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STANDARD IDEALS IN CONVOLUTION SOBOLEV ALGEBRAS ON THE HALF-LINE

BY

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Abstract. We study the relation between standard ideals of the convolution Sobolev algebra $\mathcal{T}_+^{(n)}(t^n)$ and the convolution Beurling algebra $L^1((1+t)^n)$ on the half-line $(0,\infty)$. In particular it is proved that all closed ideals in $\mathcal{T}_+^{(n)}(t^n)$ with compact and countable hull are standard.

Introduction. For a nonnegative integer n, let $\mathcal{T}_+^{(n)}(t^n)$ denote the Banach space obtained as the completion of the space $C_c^{(\infty)}[0,\infty)$ of test functions on $[0,\infty)$ in the norm

$$\|f\| := \int_0^\infty |f^{(n)}(t)|t^n\,dt, \quad f \in C_c^{(\infty)}[0,\infty).$$

This space was introduced in [AK] to study ill-posed (abstract) Cauchy problems, and in connection with integrated semigroups and distribution semigroups. When $n = 0$, it is to be understood that $\mathcal{T}_+^{(n)}(t^n)$ coincides with the space $L^1(\mathbb{R}^+)$ of (classes of) Lebesgue integrable functions on $\mathbb{R}^+ := (0,\infty)$. In general $\mathcal{T}_+^{(n)}(t^n)$ is continuously contained in $L^1(\mathbb{R}^+)$. Moreover, $\mathcal{T}_+^{(n)}(t^n)$ is a semisimple and commutative convolution Banach algebra, a subalgebra of $L^1(\mathbb{R}^+)$, with character space equal to the set $\overline{\mathbb{C}^+}$, where $\mathbb{C}^+ := \{z \in \mathbb{C} : \Re z > 0\}$, and Gelfand transform given by the Laplace transform

$$(\mathcal{L}f)(z) = \int_0^\infty f(t)e^{-zt}\,dt, \quad f \in \mathcal{T}_+^{(n)}(t^n),\ z \in \mathbb{C}^+.$$

(Here, a Banach algebra is understood as a Banach space endowed with a jointly continuous multiplication, so that the submultiplicative norm constant need not be 1.) In fact, the range $\mathcal{L}(\mathcal{T}_+^{(n)}(t^n))$ is contained and dense in the Banach algebra $\mathfrak{A}^{(n)}(\mathbb{C}^+)$ of analytic functions $F\colon \mathbb{C}^+ \to \mathbb{C}$ such that

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$z^j F^{(j)}(z)$ extends continuously up to the boundary $i\mathbb{R}$ of $\mathbb{C}^+$ and satisfies

$$\lim_{z\to 0} z^j F^{(j)}(z) = 0 \quad (1 \le j \le n), \qquad \lim_{z\to\infty} z^j F^{(j)}(z) = 0 \quad (0 \le j \le n).$$

Endowed with pointwise multiplication and the norm

$$\|F\| := \sum_{j=0}^{n} \max_{\Re z \ge 0} |z^j F^{(j)}(z)|, \qquad F \in \mathfrak{A}^{(n)}(\mathbb{C}^+),$$

the space $\mathfrak{A}^{(n)}(\mathbb{C}^+)$ is a Banach algebra. The above facts and other Banach algebra properties of $\mathcal{T}_+^{(n)}(t^n)$ may be found in [GM], [GMR1] and [GMR2], proved even for fractional derivative versions of $\mathcal{T}_+^{(n)}(t^n)$.

The problem of describing closed ideals of $\mathcal{T}_+^{(n)}(t^n)$, as those of $L^1(\mathbb{R}^+)$, is not simple. In contrast, the case of the algebra $\mathfrak{A}^{(n)}(\mathbb{C}^+)$ is well understood: In [GW1] and [GW2] the closed ideals of $\mathfrak{A}^{(n)}(\mathbb{C}^+)$ are completely determined on the basis of the classical Korenblyum's theorem for the algebra $A^n(\mathbb{D})$ of functions on the disc which are analytic in $\mathbb{D}$ and of class $C^{(n)}$ on the boundary [K]. The results of [GW1] and [GW2], sketched below, may be considered as a first step towards understanding the structure of ideals in $\mathcal{T}_+^{(n)}(t^n)$.

Let $\mathcal{E}$ be a family of subsets of $i\mathbb{R}$, $\mathcal{E} = \{E_0, E_1, \dots, E_n\}$, such that

(a) $E_n \subseteq \cdots \subseteq E_1 \subseteq E_0$;
(b) $E_j \subseteq i\mathbb{R}\backslash\{0\}$ and E_j is relatively closed in $i\mathbb{R}\backslash\{0\}$ for all $j = 1, \dots, n$, and E_0 is a closed subset of $i\mathbb{R}$.

