indecomposable left B-modules have length* 4. *Furthermore*, *all indecomposable modules of length* 2 mod 4 *and all indecomposable non-projective modules of length* 0 mod 4 *are periodic.*

Proof. Let T be the tree class of θ. Then ϕ induces a graph automorphism $f : T \to T$. Suppose f has a fixed point. Then there is an indecomposable module M in θ such that $\Omega^{2m}(M_{\nu^{-m}}) \cong \tau^m(M) \cong \Omega(M_g)$. Therefore M has complexity one and is by assumption τ-periodic, which is a contradiction. Therefore f does not have fixed points. The only Euclidean trees which admit an automorphism without fixed points are $\tilde{A}_{12}$ and $(\tilde{D}_n)_{n\geq 4}$ with n odd. We have $\theta \cong \mathbb{Z}[\tilde{A}_{1,2}]$ or $\theta \cong \mathbb{Z}[\tilde{D}_n]$ by [F2, 2.1]. Furthermore, all indecomposable modules which do not lie in $\Psi := \bigcup_{w\in G} \theta_w$ are periodic. This can be seen as follows: let Y be an indecomposable module which is not in Ψ. We have $\Omega(\Psi) = \Psi$. Therefore the function $d_Y : \Psi \to \mathbb{N}, M \mapsto \dim_F \operatorname{Ext}^1(Y, M)$, is additive by [ES, 3.2] and bounded by [W, 2.4]. Since all simple modules are contained in Ψ by 2.4 there exists an $m \in \mathbb{N}$ such that $\dim_F \operatorname{Ext}^n(Y, W) \leq m$ for all $n \geq 1$ and all simple modules W, so that Y has complexity one. Therefore Y is periodic.

To prove (3) we suppose that $T = \tilde{A}_{12}$. Then the proof is the same as in [F2, 4.6]. For the convenience of the reader we include a different proof. As S and $P/\operatorname{soc} P$ are in $\theta \cong \mathbb{Z}[\tilde{A}_{1,2}]$, all modules in θ have length -1 mod $l(P)$ or 1 mod $l(P)$ and two modules connected by an arrow have a different length modulo $l(P)$. The length function l mod $l(P)$ is additive on Auslander–Reiten sequences. Therefore -2 mod $l(P) = 2$ and it follows that $l(P) = 4$. As no modules of length 2 mod $l(P)$ and 0 mod $l(P)$ occur in θ, we see by (2) that all indecomposable modules of length 2 mod $l(P)$ and all indecomposable non-projective modules of length 0 mod $l(P)$ are periodic. ∎

Let $\mathbb{Z}[\tilde{D}_n]$ be indexed as follows: $(k, 1), \ldots, (k, n+1)$ denote the kth copy of $\tilde{D}_n$ for any $k \in \mathbb{Z}$:

$$\begin{tikzcd}
(k,1) & & & & (k,n) \\
& (k,3) \arrow[ul] \arrow[dl] \arrow[r] & \cdots \arrow[r] & (k,n-1) \arrow[ur] \arrow[dr] & \\
(k,2) & & & & (k,n+1)
\end{tikzcd}$$

With this notation $\tau((k,i)) = (k-1,i)$.

For an indecomposable non-projective module we denote by $\bar{\alpha}(M)$ the number of predecessors of M in $T_s(B)$.

In order to prove our second theorem, we need the following

LEMMA 3.4. *Assume $T_s(B)$ has a component θ which is isomorphic to $\mathbb{Z}[\tilde{D}_n]$ with n odd. Then*

(a) $\mathrm{rad}P/\mathrm{soc}P$ *is indecomposable.*
(b) $l(P)$ *is even.*
(c) *All M with $\bar{\alpha}(M) = 2$ or 3 have even length.*

Proof. (1) We assume that $l(P)$ is even and show that (c) holds in this case. Let M be a module with $\bar{\alpha}(M) = 3$. Then there exist a module N such that M is its only non-projective predecessor. Let $l(N) = a$. Then $l(M) = 2a \bmod l(P)$ by Lemma 2.1. By assumption $l(P)$ is even and therefore all modules with three predecessors have even length. Consider the following extract from $\mathbb{Z}[\tilde{D}_n]$ where $M_3 := (k,3)$ and $M_{n-1} := (k,n-1)$ denote the isomorphism type of modules with three predecessors and $M_t := (k,t)$ for $3 \leq t \leq n-1$:

$$\begin{tikzcd}
& \tau^{-1}(M_{t-3}) \arrow[dr] & & & \\
M_{t-2} \arrow[ur] \arrow[dr] & & \tau^{-1}(M_{t-2}) \arrow[dr] & & \\
& M_{t-1} \arrow[ur] \arrow[dr] & & \tau^{-1}(M_{t-1}) \arrow[dr] & \\
& & M_t \arrow[ur] \arrow[dr] & & \tau^{-1}(M_t) \\
& & & M_{t+1} \arrow[ur] &
\end{tikzcd}$$

We assume (c) is false and let t be minimal such that M_t has odd length. Suppose $t > 4$. Then M_{t-2} is of even length and $\bar{\alpha}(M_{t-1}) = 2$. Hence

$2l(M_{t-1}) = l(M_t) + l(M_{t-2}) \bmod l(P)$. This gives a contradiction as the left hand side is an even number and the right hand side is odd. Therefore $t = 4$. Then M_5 is of even length and M_6 of odd length. We can show that M_s with s odd has even length and M_s with s even has odd length. As n is odd, the module M_{n-1} has to be of odd length, which is a contradiction.

(2) The Auslander–Reiten sequence ending in $P/\operatorname{soc} P$ is given by

$$0 \to \operatorname{rad} P \to P \oplus \operatorname{rad} P/\operatorname{soc} P \to P/\operatorname{soc} P \to 0.$$

We assume that $\operatorname{rad} P/\operatorname{soc} P$ is decomposable.

Then $\bar{\alpha}(P/\operatorname{soc} P) > 1$. There exists no module N such that $0 \to \tau(N) \to S \to N \to 0$ is an Auslander–Reiten sequence. Any projective indecomposable module Q appears only as a summand of the middle term of an Auslander–Reiten sequence with middle term $Q \oplus \operatorname{rad} Q/\operatorname{soc} Q$ and $S \neq \operatorname{rad} Q/\operatorname{soc} Q$ because $l(Q) > 3$. So there exists no Auslander–Reiten sequence of the form $0 \to \tau(N) \to S \oplus Q \to N \to 0$ with Q non-zero. Therefore $\bar{\alpha}(S) \neq 3$. As ϕ maps $P/\operatorname{soc} P$ to S we have $\bar{\alpha}(S) = \bar{\alpha}(P/\operatorname{soc} P)$. Therefore $\bar{\alpha}(S) = \bar{\alpha}(P/\operatorname{soc} P) = 2$. For $n = 5$ all modules have either one or three predecessors, so that $n > 5$. Suppose $l(P)$ is even. Then (1) shows that all modules with two or three predecessors in θ are of even length. As S is not of even length, this is a contradiction. Therefore $l(P)$ is odd. Let M_1 be a module of length a with only one predecessor M_3. Then $l(M_3) = 2a \bmod l(P)$. For the other module M_2 with the only predecessor M_3 and length $\bar{a}$ we have $2\bar{a} = 2a \bmod l(P)$. As $l(P)$ is odd this gives us $\bar{a} = a \bmod l(P)$. We can deduce that $l(M_4) + 2a = 4a \bmod l(P)$ and therefore $l(M_4) = 2a \bmod l(P)$. It follows inductively that $l(M_i) = 2a \bmod l(P)$ for all modules with two predecessors. Therefore we have $-1 = 2a \bmod l(P)$ and $1 = 2a \bmod l(P)$ as $P/\operatorname{soc} P$ and S are modules with two predecessors. This is a contradiction as $l(P)$ is odd. This proves (a).

(3) The only predecessor of $P/\operatorname{soc} P$ is $\operatorname{rad} P/\operatorname{soc} P$. As $S = \phi(P/\operatorname{soc} P)$, we have $\bar{\alpha}(S) = 1$. The predecessor of S has length $2 \bmod l(P)$ and the predecessor of $P/\operatorname{soc} P$ has length $-2 \bmod l(P)$ by the argument of (2).

Suppose that $l(P)$ is odd. If $n = 5$ this is a contradiction. If $n > 5$, then by (2) all modules with two predecessors have length $2 \bmod l(P)$ and $-2 \bmod l(P)$, which is a contradiction as $l(P)$ is odd. Therefore (b) holds. Then (1) proves (c). ■

We also need the following

LEMMA 3.5. *Suppose $T_s(B)$ has a component θ of tree class $\tilde{D}_n$. Then*

(1) $l(P) > 4$.
(2) *$P/\operatorname{soc} P$ and S have one predecessor each, and the τ-orbits of their predecessors are different.*

Proof. We set $l := l(P)$. We know by Lemma 3.4 that $\operatorname{rad} P/\operatorname{soc} P$ is indecomposable.

It was shown in the proof of 2.4 that $l \geq 4$. Suppose now that $l = 4$. Then by Lemma 2.2 all indecomposable modules of length 2 are periodic. By 3.4, $\operatorname{rad} P/\operatorname{soc} P$ is indecomposable and therefore periodic. As P is attached to θ, the module $\operatorname{rad} P/\operatorname{soc} P$ also belongs to θ, which is a contradiction by [Ben, 4.16.2]. Therefore $l > 4$.

From (3) of the proof of 3.4 we know that S and $P/\operatorname{soc} P$ have only one predecessor of length 2 mod l and -2 mod l respectively. As $l \neq 4$ and by 2.1 their predecessors do not lie in the same τ-orbit. ∎

We can now deduce the length of projective indecomposable modules if the Auslander–Reiten quiver has components $\mathbb{Z}[\tilde{D}_5]$.

PROPOSITION 3.6. *Let B be as in* 3.1. *Suppose $T_s(B)$ has a component θ of tree class $\tilde{D}_5$. Then $l(P) = 8$ and all indecomposable modules of length* 2 *and of length* 4 mod $l(P)$ *are periodic.*

Proof. We set $l := l(P)$. Let x be the length modulo l of the module which has as only predecessor the module of length -2 mod l and let y the length modulo l of the module which has as only predecessor the module of length 2 mod l. We visualize this in the following diagram:

$$\begin{tikzcd}
1 \bmod l & & & -1 \bmod l \\
& 2 \bmod l \arrow[ul] \arrow[r] \arrow[dl] & -2 \bmod l \arrow[ur] \arrow[dr] & \\
y \bmod l & & & x \bmod l
\end{tikzcd}$$

Then by comparing lengths in Auslander–Reiten sequences we get the following equations:

(1) $2x + 2 = 0 \bmod l$,
(2) $x + 5 = 0 \bmod l$,
(3) $5 - y = 0 \bmod l$,
(4) $2y - 2 = 0 \bmod l$.

We can therefore deduce from (1) and (2) that l divides 8. Therefore $l = 8$ by 3.5(1).

Suppose now that there is an indecomposable module of length 2 which is not periodic. By transitivity there is an indecomposable non-periodic module M of length 2 in θ. Then by the equations (1)–(4), $\bar{\alpha}(M) = 3$ and there is an indecomposable module N which has only M as predecessor. Therefore M appears in the Auslander–Reiten sequence $0 \to \tau(N) \to M \to N \to 0$.

This means that N and $\tau(N)$ have length 1 and are simple modules. By transitivity $\tau(Q)$ is a simple B-module for any simple B-module Q. Then N has to be periodic, which is a contradiction. By the equations (1)–(4), θ does not have an indecomposable module of length 4 mod l. Therefore those modules are periodic by 3.3. ■

We can now exclude tree class $\tilde{D}_5$ for certain algebras.

THEOREM 3.7. *Let B be as in* 3.1. *Then $T_s(B)$ does not have a component of tree class $\tilde{D}_5$.*

Proof. We assume, for a contradiction, that $T_s(B)$ has a component θ of tree class $\tilde{D}_5$. Using 3.6 and 3.2, we know that B does not have a uniserial module of length 3, but has a non-periodic indecomposable module of length 3. Therefore x in the proof of 3.6 is 3. By the proof of 3.6 there is an almost split sequence $0 \to \tau(X) \xrightarrow{f} H \xrightarrow{g} X \to 0$ with $H := \mathrm{rad}(P)/\mathrm{soc}(P)$. Therefore $l(\tau(X)) = l(X) = 3$. Suppose $\tau(X)$ has an indecomposable submodule U of length 2. Then H also has an indecomposable submodule $V := f(U)$ of length 2. But then the preimage of V of the canonical surjection $\mathrm{rad}(P) \to H$ is a submodule of length 3 and is uniserial, which is a contradiction. Therefore $\tau(X)$ has a quotient W that is indecomposable of length 2. Let $h : \tau(X) \to W$ be the canonical surjection. Then by Auslander–Reiten theory h factors through f. Therefore there exists a surjective map $s : H \to W$. But then $P/\ker s$ is a uniserial module of length 3, which is a contradiction. Therefore $T_s(B)$ does not have a component of tree class $\tilde{D}_5$. ■

We define the following automorphisms of $\mathbb{Z}[\tilde{D}_n]$ as in [F2]:

$$\alpha(k,i) = \begin{cases} (k,2), & i = 1, \\ (k,1), & i = 2, \\ (k,i), & i \geq 3, \end{cases}$$

$$\beta(k,i) = \begin{cases} (k,n+1), & i = n, \\ (k,n), & i = n+1, \\ (k,i), & i \leq n-1, \end{cases}$$

$$\gamma(k,i) = \begin{cases} (k,n), & i = 1, \\ (k,n+1), & i = 2, \\ (k+i-3, n+2-i), & 3 \leq i \leq n-1, \\ (k+n-4, 1), & i = n, \\ (k+n-4, 2), & i = n+1. \end{cases}$$

LEMMA 3.8 ([F2, 2.1]). *The automorphism group of $\mathbb{Z}[\tilde{D}_n]$ is given by*

$$\{\tau^k \circ \alpha^i \circ \beta^j \circ \gamma^l \mid k \in \mathbb{Z},\ i,j,l \in \{0,1\}\}.$$

We describe the action of G on Euclidean components.

LEMMA 3.9. *Let B be as in 3.1. Let $h \in G$ and suppose h induces an automorphism $A_h : \theta \to \theta$, $M \mapsto M_h$. Suppose that B has an indecomposable non-periodic module of length 3 if θ has tree class $\tilde{D}_n$ for $n > 5$. Then A_h is the identity.*

Proof. By 3.3 we know that $\theta \cong \mathbb{Z}[\tilde{D}_n]$ or $\theta \cong \mathbb{Z}[\tilde{A}_{1,2}]$. We assume first that $\theta \cong \mathbb{Z}[\tilde{D}_n]$ with $n > 5$. Suppose A_h is not the identity. By 3.8 the automorphisms of finite order have the form $\tau^k \circ \alpha^i \circ \beta^j \circ \gamma$ for $k = n/2 - 2$ or $\alpha^i \circ \beta^j$ with $i, j \in \{0, 1\}$. As n is odd the first possibility cannot occur. Therefore A_h is equal to either α, β or $\alpha \circ \beta$.

Suppose $A_h = \alpha \circ \beta$. Then all modules with only one predecessor have length $\pm 1 \bmod l(P)$. There exists a non-periodic indecomposable module of length 3 and by transitivity there is an indecomposable length 3 module M in θ. As $l \neq 4$ by 3.5(1) we have $\bar{\alpha}(M) = 3$ or $\bar{\alpha}(M) = 2$. Therefore $M_h \cong M$. This is a contradiction because M has either a simple top or a simple radical, and the map A_h does not stabilize simple modules.

Assume that $A_h = \beta$. Then $A_h(P/\operatorname{soc} P) = P_h/\operatorname{soc} P_h \ncong P/\operatorname{soc} P$. By definition of S and ϕ, we have $S = \operatorname{soc} P_g$. Then $S = \operatorname{soc} P_g = \phi(P/\operatorname{soc} P) \ncong \phi(P_h/\operatorname{soc} P_h) = \operatorname{soc} P_{hg} = S_{g^{-1}hg}$. Therefore $A_{g^{-1}hg} = \alpha$, as by the first case no automorphism induced by an element of G is equal to $\alpha \circ \beta$. But then $A_{hg^{-1}hg} = \alpha \circ \beta$, which is a contradiction.