Let Q be an inner function on $\mathbb{C}^+$ and let F be a bounded analytic function on $\mathbb{C}^+$. We write $Q \,|\, F$ to indicate that the quotient F/Q remains analytic and bounded on $\mathbb{C}^+$. Then a (closed) ideal of $\mathfrak{A}^{(n)}(\mathbb{C}^+)$ is said to be *standard* if it is of the form

$$\mathcal{I}(Q; \mathcal{E}) := \{F \in \mathfrak{A}^{(n)}(\mathbb{C}^+) : Q \,|\, F \text{ and } F^{(j)}(z) = 0,\ \forall z \in E_j\ (0 \le j \le n)\}.$$

Given an ideal L of $\mathfrak{A}^{(n)}(\mathbb{C}^+)$, put

$$Z^k(L) := \{z \in \overline{\mathbb{C}^+} \setminus \{0\} : F^{(j)}(z) = 0 \ (1 \le j \le k)\}$$

if $1 \le k \le n$, and $Z(L) = Z^0(L) := \{z \in \overline{\mathbb{C}^+} : F(z) = 0 \text{ for all } F \in L\}$. Set $E_j(L) := Z^k(L) \cap i\mathbb{R}$ $(j = 0, 1, \dots, n)$ and $\mathcal{E}(L) = \{E_0(L), E_1(L), \dots, E_n(L)\}$. Let Q_L denote the inner factor of L, that is, the greatest inner common divisor (g.i.c.d., for short) of all nonzero functions in L (see [H]). We call $(Q_L; \mathcal{E}(L))$ the *data* of the ideal L. Then a closed ideal L is standard if and only if $L = \mathcal{I}(Q_L; \mathcal{E}(L))$. Furthermore, all closed ideals of $\mathfrak{A}^{(n)}(\mathbb{C}^+)$ are standard [GW2, Corollary 3.3].

Now, for an ideal I of $\mathcal{T}_+^{(n)}(t^n)$ and $0 \le k \le n$, set $h_0^k(I) := Z^k(L)$, $Q_I := Q_L$ and $\mathcal{E}(I) = \mathcal{E}(L)$, where $L := \overline{\mathcal{L}(I)}$. The space $\overline{\mathcal{L}(I)}$ is an ideal since

$\mathcal{L}(\mathcal{T}_+^{(n)}(t^n))$ is dense in $\mathfrak{A}^{(n)}(\mathbb{C}^+)$. We call $(Q_I; \mathcal{E}(I))$ the *data* of the ideal I, and $\mathcal{I}(Q_L; \mathcal{E}(L))$ the *standard ideal* associated with the data $(Q_I; \mathcal{E}(I))$. Then we say that a closed ideal I is *standard* when

$$I = \mathcal{L}^{-1}(\overline{\mathcal{L}(I)}).$$

The set $h(I) := h_0^0(I)$ is called the *hull* of the ideal L.

Let us consider the Beurling convolution algebra $L^1(\omega_n) := \{\varphi \in L^1(\mathbb{R}^+) : \varphi\omega_n \in L^1(\mathbb{R}^+)\}$ with the norm $\|\varphi\|_n = \int_{\mathbb{R}^+} |\varphi(t)|\omega_n(t)\,dt$, where ω_n is the weight given by $\omega_n(t) = (1+t)^n$ $(t > 0)$. For an ideal J in $L^1(\omega_n)$, let us define $h^k(J) := \{z \in \overline{\mathbb{C}^+} : (\mathcal{L}\varphi)^{(j)}(z) = 0 \ (0 \le j \le k), \forall \varphi \in J\}$ and put $Q_J := Q_{\overline{\mathcal{L}(J)}}$, $N_k(J) := h^k(J) \cap i\mathbb{R}$ for $k = 0, 1, \ldots, n$, and $\mathfrak{N}(J) = \{N_0(J), N_1(J), \ldots, N_n(J)\}$. We call $(Q_J; \mathfrak{N}(J))$ the *data* of the ideal J. Then a closed ideal J is said to be *standard* if

$$J = \{\varphi \in L^1(\omega_n) : Q_J \,|\, \mathcal{L}(\varphi) \text{ and } (\mathcal{L}\varphi)^{(k)} = 0 \text{ on } N^k(J) \ (0 \le k \le n)\}.$$

Perhaps the most general result about closed ideals in $L^1(\omega_0) = L^1(\mathbb{R}^+)$ is Gurariĭ's theorem which says that every closed ideal with countable hull is standard; see [G]. In [AZ, Theorem 3.6] a partial extension of that theorem is proven for $L^1(\omega_n)$, $n \ge 1$: Every closed ideal J of $L^1(\omega_n)$ for which the hull $h(J)$ is at most countable and *compact* is standard.

In the present note we establish a correspondence between standard ideals of $\mathcal{T}_+^{(n)}(t^n)$ and certain standard ideals of $L^1(\omega_n)$ and, as a consequence, we prove that all closed ideals of $\mathcal{T}_+^{(n)}(t^n)$ having compact and at most countable hull are standard (Theorem 2.5 below). Then we find that any closed ideal I of $\mathcal{T}_+^{(n)}(t^n)$ with empty hull is of the form $I = \mathcal{N}_a$ where $a > 0$ and $\mathcal{N}_a := \{f \in \mathcal{T}_+^{(n)}(t^n) : f = 0 \text{ a.e. on } [0, a)\}$.