Assume now that $A_h = \alpha$. Then $A_h(S) = S_h \ncong S$. We have therefore $P/\operatorname{soc} P = \phi^{-1}(S) \ncong \phi^{-1}(S_h) = P_{ghg^{-1}}/\operatorname{soc} P_{ghg^{-1}}$. Therefore $A_{ghg^{-1}} = \beta$ and $A_{hghg^{-1}} = \alpha \circ \beta$, which is a contradiction.

In the case of $\mathbb{Z}[\tilde{A}_{1,2}]$ there are no finite order automorphisms unequal to the identity, so this gives a contradiction as well. ∎

We describe the non-periodic components more precisely in the following

COROLLARY 3.10. *Let B be as in 3.9. Then B has exactly as many non-periodic Auslander–Reiten components as there are isomorphism classes of simple left B-modules.*

Proof. By Theorem 3.3 all non-periodic components are isomorphic and for every non-periodic Auslander–Reiten component Δ there exists a $g \in G$ such that $\theta_g = \Delta$. The component θ contains a simple module by 2.4 and therefore every non-periodic Auslander–Reiten component contains a simple module. By transitivity there exists for any simple module V a non-periodic Auslander–Reiten component $\mathcal{W}$ such that V belongs to $\mathcal{W}$. Suppose there is a non-periodic component which contains two simple modules V and V_r for some $r \in G$. Then r induces a non-identity automorphism of finite order on the component. This contradicts 3.9. ∎

We can now prove some necessary conditions for a component of tree class $\tilde{D}_n$ for $n > 5$. Compare the following theorem to [F2, 4.6]. We have proved that $n \neq 5$. Farnsteiner first shows that $n = 5$ and then deduces the other statements from this fact. As this step is wrong, we require a different proof.

THEOREM 3.11. *Let B be as in* 3.1. *Suppose $T_s(B)$ has a component θ of tree class $\tilde{D}_n$, $n > 5$. Suppose B contains an indecomposable non-periodic module of length* 3. *Then $l(P) = 8$ and all modules of length* 4 mod $l(P)$ *are periodic.*

Proof. The proof is in two steps. Let $l := l(P)$.

STEP 1: $l = 8$. Let M be an indecomposable length 3 module in θ. By 3.4(c), $\bar{\alpha}(M) = 1$. Suppose M shares a predecessor with the module of length 1 mod l. Then the predecessor has length 2 mod l and 6 mod $l = 2$ mod l, which is a contradiction as $l \neq 4$. It must therefore share a predecessor with the module of length -1 mod l. This gives us $6 = -2$ mod l and therefore $l = 8$.

We know from 3.5(2) that the modules with three predecessors have length 2 mod 8 and -2 mod 8. The modules with one predecessor have therefore length ± 1 mod 8 or ± 3 mod 8.

STEP 2: The indecomposable modules of length 4 mod 8 are periodic. Let W be a module with one predecessor and length 1 mod 8. We take W corresponding to $(k, 1)$ and use the notation of the proof of 3.4. Then $l(M_3) = 2$ mod 8. Let $\bar{W}$ be the other module with only predecessor M_3. Then $l(\bar{W}) = 4x + 1$. The module $l(M_4)$ satisfies $1 + l(\bar{W}) + l(M_4) = 4$ mod 8. Therefore $l(M_4) = 2(1 - 2x)$ mod 8. In the same way we obtain $l(M_5) = 2$ mod 8, $l(M_6) = 2(1 + 2x)$ mod 8, $l(M_7) = 2$ mod l, $l(M_8) = 2(1 - 2x)$ mod 8. The calculation shows that $l(M_t) = 2$ mod 8 if t is odd, $l(M_t) = 2(1+2x)$ mod 8 if $t = 4m + 2$ and $l(M_t) = 2(1 - 2x)$ mod 8 if $t = 4m$ for any $m \in \mathbb{N}$.

Thus modules of length 4 mod 8 in θ do not have two predecessors. By the remark before Step 2 they do not have one or three predecessors. Therefore no module of length 4 mod 8 belongs to θ. As no module of length 4 mod 8 appears in θ, they have to be periodic by 3.3(2). ∎

Note also that by the proof of 3.7, B has a uniserial module of length 3.

4. Application to Auslander–Reiten components of enveloping algebras of restricted p-Lie algebras. Let L be a finite-dimensional restricted p-Lie algebra and χ a linear form on L. We denote by $u(L, \chi)$ the universal enveloping algebra of (L, χ). If $\chi = 0$ we set $u(L, \chi) = u(L)$.

We denote by $G(L)$ the set of group-like elements of the dual Hopf algebra $u(L)^*$. The group-like elements are the homomorphisms of $u(L)$. The comultiplication on $u(L)$ induces an algebra homomorphism $\Delta : u(L,\chi) \to u(L) \otimes u(L,\chi), x \mapsto x \otimes 1 + 1 \otimes x$ for all $x \in L$. We write $\Delta(u) = u_1 \otimes u_2$ for $u \in u(L,\chi)$. This defines a left $u(L)$-comodule algebra structure and right $u(L)^*$-module algebra structure on $u(L,\chi)$. Therefore $G(L)$ acts on the automorphism group of $u(L,\chi)$ via $(g \cdot \psi)(u) = \psi(u \cdot g) = g(u_1)\psi(u_2)$ for all $\psi \in \mathrm{Aut}(u(L,\chi))$, $g \in G(L)$ and $u \in u(L,\chi)$. We embed $G(L)$ into $\mathrm{Aut}(u(L,\chi))$ via the injective group homomorphism $f : G(L) \to \mathrm{Aut}(u(L,\chi)), w \mapsto w \cdot \mathrm{id}_{u(L,\chi)}$. For an $u(L,\chi)$-module M and $w \in G(L)$ we denote by M_w the twisted module $M_{f(w)}$. Note that every element of $G(L) \setminus \{1\}$ has order p.

By [FS2, 1.2] the Nakayama automorphism of $u(L,\chi)$ has order 1 or p and all modules of complexity one are 2-periodic by [F1, 2.5]. Furthermore, $u(L,\chi)$ has a non-periodic indecomposable module of order 3 by [F2, 4.5]. Therefore the assumptions of 3.1 are satisfied for $G(L)$-transitive blocks or G-transitive blocks, where G is a finite subgroup of $\mathrm{Aut}(u(L,\chi))$. The next corollary follows directly from 3.7.

Corollary 4.1. *Let $B \subset u(L,\chi)$ be a G-transitive block. Then $T_s(B)$ does not have a component of tree class $\tilde{D}_5$.*

More generally, we have

Lemma 4.2. *Let $B \subset u(L,\chi)$ be a G-transitive block and let S be a simple module in B. Then $T_s(B)$ admits a Euclidean component only if $p = 2$ or* $\dim S = 0 \bmod p$.

Proof. By 3.3 and 3.11 all indecomposable modules of length 2 or 4 are periodic. By [F1, 2.5] all periodic indecomposable modules have dimension 0 mod p. As B is G-transitive all simple modules in B have the same dimension. Therefore $2\dim S = 0 \bmod p$. ∎

As a G-transitive principal block of $u(L,\chi)$ has only one-dimensional simples, we get the following corollary immediately from the preceding lemma.

Corollary 4.3. *Let $B \subset u(L,\chi)$ be the principal block, and assume B is G-transitive. Then $T_s(B)$ admits a Euclidean component only if $p = 2$.*

We call a block B *primary* if it only contains one isomorphism type of simple modules. Note that [F2, 4.7] remains true for primary blocks of $u(L,\chi)$ under the additional assumption that all indecomposable modules of length 2 are periodic.

We recall the definition of supersolvable Lie algebras:

Definition 4.4 ([FS2, I]). Let $(L^i)_{i\in\mathbb{N}}$ with $L^i = [L^{i-1}, L]$ and $L^0 = L$ be a sequence of ideals in L. Then L is *nilpotent* if there is an $n \in \mathbb{N}$ such that $L^n = 0$. The sequence $(L^{(i)})_{i\in\mathbb{N}}$ with $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$ and $L^{(0)} = L$ is the *derived series*. We call L *solvable* if there is an $n \in \mathbb{N}$ such that $L^{(n)} = 0$, and *supersolvable* if L^1 is nilpotent.

Since projective modules of restricted universal enveloping algebras of supersolvable Lie algebras have p-power length by [F3, 2.10], the result of [F2, 4.1] remains true by applying 3.11.

LEMMA 4.5. *Let L be a supersolvable finite-dimensional restricted p-Lie algebra and $p > 2$. Then $T_s(u(L,\chi))$ does not have a component of Euclidean tree class.*

This result cannot be extended to $p = 2$ as the following example shows.

EXAMPLE 4.6. Let $A = k[x,y]/(x^2,y^2)$ be the Kronecker algebra. Then $A \cong u(L)$ where $L = \mathrm{Span}\{x,y\}$ is the restricted 2-Lie algebra given by $[x,y] = 0$ and $x^{[2]} = y^{[2]} = 0$. Then L is supersolvable and the component containing the trivial module k is isomorphic to $\mathbb{Z}[\tilde{A}_{1,2}]$. This is well known: see for example [E, 2.3].

5. Euclidean components of smash products. The goal of this section is to determine conditions so that the smash product of a basic simple algebra and a semisimple commutative group algebra have an Auslander–Reiten component of Euclidean tree class. We assume that k is algebraically closed.

We start by describing the simple and indecomposable projective modules of certain smash products.

LEMMA 5.1. *Let Γ be a local and basic algebra with simple module S and let G be a finite group such that $G < \mathrm{Aut}(\Gamma)$. Let $\{e_1,\dots,e_m\}$ be a full set of primitive orthogonal idempotents in kG, let $\bigoplus_{i=1}^m P_i$ be a decomposition of kG into projective indecomposable kG-modules $P_i := kGe_i$ and let $S_i := \operatorname{soc} P_i$ for $1 \le i \le m$. Then for every simple $\Gamma \rtimes kG$-module V there exists an $1 \le i \le m$ such that $V \cong S \otimes S_i$. A complete set of primitive orthogonal idempotents of $\Gamma \rtimes kG$ is given by $\{1 \rtimes e_i \mid 1 \le i \le m\}$, and $\Gamma \rtimes kG$ has a decomposition $\bigoplus_{i=1}^m \Gamma \rtimes P_i$ into projective indecomposable modules $\Gamma \rtimes P_i$.*

Proof. As g induces an automorphism on Γ for all $g \in G$, we have $G(J(\Gamma)) = J(\Gamma)$ and $J(kG)\Gamma \subset J(\Gamma)$. Therefore $J(\Gamma) \rtimes kG + \Gamma \rtimes J(kG) \subset J(\Gamma \rtimes kG)$. Furthermore, $\Gamma \rtimes kG/(J(\Gamma) \rtimes kG + \Gamma \rtimes J(kG)) \cong \Gamma/J(\Gamma) \otimes kG/J(kG) \cong \bigoplus_{i=1}^m S \otimes S_i$, which is semisimple. This proves $J(\Gamma) \rtimes kG + \Gamma \rtimes J(kG) = J(\Gamma \rtimes kG)$ and all simples are given by $S \otimes S_i$. Clearly $\{1 \rtimes e_i \mid 1 \le i \le m\}$ is a set of orthogonal idempotents and $\bigoplus_{i=1}^m \Gamma \rtimes P_i$ is a decomposition of $\Gamma \rtimes kG$ into projective modules $\Gamma \rtimes P_i = (\Gamma \rtimes kG)(1 \rtimes e_i)$. The projective modules are indecomposable as $\operatorname{soc}(\Gamma \rtimes P_i) = S \otimes S_i$ is simple and therefore $\{1 \rtimes e_i \mid 1 \le i \le m\}$ is a complete set of primitive idempotents. ∎

From now on let Γ be a basic local algebra with simple module S. Let G be an abelian group such kG is semisimple, and G is a subgroup of $\mathrm{Aut}(\Gamma)$. Then the smash product $R := \Gamma \rtimes kG$ is well defined.

By Gabriel's lemma [Ben, 4.1.7], there exists a quiver Q such that $\Gamma \cong kQ/I$ for an admissible ideal $I \subset kQ$. As G is abelian and kG semisimple, the set of irreducible characters of kG forms a multiplicative group isomorphic to G. We index the characters by elements of G via a fixed isomorphism and index the primitive orthogonal idempotents by the same group element as its corresponding character. So let $\{\chi_g \,|\, g \in G\}$ be the set of irreducible characters and $\{e_g \,|\, g \in G\}$ the set of primitive orthogonal idempotents such that $he_g = \chi_g(h)e_g$ for all $g, h \in G$. Suppose $G \leq \operatorname{Aut}(\Gamma)$. Then kG acts on $J(\Gamma)$ and $J^2(\Gamma)$. As kG is semisimple, $J(\Gamma)/J^2(\Gamma)$ splits as a direct sum of one-dimensional kG-modules. Let $\alpha_1, \dots, \alpha_m$ be the simultaneous eigenvectors of the action of G on $J(\Gamma)/J^2(\Gamma)$. Let χ_{n_i}, $n_i \in G$, $i = 1, \dots, m$, be the corresponding irreducible characters. By 5.1 we know that $\Gamma \rtimes kG$ is a basic algebra with projective indecomposable modules $\Gamma \rtimes ke_g$ for $g \in G$ which have simple quotients $S \rtimes ke_g$. We have the following presentation of $\Gamma \rtimes kG$. Take the quiver where vertices are labelled by $1 \rtimes e_g$ and where arrows are $\alpha_i \rtimes e_g$. Note that

$$(\alpha_i \rtimes e_h)(\alpha_j \rtimes e_g) = (\alpha_i\alpha_j) \rtimes \chi_{n_j g}(e_h)e_g = (\chi_{n_j g}, \chi_h)(\alpha_i\alpha_j \rtimes e_g)$$

where $(-,-)$ is the usual inner product of characters. Therefore the arrow $\alpha_i \rtimes e_g$ ends in $1 \rtimes e_g$ and starts in $1 \rtimes e_q$ with $q = gn_i$. We can obtain the relations that generate T via the relations that generate I in Γ.

Note that the construction of W coincides with the Mc Kay quiver (see [SSS, 2] for the definition) where $V := J(\Gamma)/J^2(\Gamma)$.

We will illustrate this construction on a small example.

EXAMPLE 5.2. Let $\Gamma = k[x,y]/\langle x^2, y^2\rangle$ the Kronecker algebra and let $G = \langle g\rangle$ be a cyclic group of order 3. Then $\Gamma \cong kQ/I$ with

$$Q = x \circlearrowright \bullet \circlearrowleft y$$

and $I = \langle x^2, y^2, xy - yx\rangle$. The algebra Γ is a kG-module algebra via the action $gx = q^{-1}x$, $gy = qy$ and $gxy = xy$ for a primitive third root of unity q. We label the character χ with $\chi(g) = q$ as $\chi = \chi_g$. Then $n_x = g^2$ and $n_y = g$. Let e_1, e_g and e_{g^2} denote the primitive idempotents in G such that $ge_1 = e_1$, $ge_g = qe_g$ and $ge_{g^2} = q^{-1}e_{g^2}$. Then the primitive idempotents are given by $1 \rtimes e_i$. We construct the quiver W as described in the previous example.

$$\begin{tikzcd}
1 \rtimes e_{g^2} \arrow[rr, shift right, "y\rtimes e_g"'] \arrow[dr, shift right, "x\rtimes e_1"'] & & 1 \rtimes e_g \arrow[ll, shift right, "x\rtimes e_{g^2}"'] \arrow[dl, shift right, "y\rtimes e_1"'] \\
& 1 \rtimes e_1 \arrow[ul, shift right, "y\rtimes e_{g^2}"'] \arrow[ur, shift right, "x\rtimes e_g"'] &
\end{tikzcd}$$

The relations are given by

$$T := \langle (y \rtimes e_i)(y \rtimes e_j), (x \rtimes e_i)(x \rtimes e_j),\ (x \rtimes e_{gj})(y \rtimes e_j) - (y \rtimes e_{g^2 j})(x \rtimes e_j) \mid i, j \in \{1, g, g^2\}\rangle.$$

By the previous example, $\Gamma \rtimes kG \cong kW/T$.