1. Closed ideals in Sobolev algebras and Beurling algebras. Let $L^1(t^n)$ be the Banach space of (classes of) Lebesgue measurable functions φ on $(0, \infty)$ such that $\varphi(t)t^n$ belongs to $L^1(\mathbb{R}^+)$ with the usual norm. By the definition of $\mathcal{T}_+^{(n)}(t^n)$ the derivation operator $W^n := (-1)^n d^n/dt^n$ is a surjective isometry $W^n \colon \mathcal{T}_+^{(n)}(t^n) \to L^1(t^n)$ whose inverse operator, say W^{-n}, is given by the Weyl-type integral

$$W^{-n}\varphi(t) = \frac{1}{(n-1)!} \int_t^\infty (x-t)^{n-1}\varphi(x)\,dx \qquad (\varphi \in L^1(t^n);\ t > 0).$$

Note that $L^1(t^n)$ is not a convolution algebra, and $L^1(\omega_n)$ is formed by the elements of $L^1(t^n)$ which are integrable near 0.

In general, for $f \in \mathcal{T}_+^{(n)}(t^n)$, the values $\lim_{t\to 0^+} f^{(k)}(t)$, $k = 0, 1, \ldots, n$, need not exist. If they do, we denote them by $f^{(k)}(0)$.

Define the subspace $\mathfrak{T}_n$ of $\mathcal{T}_+^{(n)}(t^n)$ by

$$\mathfrak{T}_n := \{f \in W^{-n}L^1(\omega_n) : f^{(k)}(0) = 0\,(0 \le k \le n-1)\}.$$

Let $\mathcal{S}$ be the space of all restrictions to $[0,\infty)$ of members of the Schwartz test space $\mathcal{S}(\mathbb{R})$.

Lemma 1.1. *Let f in $\mathcal{T}_+^{(n)}(t^n)$ be such that $f^{(j)}(0)$ exists up to order k, with $k \le n-1$, and $f(0) = \cdots = f^{(k)}(0) = 0$. Then*

$$(f * g)^{(j)} = f^{(j)} * g$$

for every $1 \le j \le k+1$ and $g \in \mathcal{S}$.

Proof. For g and f as in the statement and $x > 0$,

$$g * f(x) = \int_0^x g(y)f(x-y)\,dy,$$

whence $(g*f)' = f(0)g + g*f' = g*f'$. Now the conclusion follows by simple induction. ∎

Proposition 1.2. *The space $\mathfrak{T}_n$ has the following properties:*

(i) $\mathfrak{T}_n * \mathcal{S} \subseteq \mathfrak{T}_n$.
(ii) $W^n(f*g) = (W^n f) * g$ *for all $f \in \mathfrak{T}_n$ and $g \in \mathcal{S}$.*
(iii) $I \cap \mathfrak{T}_n$ *is dense in I for every closed ideal I of $\mathcal{T}_+^{(n)}(t^n)$; in particular $\mathfrak{T}_n$ is dense in $\mathcal{T}_+^{(n)}(t^n)$.*

Proof. Properties (i) and (ii) are straightforward consequences of Lemma 1.1. To prove (iii) we first show that $\mathfrak{T}_n$ is dense in $\mathcal{T}_+^{(n)}(t^n)$. For $a > 0$, let $\mathcal{N}_a$ be as at the end of the Introduction. Put

$$\mathfrak{D} := \bigcup_{a>0} \mathcal{N}_a.$$

Clearly, $\mathfrak{D} \subseteq \mathfrak{T}_n$ and $\mathfrak{D}$ is an ideal of $\mathcal{T}_+^{(n)}(t^n)$. Moreover, $\mathcal{T}_+^{(n)}(t^n)$ possesses bounded approximate identities (b.a.i., for short) of the form $\psi_\varepsilon(x) = \varepsilon^{-1}\psi(\varepsilon^{-1}x)$ $(x > 0,\ \varepsilon > 0)$, where one can take ψ in $C_c^{(n)}(0,\infty) \subseteq \mathfrak{D}$; see for instance [GMR1, Proposition 2.3]. Hence $\lim_{\varepsilon\to 0^+} f * \psi_\varepsilon = f$ for every $f \in \mathcal{T}_+^{(n)}(t^n)$, with $f * \psi_\varepsilon \in \mathfrak{D}$. In particular $\mathfrak{T}_n$ is a dense subspace of $\mathcal{T}_+^{(n)}(t^n)$.

Let now I be any closed ideal of $\mathcal{T}_+^{(n)}(t^n)$. Take a b.a.i. ψ_ε in $\mathfrak{D}$ as above. Since for each $g \in I$ we have $g * \psi_\varepsilon \in I \cap \mathfrak{D} \subset I \cap \mathfrak{T}_n$ and $\lim_{\varepsilon\to 0^+} g * \psi_\varepsilon = g$ it follows that $I \cap \mathfrak{T}_n$ is dense in I and the proof is complete. ∎

Next, we consider the companion set of $\mathfrak{T}_n$ in $L^1(\omega_n)$. Define

$$\mathfrak{M}_n := \Big\{\varphi \in L^1(\omega_n) : \int_0^\infty x^k\varphi(x)\,dx = 0 \ (0 \le k \le n-1)\Big\}.$$

PROPOSITION 1.3. *The space $\mathfrak{M}_n$ has the following properties:*

(a) $\mathfrak{M}_n = W^n\mathfrak{T}_n$ *and therefore* $\mathfrak{M}_n$ *is a dense subspace of* $L^1(t^n)$.
(b) $\mathfrak{M}_n$ *is a closed ideal of* $L^1(\omega_n)$.

Proof. (a) Let $f \in \mathcal{T}_+^{(n)}(t^n)$ and suppose that $f = W^{-n}\varphi$ with $\varphi \in L^1(\omega_n)$. Then

$$f(t) = \int_t^\infty \int_{t_{n-1}}^\infty \ldots \int_{t_1}^\infty \varphi(t_0)\,dt_0\,dt_1 \ldots dt_{n-1} \qquad (t > 0).$$

Hence, for $0 \le m \le n-1$, we have

$$\begin{aligned} f^{(m)}(t) &= (-1)^m \int_t^\infty \int_{t_{n-m-1}}^\infty \ldots \int_{t_1}^\infty \varphi(t_0)\,dt_0\,dt_1 \ldots dt_{n-m-1} \\ &= \frac{(-1)^m}{(n-m-1)!}\int_t^\infty (x-t)^{n-m-1}\varphi(t_0)\,dt_0 \qquad (t>0). \end{aligned}$$

It follows that $f \in \mathfrak{T}_n$ if and only if $\varphi \in \mathfrak{M}_n$. Equivalently $\mathfrak{M}_n = W^n\mathfrak{T}_n$.

(b) The functional $\varphi \mapsto \int_0^\infty x^k\varphi(x)\,dx$ is continuous on $L^1(\omega_n)$ for every $0 \le k \le n-1$, so $\mathfrak{M}_n$ is a closed subspace of $L^1(\omega_n)$. Also, by (i) and (ii) of Proposition 1.2 and (a) above we have $\mathfrak{M}_n * \mathcal{S} \subseteq \mathfrak{M}_n$. By density we infer that $\mathfrak{M}_n$ is an ideal of $L^1(\omega_n)$, too. ∎

The following result is central to this paper.

THEOREM 1.4. *For every closed ideal I in $\mathcal{T}_+^{(n)}(t^n)$, the subspace*

$$\Omega(I) := W^n(I \cap \mathfrak{T}_n) = (W^nI) \cap \mathfrak{M}_n$$

is a closed ideal of $L^1(\omega_n)$.

Proof. The operator $W^{-n}\colon \mathfrak{M}_n \to \mathfrak{T}_n$ is bijective, and it is continuous if we endow $\mathfrak{M}_n$ with the $L^1(\omega_n)$-norm and $\mathfrak{T}_n$ with the relative topology induced by the one of $\mathcal{T}_+^{(n)}(t^n)$. For the last topology the ideal $I \cap \mathfrak{T}_n$ is closed in $\mathfrak{T}_n$ because I is closed. Now, $W^n(I \cap \mathfrak{T}_n)$ is the inverse image of $I \cap \mathfrak{T}_n$ under W^{-n}, so it is closed in $\mathfrak{M}_n$ and consequently in $L^1(\omega_n)$ by Proposition 1.3(b).

By Proposition 1.2(i), and since I is an ideal, $I \cap \mathfrak{T}_n$ is invariant under convolution with $\mathcal{S}$. From Proposition 1.2(ii) it follows that $\Omega(I)$ is also $\mathcal{S}$-invariant for convolution and so an ideal of $L^1(\omega_n)$. ∎

According to the theorem, the mapping $\Omega\colon I \mapsto \Omega(I)$ defines a correspondence between closed ideals of $\mathcal{T}_+^{(n)}(t^n)$ and closed ideals of $L^1(\omega_n)$ contained in $\mathfrak{M}_n$. Since $W^{-n}(\Omega(I)) = I \cap \mathfrak{T}_n$ and this ideal is dense in I, Ω is injective. In the next section we use Ω to establish a relationship between standard ideals of $\mathcal{T}_+^{(n)}(t^n)$ and $L^1(\omega_n)$.

2. Standard ideals in Sobolev algebras. To each closed ideal I in $\mathcal{T}_+^{(n)}(t^n)$ we can associate the closed ideals $L = \overline{\mathcal{L}(I)}$ in $\mathfrak{A}^{(n)}(\mathbb{C}^+)$ and $J = \Omega(I)$ in $L^1(\omega_n)$ with respective data $(Q_I; \mathcal{E}(I))$ and $(\mathfrak{Q}_I; \mathfrak{N}(I))$ where $\mathfrak{Q}_I := Q_{\Omega(I)}$, $\mathfrak{N}(I) := \mathfrak{N}(\Omega(I))$ (see Introduction). In order to compare these two sets of data, we need a couple of lemmas. The first one tells us that the Laplace transform intertwines the operator W^n and multiplication by z^n.