We see that R is G-transitive via $g(a \rtimes h) = a \rtimes \chi_{g^{-1}}(h)h$ or $g(a \rtimes e_h) = a \rtimes e_{gh}$ for all $a \in \Gamma$ and $g, h \in G$. With this action, G is a subgroup of $\mathrm{Aut}(R)$. Note that $1 \rtimes G \cong G$ is a subgroup of R and $k \rtimes G \cong kG$ is a subalgebra of R. We first define the following notation. Let C be an R-module. Then C is a kG-module via $g \cdot c := (1 \rtimes g)c$ for all $c \in C$ and $g \in G$. We denote by C_g the R-module with $(a \rtimes h) * c := \chi_{g^{-1}}(h)(a \rtimes h)c$ for all $c \in C$, $g, h \in G$ and $a \in \Gamma$. If C is a Γ-module, we denote by C_g the module with $t * c := g(t)c$ for all $c \in C$ and $t \in \Gamma$.

If C is a Γ-module or an R-module, then we set $S(C) := \{g \in G \mid C_g \cong C\}$ with the respective actions of G on Γ-modules and on R-modules. Let $T(C)$ be a transversal of $G/S(C)$.

We determine how R-modules or Γ-modules decompose if restricted to Γ or respectively lifted to R.

Lemma 5.3.

(1) *Let M be an R-module. Then* $M_\Gamma^R \cong \bigoplus_{g \in G} M_g$.
(2) *Let N be a Γ-module. Then* $N_\Gamma^R \cong \bigoplus_{g \in G} N_g$ *where N_g denotes the twist of N by the element* $g \in \mathrm{Aut}(\Gamma)$.
(3) *Let M be an indecomposable R-module and N an indecomposable Γ-module such that $N \mid M_\Gamma$. Then* $M_\Gamma = q\bigoplus_{g \in T(N)} N_g$, $N^R = n\bigoplus_{g \in T(M)} M_g$ *and* $qn|T(N)|\,|T(M)| = |G|$ *for some* $n, m \in \mathbb{N}$.

Proof. (1) Let $\psi_g : M_g \to M_\Gamma^R$, $m \mapsto |G|^{-1}\sum_{l \in G}\chi_g(l)(1 \rtimes l \otimes l^{-1}m)$ and let $\phi_g : M_\Gamma^R \to M_g$, $r \otimes m \mapsto g(r)m$ for all $r \in R$, $m \in M$ and $g \in G$. Let $a \in \Gamma$ and $h \in G$. Then

$$\begin{aligned}
\psi_g((a \rtimes h) * m) &= \psi_g(\chi_{g^{-1}}(h)(a \rtimes h)m) \\
&= |G|^{-1}\sum_{l \in G}\chi_{g^{-1}}(h)\chi_g(l)(1 \rtimes l \otimes l^{-1}(a \rtimes h)m) \\
&= |G|^{-1}\sum_{l \in G}\chi_{g^{-1}}(h)\chi_g(l)(1 \rtimes l)(l^{-1}(a) \rtimes 1 \otimes l^{-1}hm) \\
&= |G|^{-1}(a \rtimes 1)\sum_{l \in G}\chi_{g^{-1}}(h)\chi_g(l)(1 \rtimes l \otimes l^{-1}hm) \\
&= (a \rtimes h)|G|^{-1}\sum_{s \in G}\chi_g(s)(1 \rtimes s \otimes s^{-1}m) = (a \rtimes h)\psi_g(m),
\end{aligned}$$

by substituting $l^{-1}h = s^{-1}$, and therefore ψ_g is an R-module homomorphism.

It is clear that ϕ_g is an R-module homomorphism and $\phi_g \circ \psi_g = \mathrm{id}_{M_g}$. Let $\{m_1, \ldots, m_n\}$ be a k-basis of M. A basis of M_Γ^R is given by

$$\{(1 \rtimes l) \otimes m_i \mid 1 \leq i \leq n \text{ and } l \in G\}.$$

Using this basis we have $\psi_g(m) = \psi_h(\bar{m})$ for some $m, \bar{m} \in M$ if and only if $\chi_g(l)m = \chi_h(l)\bar{m}$ for all $l \in G$. Therefore $\psi_g(M_g) \cap \psi_h(M_h) = 0$ for $g \neq h$. Finally, by comparing dimensions we have $M_\Gamma^R \cong \bigoplus_{g\in G} M_g$.

(2) We have $N_\Gamma^R = \bigoplus_{g\in G} 1 \rtimes g \otimes N$. Furthermore, $1 \rtimes g \otimes N \cong N_{g^{-1}}$ as Γ-module, which proves the statement.

(3) Suppose Q is an indecomposable module with $Q \mid M_\Gamma$. Then Q^R and N^R are direct summands of $M_\Gamma^R = \bigoplus_{g\in G} M_g$. As $(M_g)_\Gamma \cong M_\Gamma$ for all $g \in G$, we deduce that $N \mid Q_\Gamma^R = \bigoplus_{g\in G} Q_g$. Therefore $Q \cong N_g$ for some $g \in G$. Furthermore, $(M_\Gamma)_g \cong M_\Gamma$ via the Γ-module isomorphism $\psi : M_\Gamma \to (M_\Gamma)_g$, $m \mapsto gm$ for all $m \in M$ and $g \in G$. Therefore M_Γ is G-invariant. This proves the first identity.

By the first identity, we know that all indecomposable direct summands of N^R are isomorphic to M_g for some $g \in G$. Note that N^R is G-invariant via the R-module isomorphism $\phi : N^R \to N_g^R$, $r \otimes n \mapsto g(r) \otimes n$ for all $g \in G$, $r \in R$ and $n \in N$. This map is well defined as G acts on $\Gamma \rtimes 1 \subset R$ as the identity. Therefore the second identity holds.

Finally, we compare the multiplicity of N as a direct summand of N_Γ^R. The first and second identities of (3) give a multiplicity of $n|T(M)|q$ and (2) gives multiplicity $|S(N)|$. ∎

By standard arguments we deduce the next two lemmas.

Lemma 5.4. *Every R-module M is relatively Γ-projective.*

Proof. Suppose A, B are R-modules and $h : A \to B$, $f : M \to B$ are R-module homomorphisms. Suppose there is a Γ-module homomorphism $v : M_\Gamma \to A$ such that $h \circ v = f$. Then $\bar{v} : M \to A$, $m \mapsto |G|^{-1} \sum_{g\in G} gv(g^{-1}m)$, is an R-module homomorphism. This can be seen as follows: let $t \in \Gamma$, $h \in G$; then

$$\begin{aligned}
\bar{v}((t \rtimes h)m) &= |G|^{-1} \sum_{g\in G} gv(g^{-1}(t \rtimes h)m) \\
&= |G|^{-1} \sum_{g\in G} gv(g^{-1}(t) \rtimes g^{-1}h)m) \\
&= |G|^{-1}(t \rtimes 1) \sum_{g\in G} gv(g^{-1}hm) \\
&= (t \rtimes h)|G|^{-1} \sum_{s\in G} sv(s^{-1}m) = (t \rtimes h)\bar{v}(m),
\end{aligned}$$

if we substitute $s^{-1} = g^{-1}h$. Furthermore, $\bar{v}$ satisfies $h \circ \bar{v} = f$. ∎

LEMMA 5.5 (Frobenius reciprocity). *Let V be a Γ-module and M an R-module. Then there is a bijection of vector spaces between* $\mathrm{Hom}_\Gamma(V, M_\Gamma)$ *and* $\mathrm{Hom}_R(V^R, M)$.