LEMMA 2.1. *For every $f \in \mathfrak{T}_n$,*

$$\mathcal{L}(W^n f)(z) = (-1)^n z^n (\mathcal{L} f)(z), \quad \Re z \geq 0.$$

Proof. For $0 \leq j \leq n-1$, $f^{(j)}(x) = \int_x^\infty f^{(j+1)}(y)\,dy$ with

$$\int_1^\infty |f^{(j+1)}(y)|\,dy \leq \int_1^\infty |f^{(j+1)}(y) y^{j+1}|\,dy < \infty.$$

Hence, $\lim_{x\to\infty} f^{(j)}(x) = 0$. Then the statement follows from the equality $\mathcal{L}(W^n f)(z) = (-1)^n \int_0^\infty f^{(n)}(t) e^{-zt}dt$ and integration by parts. ∎

Let F, G be complex functions on $\overline{\mathbb{C}^+} \setminus \{0\}$ such that $F(z) = z^{-n} G(z)$.

LEMMA 2.2. *For F, G as above,*

$$F \in \mathfrak{A}^{(n)}(\mathbb{C}^+) \Leftrightarrow G \in C^{(n)}(\overline{\mathbb{C}^+}) \cap \mathrm{Hol}(\mathbb{C}^+),$$

with

$$G^{(j)}(0) = 0 \quad (0 \leq j \leq n-1), \qquad \lim_{z\to\infty} z^{j-n} G^{(j)}(z) = 0 \quad (0 \leq j \leq n).$$

In this case, $G^{(n)}(0) = n!\,F(0)$.

Proof. Suppose $F \in \mathfrak{A}^{(n)}(\mathbb{C}^+)$. For $0 \leq m \leq n$,

$$\begin{aligned} G^{(m)}(z) = (z^n F)^{(m)}(z) &= \sum_{k=0}^m \binom{m}{k} (z^n)^{(k)} F^{(m-k)}(z) \\ &= \sum_{k=0}^m \binom{m}{k}\binom{n}{k} k!\; z^{n-k} F^{(m-k)}(z), \end{aligned}$$

from which we see that $G \in C^{(n)}(\overline{\mathbb{C}^+}) \cap \mathrm{Hol}(\mathbb{C}^+)$, with $G^{(m)}(0) = 0$ if $0 \leq m \leq n-1$, $G^{(n)}(0) = n!\; F(0)$, and $\lim_{z\to\infty} z^{m-n} G^{(m)}(z) = 0$ $(0 \leq m \leq n)$.

Conversely, assume now that G is as above. Then, for $0 \leq m \leq n$,

$$\lim_{z\to 0} z^{m-n} G^{(m)}(z) = G^{(n)}(0)/(n-m)!. \tag{1}$$

This is a consequence of the formula

$$G^{(m)}(z) = \frac{1}{(n-m-1)!} \int\limits_{[0,z]} (z-\lambda)^{n-m-1} G^{(n)}(\lambda)\, d\lambda,$$

which is valid for $0 \le k \le n-1$ and $z \in \overline{\mathbb{C}^+} \setminus \{0\}$, and holds because $G^{(k)}(0) = 0$ if $0 \le k \le n-1$.

Then for $F(z) = z^{-n} G(z)$ and $1 \le m \le n$ we have

$$z^m F^{(m)}(z) = \sum_{k=0}^{m} \binom{m}{k} \binom{-n}{k} k!\, z^{m-k-n} G^{(m-k)}(z) \tag{2}$$

whence, by (1),

$$\begin{aligned}\lim_{z\to 0} z^m F^{(m)}(z) &= \sum_{k=0}^{m} \binom{m}{k}\binom{-n}{k} \frac{k!}{(n-m+k)!} G^{(n)}(0) \\ &= \frac{m!}{n!} G^{(n)}(0) \sum_{k=0}^{m} \binom{n}{m-k}\binom{-n}{k} = 0.\end{aligned}$$

The last equality is well known. However, for completeness, let us point out that it can be shown by noticing that if $|z| < 1$ and $c_n(k) = \chi_{[1,n]}(k)$ where $\chi_{[1,n]}$ is the indicator function of $[1,n]$, then

$$\begin{aligned} 1 = (1+z)^n (1+z)^{-n} &= \Big(\sum_{k=0}^{n} \binom{n}{k} z^k\Big)\Big(\sum_{k=0}^{\infty} \binom{-n}{k} z^k\Big) \\ &= \sum_{k=0}^{\infty}\sum_{k=0}^{m} c_n(k) \binom{n}{k}\binom{-n}{m-k} z^m, \end{aligned}$$

whence in particular

$$\sum_{k=0}^{m} \binom{n}{k}\binom{-n}{m-k} = 0 \quad \text{if } 1 \le m \le n.$$

Finally, from formula (2) it follows readily that $\lim_{z\to\infty} z^m F^{(m)}(z) = 0$ for all $0 \le m \le n$, and so $f \in \mathfrak{A}^{(n)}(\mathbb{C}^+)$. ∎

Let I be a closed ideal in $\mathcal{T}_+^{(n)}(t^n)$. Let consider the two (closed) ideals $\overline{\mathcal{L}(I)}$ in $\mathfrak{A}^{(n)}(\mathbb{C}^+)$ and $\Omega(I)$ in $L^1(\omega_n)$, with data $(Q_I; \mathcal{E}(I))$ and $(\mathfrak{Q}_I; \mathfrak{N}(I))$, respectively, as in Section 1. Next, we establish a relation between $(Q_I; \mathcal{E}(I))$ and $(\mathfrak{Q}_I; \mathfrak{N}(I))$. Note that the g.c.i.d. of a family $\mathcal{F}$ of functions in $\mathfrak{A}^{(n)}(\mathbb{C}^+)$ is the same as the g.c.i.d. of the closure $\overline{\mathcal{F}}$ since norm convergence in $\mathfrak{A}^{(n)}(\mathbb{C}^+)$ implies uniform convergence on $\overline{\mathbb{C}^+}$.