Proof. The bijection is given by $\psi : \mathrm{Hom}_\Gamma(V, M_\Gamma) \to \mathrm{Hom}_R(V^R, M)$ where $\psi(f)(r \otimes v) = rf(v)$ and $\phi : \mathrm{Hom}_R(V^R, M) \to \mathrm{Hom}_\Gamma(V, M_\Gamma)$ with $\phi(g)(v) = g(1 \otimes v)$ for all $r \in R$, $v \in V$, $f \in \mathrm{Hom}_\Gamma(V, M_\Gamma)$ and $g \in \mathrm{Hom}_R(V^R, M)$. ∎

Let $G(A)$ denote the free abelian group of an algebra A with free generators $[V_i]$, the representatives of the isomorphism classes of all indecomposable A-modules V_i. If $M = \bigoplus a_i V_i$ where the $a_i \geq 0$ then we write $[M] := \sum a_i [V_i]$. We denote by $(-,-)_A : G(A) \times G(A) \to k$ the bilinear form $\dim_k \mathrm{Hom}_A(-,-)$.

Let $Q : 0 \to B \to C \to D \to 0$ be an exact sequence. Then we set $[[Q]] := [B] + [D] - [C] \in G(A)$. Let $A(V_i)$ denote the Auslander–Reiten sequence starting in V_i for V_i non-projective. Furthermore, we set $X_i := [[A(V_i)]]$ for V_i non-projective and $X_i := [V_i] - [\mathrm{rad}(V_i)]$ if V_i is projective.

The first part of the next theorem is the general version of [BP, 3.4], which was only proven for group algebras.

THEOREM 5.6. *Let A be a finite-dimensional algebra over an algebraically closed field k.*

(1) *We have $([V_i], X_j) = \delta_{i,j}$. Therefore $(-,-)_A$ is non-degenerate. Furthermore, $([V_i], [[E]]) \geq 0$ for any exact sequence E.*
(2) *Suppose $Q := 0 \to C \to B \to V_j \to 0$ is an exact non-split sequence with $[[Q]] \neq X_j$. Then there is a V_i with $i \neq j$ such that $([V_i], [[Q]]) \geq 1$.*

Proof. (1) Take the almost split sequence

$$0 \to \tau(V_j) \xrightarrow{l} M_j \xrightarrow{s} V_j \to 0.$$

This gives an exact sequence

$$0 \to \mathrm{Hom}_A(V_i, \tau V_j) \to \mathrm{Hom}_A(V_i, M_j) \xrightarrow{\psi} \mathrm{Hom}_A(V_i, V_j).$$

If $i \neq j$ then by the Auslander–Reiten property, the map ψ is onto and it follows that $([V_i], X_j) = 0$. If $i = j$ then by the Auslander–Reiten property, $\mathrm{Im}(\psi)$ is the radical of $\mathrm{End}(V_i)$. Therefore

$$\begin{aligned}([V_i], X_i) &= (V_i, \tau(V_i)) + (V_i, V_i) - (V_i, M_i) = (V_i, V_i) - \dim \mathrm{Im}(\psi)\\ &= \dim \mathrm{End}(V_i)/\mathrm{rad}(\mathrm{End}(V_i)).\end{aligned}$$

Since k is algebraically closed and V_i is indecomposable, this number is equal to 1.

Let $E := 0 \to S \to T \to U \to 0$ be an exact sequence. Then $0 \to \mathrm{Hom}_A(V_i, S) \to \mathrm{Hom}_A(V_i, T) \to \mathrm{Hom}_A(V_i, U)$ is exact. Therefore $([V_i], [[E]]) \geq 0$.

(2) Let $Q := 0 \to C \xrightarrow{\delta} B \xrightarrow{\sigma} V_j \to 0$ be an exact sequence. Suppose that $[[Q]] \neq [[A(V_j)]]$. Then we get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "\delta"] \arrow[d, "g"'] & B \arrow[r, "\sigma"] \arrow[d, "h"'] & V_j \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \\
0 \arrow[r] & \tau(V_j) \arrow[r, "l"] & M^j \arrow[r, "s"] & V_j \arrow[r] & 0
\end{tikzcd}$$

where the existence of h follows from the Auslander–Reiten property since the map $\sigma : B \to V_j$ is non-split; and g is the restriction of h to C.

This diagram induces a short exact sequence

$$Z := 0 \to C \xrightarrow{\binom{\delta}{g}} B \oplus \tau(V_j) \xrightarrow{(h,l)} M_j \to 0.$$

Suppose that this sequence is split. Then there is a map $\binom{f_1}{f_2} : M_j \to B \oplus \tau(V_i)$ such that $h \circ f_1 + l \circ f_2 = \mathrm{id}_{M_j}$. Then $s \circ h f_1 = s$. As s is minimal right almost split, hf_1 is an automorphism. Moreover, $\sigma f_1 = s$, so let $g_1 : \tau(V_j) \to C$ be such that $\delta g_1 = f_1 l$. Then also gg_1 is an automorphism. So $B = M_j \oplus \mathrm{Ker}(h)$ and $C = \tau(V_j) \oplus \mathrm{Ker}(g)$. By the Snake Lemma, $\mathrm{Ker}(g) \cong \mathrm{Ker}(h)$. But then $[[Q]] = [[A(V_j)]]$. So we have a contradiction, therefore Z is non-split. Then, working in $G(A)$, we have $[[Q]] = [[Z]] + [[A(V_j)]]$ and hence $([M_j], [[Q]]) = ([M_j], [[Z]] + [[A(V_j)]]) = ([M_j], [[Z]]) \geq 1$ as the image of the map $\mathrm{Hom}(M_j, M_j) \to \mathrm{Ext}^1(M_j, C)$ induced by Z contains the non-split sequence Z. As $(-,-)_A$ is biadditive and V_j is not a summand of M_j, the second statement is proven. ∎

We can write the element $[[Q]] \in G(A)$ for any exact sequence Q ending in W as a sum in $G(A)$ of $[[Q_1]]$ and $[[Q_2]]$ for two short exact sequences Q_1 and Q_2 ending in direct summands of W.

LEMMA 5.7. *Suppose that $Q := 0 \to U \to V \xrightarrow{\pi} W \to 0$ is an exact sequence and $W = W_1 \oplus W_2$ for two non-trivial A-modules W_1 and W_2. Then there is an exact sequence Q_1 ending in W_1 and an exact sequence Q_2 ending in W_2 such that $[[Q]] = [[Q_1]] + [[Q_2]]$.*

Proof. Let $p_i : W \to W_i$ be the natural projection for $i = 1, 2$. Then

$$Q_1 := 0 \to \pi^{-1}(W_1) \to V \xrightarrow{p_2\pi} W_2 \to 0$$

and

$$Q_2 := 0 \to U \to \pi^{-1}(W_1) \xrightarrow{p_1\pi} W_1 \to 0$$

are exact sequences and $[[Q]] = [[Q_1]] + [[Q_2]]$ in $G(A)$. ∎

Furthermore, we have $([V],[[Q]]) \geq ([V],[[Q_1]])$ for any module V, as $([V],[[E]]) \geq 0$ for any exact sequence E by 5.6(1).

We can prove the next result.

THEOREM 5.8. *Let M be an indecomposable R-module and C an indecomposable Γ-module such that M is a direct summand of C^R with multiplicity n. Then $[[A(M)_\Gamma]] = n\sum_{g\in T(C)}[[A(C_g)]]$.*

Proof. We first show that $([V],[[A(M)_\Gamma]]) - n\sum_{g\in T(C)}([V],[[A(C_g)]]) = 0$ for all indecomposable Γ-modules V. Using Frobenius reciprocity 5.5, we have $([V],[[A(M)_\Gamma]]) = ([V^R],[[A(M)]])$ which is equal to the multiplicity of M as a direct summand in V^R. By 5.3 we know that $M_\Gamma \mid C_\Gamma^R = \bigoplus_{g\in G} C_g$. Therefore M is a direct summand of V^R if and only if $V \cong C_g$ for some $g \in G$. But in this case $V^R = C^R$ and therefore the multiplicity of M as a direct summand of V^R is n.

It remains to show that $[[A(M)_\Gamma]]$ is a linear combination of Auslander–Reiten sequences X_i. We have $M_\Gamma = q\sum_{g\in T(C)} C_g$. By 5.7 we know that $[[A(M)_\Gamma]]$ can be written as the sum of $[[Q_g^i]]$ where the Q_g^i are exact sequences starting in C_g for $g \in G$ for $1 \leq i \leq q$. Suppose one of them is non-split and not an Auslander–Reiten sequence X_i. By 5.6(2) there exists an indecomposable direct summand L of the middle term of some $A(C_g)$ such that $([L],[[A(M)_\Gamma]]) \geq 1$. There is no irreducible map from C_l to C_h for any $l,h \in G$, as both modules have the same dimension. Therefore $L \ncong C_l$ for all $l \in G$. By 5.6 we have $([L],[[A(C_l)]]) = 0$ for all $l \in G$, which contradicts the first part. ∎

Clearly if $A(M)$ is an Auslander–Reiten sequence and M a non-projective indecomposable module, then $\tau(M)$, M and the middle term of $A(M)$ have no direct summand in common. The same is true for the restriction of the Auslander–Reiten sequence to Γ.

LEMMA 5.9. *Let M be an indecomposable R-module and $A(M) := 0 \to \tau(M) \to X \to M \to 0$. Then the pair $\tau(M)_\Gamma$, X_Γ and the pair M_Γ, X_Γ each have no direct summand in common. If $M_g \ncong \tau(M)$, then $\tau(M)_\Gamma$ and M_Γ have no direct summand in common.*

Proof. Suppose there is an indecomposable Γ-module Q such that $Q \mid M_\Gamma$ and $Q \mid X_\Gamma$. By 5.3 there exists an indecomposable direct summand E of X and a $g \in G$ such that $E \cong M_g$. As M and M_g have the same dimension there is no irreducible map from E to M, which is a contradiction. By an analogous argument it is clear that $\tau(M)_\Gamma$ and X_Γ have no direct summand in common. Suppose now that M_Γ and $\tau(M)_\Gamma$ have a common direct summand. Then there exists a $g \in G$ such that $\tau(M) \cong M_g$, which is a contradiction. ∎

This gives the next

Corollary 5.10. *Let M be an indecomposable, non-projective R-module. Let C be an indecomposable direct summand of M_Γ with multiplicity n. Then M is a direct summand of C^R with multiplicity n. Furthermore, if N is the middle term of $A(M)$ and Q the middle term of $A(C)$, then*

$$N_\Gamma = n \bigoplus_{g\in T(C)} Q_g \quad \textit{and} \quad \tau(M)_\Gamma = n \bigoplus_{g\in T(C)} \tau(C)_g.$$

Proof. Let

$$A(M) := 0 \to \tau(M) \to \bigoplus_{1\le i\le t} d_i N_i \to M \to 0,$$
$$A(C) := 0 \to \tau(C) \to \bigoplus_{1\le i\le s} f_i Q_i \to C \to 0$$

for Q_i, N_i indecomposable and $d_i, f_i \in \mathbb{N}$. We set $Q := \bigoplus_{1\le i\le s} f_i Q_i$ and $N := \bigoplus_{1\le i\le t} d_i N_i$. Then $d_i N^i_\Gamma = b_i \sum_{g\in T(E^i)} E^i_g$, $\tau(M)_\Gamma = a \sum_{g\in T(L)} L_g$ and $M_\Gamma = q \sum_{g\in T(C)} C_g$ for some indecomposable $L, E^i \in \Gamma$-mod and $a, q, b_i \in \mathbb{N}$. Then

$$[[A(M)_\Gamma]] = q \sum_{g\in T(C)} [C_g] + a \sum_{g\in T(L)} [L_g] - \sum_{1\le i\le t} b_i \sum_{g\in T(E^i)} [E^i_g].$$

By 5.8 we also have

$$[[A(M)_\Gamma]] = n \sum_{g\in T(C)} [C_g] + n \sum_{g\in T(C)} [\tau(C)_g] - n \sum_{1\le i\le s} f_i \sum_{g\in T(C)} [Q^i_g].$$

We assume that $M_g \ncong \tau(M)$. Then by 5.9 the set

$$\{[C_g] \mid g\in T(C)\} \cup \bigcup_{1\le i\le t} \{[E^i_g] \mid g\in T(E^i)\} \cup \{[L_g] \mid g \in T(L)\}$$

is linearly independent. Similarly the set

$$\{[C_g] \mid g\in T(C)\} \cup \bigcup_{1\le i\le s} \{[Q^i_g] \mid g\in T(Q^i)\} \cup \{[\tau(C)_g] \mid g \in T(\tau(C))\}$$

is linearly independent. We can see this as follows: if $\tau(C) \cong C_h$ for some $h\in G$, then $\sum_{g\in T(C)} \tau(C)_g = \sum_{g\in T(C)} C_g$, which is a contradiction, as $[L]$ would not appear as a summand of $[[A(M)_\Gamma]]$ in the second presentation. Also $Q^i_h \ncong \tau(C)$ and $Q^i_h \ncong C$ for some $h \in G$ because there is no irreducible map between elements of the same dimension. We now compare the two presentations and use the linear independence of the indecomposable Γ-modules in $G(\Gamma)$. Then

$$(*) \qquad q = n, \quad \tau(M)_\Gamma = n \sum_{g\in T(C)} \tau(C)_g, \quad N_\Gamma = n \sum_{g\in T(C)} Q_g.$$

Assume now that $M_g \cong \tau(M)$. Then $\tau(M)_\Gamma = q\sum_{g\in T(C)} C_g$. Therefore

$$[[A(M)_\Gamma]] = 2q \sum_{g\in T(C)} [C_g] - \sum_{1\le i\le t} b_i \sum_{g\in T(E^i)} [E^i_g].$$

By 5.9 we see that the set

$$\{[C_g] \mid g\in T(C)\} \cup \bigcup_{1\le i\le t} \{[E^i_g] \mid g \in T(E^i)\}$$

is linearly independent in $G(\Gamma)$. We have $\tau(C) = C_h$ for some $h\in G$ by comparing summands in the two presentations of $[[A(M)_\Gamma]]$. Then $\sum_{g\in T(C)}\tau(C)_g = \sum_{g\in T(C)} C_g$. Therefore we have $(*)$ again. ∎

We can now investigate the relation between periodic R-modules and periodic Γ-modules.

Lemma 5.11 (periodic modules). *The indecomposable R-module M is periodic if and only if M_Γ contains a periodic direct summand.*

Proof. Let Q be an indecomposable direct summand of M_Γ. Then $M_\Gamma = n\bigoplus_{g\in T(Q)} Q_g$ and $\tau(M)_\Gamma = n\bigoplus_{g\in T(Q)} \tau(Q)_g$ by 5.10. Suppose now that Q is τ-periodic with period m. Then Q_g is τ-periodic with period m and $\tau^m(M)_\Gamma \cong M_\Gamma$, as by 5.10, τ and the restriction to Γ commute. This implies that $M^g \cong \tau^m(M)$. As G has finite order, M is periodic. Similarly, if M is τ-periodic with period m, we have $n\bigoplus_{g\in T(Q)} Q_g = M_\Gamma = \tau^m(M)_\Gamma = n\bigoplus_{g\in T(Q)} \tau^m(Q)_g$. Therefore $\tau^m(Q) \cong Q_g$ for some $g\in G$. As G has finite order, Q is τ-periodic. ∎

Next, we summarize the properties that we need for the following theorems.

Assumption 5.12. In the following we assume that Γ and R are Frobenius algebras that satisfy (C′).

The following lemma shows that this could be slightly weakened.

Lemma 5.13. *The algebra R satisfies* (C′) *if and only if the algebra Γ satisfies* (C′).