PROPOSITION 2.3. *Let I, $(Q_I; \mathcal{E}(I))$, $(\mathfrak{Q}_I; \mathfrak{N}(I))$ be as above, with $\mathcal{E}(I) = \{E_0, E_1, \dots, E_n\}$ and $\mathfrak{N}(I) = \{N_0, N_1, \dots, N_n\}$. Then*

(i) $Q_I = \mathfrak{Q}_I$.
(ii) $0 \in E_0 \Rightarrow N_j = E_j \cup \{0\}$ *for all* $0 \le j \le n$.
(iii) $0 \notin E_0 \Rightarrow N_j = E_j \cup \{0\}$ *for all* $0 \le j \le n-1$, *and* $N_n = E_n$.

Proof. Note that

$$\overline{\mathfrak{L}(I \cap \mathfrak{T}_n)} = \overline{\mathfrak{L}(I)} \tag{3}$$

since $I \cap \mathfrak{T}_n$ is dense in I. Also, from Lemma 2.1 we have in particular

$$\mathfrak{L}(\Omega(I)) = z^n \mathfrak{L}(I \cap \mathfrak{T}_n). \tag{4}$$

Then (i) follows immediately from (3), (4) and the remark prior to the proposition, since $z \mapsto z^n$ is an outer function on $\mathbb{C}^+$; see [H].

Now, let $F = \mathfrak{L}(f)$ and $G = \mathfrak{L}(\varphi)$, for $f \in I \cap \mathfrak{T}_n$ and $\varphi \in \Omega(I)$, satisfy $G(z) = z^n F(z)$. For $0 \le j \le n$ and $\lambda \ne 0$,

$$\begin{aligned}\lambda \in E_j \;&\Leftrightarrow\; F^{(k)}(\lambda) = 0 \;\forall F \in \overline{\mathfrak{L}(I)} \;(0 \le k \le j)\\ &\Leftrightarrow\; F^{(k)}(\lambda) = 0 \;\forall F \in \mathfrak{L}(I \cap \mathfrak{T}_n) \;(0 \le k \le j)\\ &\Leftrightarrow\; G^{(k)}(\lambda) = 0 \;\forall G \in \mathfrak{L}(\Omega(I)) \;(0 \le k \le j) \;\Leftrightarrow\; \lambda \in N_j,\end{aligned}$$

where the next-to-last equivalence is due to Lemma 2.2. Moreover, it is clear that $G^{(k)}(0) = 0$ for every $0 \le k \le n-1$, and since $0 \in E_0$ we have $G^{(n)}(0) = 0$ by Lemma 2.2 again. In conclusion, $0 \in N_j$ for all $0 \le j \le n$. This proves part (ii).

For (iii), if $\lambda \ne 0$ then $\lambda \in N_j \Leftrightarrow \lambda \in E_j$ for every $0 \le j \le n$, as in (ii) above. Also, $0 \in N_j$ for all $0 \le j \le n-1$; but $G^{(n)}(0) = n!\,F(0) \ne 0$ because $0 \notin E_0$. Hence $N_n = E_n$ and the proof is complete. ■

We are in a position to prove the main result of this section. Given a closed ideal J in $L^1(\omega_n)$ we say that J is *standard in* $\mathfrak{M}_n$ if it is the intersection of $\mathfrak{M}_n$ with a standard ideal of $L^1(\omega_n)$.

THEOREM 2.4. *Let I be a closed ideal of $\mathcal{T}_+^{(n)}(t^n)$. Then I is standard in $\mathcal{T}_+^{(n)}(t^n)$ if and only if $\Omega(I)$ is standard in $\mathfrak{M}_n$.*

Proof. Suppose that I is standard in $\mathcal{T}_+^{(n)}(t^n)$. Thus there exists data $(Q, \mathcal{E})$ with $\mathcal{E} = \{E_0, E_1, \ldots, E_n\}$ such that

$$I \cap \mathfrak{T}_n = \{f \in \mathfrak{T}_n : Q \,|\, \mathfrak{L}f \text{ and } (\mathfrak{L}f)^{(k)} = 0 \text{ on } E_k \ (0 \le k \le n)\}.$$

Hence

$$\Omega(I) = \{W^n f \in \mathfrak{M}_n : Q \mid \mathfrak{L}f \text{ and } (\mathfrak{L}f)^{(k)} = 0 \text{ on } E_k \ (0 \le k \le n)\}.$$

Let N_k be related with E_k as in Proposition 2.3. Then it follows readily by Lemma 2.1, Lemma 2.2 and Proposition 2.3 that

$$\Omega(I) = \{\varphi \in \mathfrak{M}_n : Q \,|\, \mathfrak{L}\varphi \text{ and } (\mathfrak{L}\varphi)^{(k)} = 0 \text{ on } N_k \ (0 \le k \le n)\}.$$

This means that $\Omega(I)$ is standard in $\mathfrak{M}_n$.

Conversely, suppose that $\Omega(I)$ is standard in $\mathfrak{M}_n$ and set $\widetilde{I} = \mathfrak{L}^{-1}(\overline{\mathfrak{L}(I)})$. Then $L := \mathfrak{L}(\widetilde{I}) \subseteq \overline{\mathfrak{L}(I)} =: \widetilde{L}$, and $\mathfrak{L}(I) \subseteq \mathfrak{L}(\widetilde{I})$ since $I \subseteq \widetilde{I}$. Thus we have $L = \widetilde{L}$ and so the data of $\Omega(I)$ and $\Omega(\widetilde{I})$ in $\mathfrak{M}_n$ coincide. Since $\Omega(I)$ is standard it means that $\Omega(\widetilde{I}) \subseteq \Omega(I)$. Also, $\Omega(I) \subseteq \Omega(\widetilde{I})$ since $I \subseteq \widetilde{I}$. Hence $\Omega(I) = \Omega(\widetilde{I})$. Finally note that, for each closed ideal H in $\mathcal{T}_+^{(n)}(t^n)$, we have $W^{-n}(\Omega(H)) = H \cap \mathfrak{T}_n$ and therefore $H = \overline{H \cap \mathfrak{T}_n} = \overline{W^{-n}(\Omega(H))}$. Applying this identity to $H = I$ and $H = \widetilde{I}$ we get

$$\widetilde{I} = \overline{W^{-n}\Omega(\widetilde{I})} = \overline{W^{-n}\Omega(I)} = I,$$

that is, $I = \mathfrak{L}^{-1}(\overline{\mathfrak{L}(I)})$, so I is standard, as we wanted to show. ∎

Remark. The above is an interesting characterization, even though what we really need to prove the result below is only the fact that if $\Omega(I)$ is standard then I is standard.

Theorem 2.5. *Let I be a closed ideal in $\mathcal{T}_+^{(n)}(t^n)$ with hull $h(I)$ compact and at most countable. Then I is standard.*

Proof. Let Q_I be the greatest inner common divisor of the ideal I. Using previous notation, we have

$$h_0(I) = [Z(\mathfrak{Q}_I) \cap \overline{\mathbb{C}^+}] \cup E_0(I).$$

Hence, from Proposition 2.3 we deduce that the hull $h(\Omega(I))$ in $L^1(\omega_n)$,

$$\begin{aligned} h(\Omega(I)) &= [Z(\mathfrak{Q}_I) \cap \overline{\mathbb{C}^+}] \cup N_0(I) \\ &= [Z(Q_I) \cap \overline{\mathbb{C}^+}] \cup E_0(I) \cup \{0\}, \end{aligned}$$

is compact and at most countable. So I is standard by [AZ, Theorem 3.6]. ∎

3. Ideals with empty hull. Let I be a not necessarily closed ideal of $L^1(\mathbb{R}^+)$. Define $\gamma(I) := \inf\{\gamma(g) : g \in I\}$ where $\gamma(g) = \inf(\operatorname{supp} g)$ $(g \in I)$. Then the celebrated Nyman's theorem says that every ideal of $L^1(\mathbb{R}^+)$ such that $h(I) = \emptyset$ and $\gamma(I) = 0$ must be dense in $L^1(\mathbb{R}^+)$; see [D, p. 197], for instance. As a corollary, for a closed ideal I in $L^1(\mathbb{R}^+)$ with $h(I) = \emptyset$ there exists $a > 0$ such that $I = M_a$, where $M_a = \{g \in L^1(\mathbb{R}^+) : \gamma(g) \geq a\}$. This follows from the fact that the translation

$$\delta_a \colon g \mapsto \delta_a * g, L^1(\mathbb{R}^+) \to M_a$$

is bijective and continuous with inverse δ_{-a}. In fact, if $h(I) = \emptyset$ with $\gamma(I) = a$ then $J = \delta_{-a} * I$ is a closed ideal of $L^1(\mathbb{R}^+)$ such that $h(J) = \emptyset$ and $\gamma(J) = 0$. Hence, by Nyman's theorem, $J = L^1(\mathbb{R}^+)$. Finally, $I = \delta_a * J = \delta_a * L^1(\mathbb{R}^+) = M_a$.

Although Nyman's theorem has recently been extended to the Sobolev algebra $\mathcal{T}_+^{(n)}(t^n)$ (even for fractional derivation), the above argument does not work in this case because $\mathcal{T}_+^{(n)}(t^n)$ is not invariant under translations; see [GMR1] for both results.

Here, we apply Theorem 2.5 to show that all closed ideals in $\mathcal{T}_+^{(n)}(t^n)$ having empty hull are of the form $\mathcal{N}_a := M_a \cap \mathcal{T}_+^{(n)}(t^n)$.