Proof. The proof is analogous to 5.11. ∎

We also have

Lemma 5.14. *Suppose $\Gamma \ne S$. Let E be a non-projective Γ-module and N a simple R-module. Then $\underline{\mathrm{Hom}}(S,E)\ne 0$ and $\underline{\mathrm{Hom}}(N,E^R)\ne 0$.*

Proof. We consider the map which maps S to $\operatorname{soc} E$. This map does not factor through Γ. Therefore $\underline{\mathrm{Hom}}(S,E)\ne 0$. We have $\mathrm{Hom}(N,E^R)\cong \mathrm{Hom}(N_\Gamma,E)\cong \mathrm{Hom}(S,E)$ by 5.5. As the restriction to Γ and lifting to R preserve projectivity, we have $\underline{\mathrm{Hom}}(N,E^R) = \underline{\mathrm{Hom}}(S,E)\ne 0$. ∎

We can now investigate the relation between the Auslander–Reiten components of $T_s(\Gamma)$ and of $T_s(R)$. We assume for the next two theorems that 5.12 is satisfied. We denote by $\mathrm{Obj}(\theta)$ the set of all indecomposable modules in an Auslander–Reiten component θ.

THEOREM 5.15.

(1) *$T_s(\Gamma)$ has a component of tree class $\tilde{D}_n$ if and only if $T_s(R)$ does.*
(2) *If $T_s(R)$ has a component of tree class $\tilde{A}_{1,2}$, then so does $T_s(\Gamma)$.*

Proof. (a) Let $\theta \cong \mathbb{Z}[\tilde{D}_n]$ or $\theta \cong \mathbb{Z}[\tilde{A}_{1,2}]$ be a component of $T_s(R)$. We denote the tree class of θ by $\mathcal{T}$. Then by 2.4, θ contains a simple module M, and a projective module P is attached to θ. Let Δ be the component in $T_s(\Gamma)$ containing the simple module $S = M_\Gamma$. Then by 5.8 we have $\mathrm{Obj}(\theta_\Gamma) \subset \bigcup_{g\in G} \mathrm{Obj}(\Delta_g)$; this is proved by induction on the distance of a module in θ to M.

As S is contained in Δ and G acts trivially on S, we have $\Delta_g = \Delta$ for all $g \in G$ and therefore $\mathrm{Obj}(\theta_\Gamma) \subset \mathrm{Obj}(\Delta)$. As P is attached to θ, Γ is attached to Δ. Then Ω induces a fixed point free automorphism on the tree class T of Δ.

By 3.3 and 3.11, $\mathrm{mod}\, R$ has a periodic module that does not lie in θ, and therefore by 5.11, $\mathrm{mod}\, \Gamma$ also contains a periodic module that does not lie in Δ. Using 5.14 we can define a subbaditive, non-zero function on the component Δ as in [ES, 3.2]. The tree class of Δ is therefore in the HPR-list (a list of trees in [HPR, p. 286]).

As we have a fixed point free automorphism operating on T, this gives either $T = A_\infty^\infty$, $T = \tilde{D}_m$ for m odd, or $T = \tilde{A}_{1,2}$.

Note that S and Γ/S do not lie in the same τ-orbit, because otherwise by 2.1 we would have $2 = l(\Gamma)$; but then S would be periodic, contrary to the fact that M is not periodic.

Twisting with $g \in G$ also induces an automorphism of finite order on Δ that fixes S and Γ/S. If $T = \tilde{D}_m$, then S and Γ/S have only one predecessor, and their predecessors do not lie in the same τ-orbit by 3.5. Therefore g acts as the identity and there are no modules contained in Δ that are twists of each other.

This means that we can embed Δ into θ using induction on the distance of a module in Δ to S. We give a sketch of how to construct this injection: We map S to M. Let $W \in \mathrm{Obj}(\Delta)$ and suppose that there is an arrow from W to S in Δ. Then by 5.10 there exists an R-module J such that J is a summand of the middle term of $A(M)$ and $W \mid J_\Gamma$. We map W to J. This gives an injection, as for two indecomposable Γ-modules W^1 and W^2 with $W^1 \mid Z_\Gamma$ and $W^2 \mid Z_\Gamma$ for an indecomposable R-module Z, we have $W^1 \cong (W^2)_g$ for some $g \in G$. So if W^1 and W^2 lie in Δ we have $W^1 \cong W^2$.

As this embedding respects τ, it induces an embedding $T \subset \mathcal{T}$. Therefore $T = \mathcal{T}$. This proves the first direction of (1), and also (2).

(b) Suppose now that $\varDelta \cong \mathbb{Z}[\tilde{D}_n]$. Then $S \in \varDelta$ and $\varGamma$ is attached to $\varDelta$. Let θ be a component of $T_s(R)$ that contains a simple module M. As in (a), we have $\mathrm{Obj}(\theta_\varGamma) \subset \bigcup_{g\in G} \mathrm{Obj}(\varDelta_g)$.

By 3.11, $\operatorname{mod}\varGamma$ contains a periodic module E that is not in $\varDelta$, and by 5.11 all direct summands of E^R are periodic and do not lie in θ. As in the first part of the proof, this shows that the tree class T of θ is in the HPR-list.

As $\bar{\alpha}(S) = \bar{\alpha}(\varGamma/S) = 1$ by 3.5, and do not have the same predecessor, g acts as the identity on $\varDelta$. That means $\mathrm{Obj}(\theta_\varGamma) \subset \mathrm{Obj}(\varDelta)$.

As in (a), we can embed $\varDelta$ into θ. As τ and the restriction to $\varGamma$ commute by 5.10, this gives an embedding of $\tilde{D}_n$ into T. Therefore $T = \tilde{D}_n$. ∎

Note that the embedding of $\varDelta$ into θ does not respect labels of arrows. The next example shows that the converse of part (2) of the previous theorem does not hold.

EXAMPLE 5.16. Let k be a field of characteristic 2, $\varGamma = kV_4$ and $R = kA_4 \cong kV_4 \rtimes kC_3$. Then $\varGamma$ has a component with tree class $\tilde{A}_{1,2}$ and R has a component with reduced graph $\tilde{A}_5$ which corresponds to a tree class A_∞^∞. By 3.3, $T_s(R)$ has no component of Euclidean tree class.

We can also show the following

THEOREM 5.17. *Suppose $T_s(\varGamma)$ has a component $\varDelta$ of tree class $\tilde{A}_{1,2}$. Then $T_s(R)$ also has a component θ of tree class $T = \tilde{A}_{1,2}$ or $T = A_\infty^\infty$. In the second case $\theta \cong \mathbb{Z}[\tilde{A}_n]$, where $(n+1)/2$ divides the order of G.*

Proof. Let M be a simple R-module and let θ be the component that contains M. As in the proof of the previous theorem, the tree class T of θ is from the HPR-list and θ is not a periodic component. Let Q be the middle term of $A(M)$. Then $Q_\varGamma = N \oplus N$ for some indecomposable $\varGamma$-module N.

Suppose first that $Q \cong L \oplus L_g$ for L an indecomposable R-module and $g \in G$ such that $L_\varGamma \cong N$ and $L_g \ncong L$. Then $\bar{\alpha}(M) = 2$ and g acts as a graph automorphism on T of finite order and is not the identity. Also this automorphism commutes with τ. Furthermore, the middle term of $A(\tau^{-1}(L))$ is $M \oplus M_{g^{-1}}$. As $M \ncong M_{g^{-1}}$ we have $\theta \cong \mathbb{Z}[\tilde{A}_n]$, where $n = 2|g| - 1$.

Suppose now that Q is indecomposable. Then either $M \oplus M$ or $M \oplus M_g$ is the non-projective summand in the middle term of $A(\tau^{-1}(Q))$. In the first case, we have $\bar{\alpha}(M) = \bar{\alpha}(Q) = 1$. Therefore $T = Q \xrightarrow{(1,2)} M$. This contradicts the HPR-list. In the second case we have $\bar{\alpha}(Q) = 2$ and $\bar{\alpha}(M) = \bar{\alpha}(M_g) = 1$. As all three are in different τ-orbits, we have $T = M \to Q \leftarrow M_g$, which is also a contradiction, as $\varDelta$ is not of finite type.

Suppose now that $Q \cong L \oplus L$. Then either $M \oplus M$ or $M \oplus M_g$ is the non-projective summand in the middle term of $A(\tau^{-1}(L))$. The first case gives $\theta \cong \mathbb{Z}[A_{1,2}]$. In the second case we have $T = \tilde{B}_2$, which we can exclude using 3.3. ∎

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