LEMMA 3.1. *If $f \in \mathcal{T}_+^{(n)}(t^n)$ is such that $\gamma(f) \geq a > 0$ then $\delta_{-a} * f \in \mathcal{T}_+^{(n)}(t^n)$. Consequently,*

$$\mathcal{N}_a = (\delta_a * \mathcal{T}_+^{(n)}(t^n)) \cap \mathcal{T}_+^{(n)}(t^n).$$

Proof. Set $g(x) := f(x+a) = \delta_{-a} * f(x)$ for $x > 0$. We know that there exists $F \in L^1(t^n)$ such that

$$f(x) = \frac{1}{(n-1)!} \int_0^\infty (y-x)^{(n-1)} F(y)\, dy \quad (x > 0).$$

Therefore

$$\begin{aligned} g(x) &= \frac{1}{(n-1)!} \int_{x+a}^\infty (y-x-a)^{n-1} F(y)\, dy \\ &= \frac{1}{(n-1)!} \int_x^\infty (u-x)^{n-1} F(u+a)\, du \quad (x > 0), \end{aligned}$$

with $F(\,\cdot + a) \in L^1(t^n)$ since $\int_0^\infty |F(u+a)| u^n\, du \leq \int_0^\infty |F(t)| t^n\, dt < \infty$. So $g \in \mathcal{T}_+^{(n)}(t^n)$.

Set now $T_a = (\delta_a * \mathcal{T}_+^{(n)}(t^n)) \cap \mathcal{T}_+^{(n)}(t^n)$. If $f \in \mathcal{N}_a \subseteq \mathcal{T}_+^{(n)}(t^n)$ then $f = \delta_a * (\delta_{-a} * f)$ with $\delta_{-a} * f \in \mathcal{T}_+^{(n)}(t^n)$, whence $f \in T_a$. Conversely, if $f \in T_a$ then $f = \delta_a * g$ with $f, g \in \mathcal{T}_+^{(n)}(t^n)$, and $\gamma(f) = a + \gamma(g) \geq a$ by Titchmarsh's theorem (see a proof in [D, p. 188]). This means that $f \in \mathcal{N}_a$, and the proof is complete. ∎

THEOREM 3.2. *Let I be a closed ideal of $\mathcal{T}_+^{(n)}(t^n)$ such that $h(I) = \emptyset$. Then $I = \mathcal{N}_a$ for some $a \geq 0$.*

Proof. By [GW2, Corollary 3.3], the closed ideal $\overline{\mathcal{L}(I)}$ is standard in $\mathfrak{A}^{(n)}(\mathbb{C}^+)$ and so $\overline{\mathcal{L}(I)} = Q\mathfrak{A}^{(n)}(\mathbb{C}^+)$ where Q is the g.i.c.d. of $\overline{\mathcal{L}(I)}$. Since $h(I) = \emptyset$ it follows that Q is an inner function on $\mathbb{C}^+$ without zeros and so $Q(z) = e^{-bz}$ for some $b \geq 0$. On the other hand, $I = \delta_a * (\delta_{-a} * I)$ for $a := \gamma(I) \geq 0$. Hence, $\mathcal{L}(I) = e^{-az}\mathcal{L}(\delta_{-a} * I)$ where the ideal $\mathcal{L}(\delta_{-a} * I)$ has no common zeros in $\mathbb{C}^+$. Thus $b = a$.

In the following, the symbol $\mathcal{L}^{-1}$ refers to preimages in $\mathcal{T}_+^{(n)}(t^n)$.

Since $\gamma(I) = a$ we have $I \subseteq \mathcal{N}_a = (\delta_a * \mathcal{T}_+^{(n)}(t^n)) \cap \mathcal{T}_+^{(n)}(t^n)$. For the converse inclusion, note that $\mathcal{T}_+^{(n)}(t^n) = \mathcal{L}^{-1}(\mathfrak{A}^{(n)}(\mathbb{C}^+))$ and so

$$\begin{aligned}\mathcal{N}_a &= (\delta_a * \mathcal{T}_+^{(n)}(t^n)) \cap \mathcal{T}_+^{(n)}(t^n)\\ &= \big(\delta_a * \mathcal{L}^{-1}(\mathfrak{A}^{(n)}(\mathbb{C}^+))\big) \cap \mathcal{T}_+^{(n)}(t^n) \subseteq \mathcal{L}^{-1}\big(\mathcal{L}(\delta_a * \mathcal{L}^{-1}(\mathfrak{A}^{(n)}(\mathbb{C}^+)))\big)\\ &\subseteq \mathcal{L}^{-1}(e^{-az}\mathfrak{A}^{(n)}(\mathbb{C}^+)) = \mathcal{L}^{-1}(\overline{\mathcal{L}(I)}) = I\end{aligned}$$

where the last equality follows because I is standard. ■

Let us notice that the above statement includes the case $a = 0$, which we next write down explicitly because it gives a proof of Nyman' theorem for $\mathcal{T}_+^{(n)}(t^n)$ different from the one given in [GMR2].

Corollary 3.3. *Let I be an ideal in $\mathcal{T}_+^{(n)}(t^n)$ such that $h(I) = \emptyset$ and $\gamma(I) = 0$. Then I is dense in $\mathcal{T}_+^{(n)}(t^n)$.*

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