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ON RESTRICTIONS OF INDECOMPOSABLES OF TAME ALGEBRAS

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Abstract. We continue the study of ditalgebras, an acronym for "differential tensor algebras", and of their categories of modules. We examine extension/restriction interactions between module categories over a ditalgebra and a proper subditalgebra. As an application, we prove a result on representations of finite-dimensional tame algebras Λ over an algebraically closed field, which gives information on the extension/restriction interaction between module categories of some special algebras Λ_0, called convex in Λ.

1. Introduction. In the representation theory of finite-dimensional algebras, the notions of tame and wild representation type play a central role. An algebra is called wild if the question of classifying its indecomposable modules contains the problem of finding a normal form for pairs of square matrices over a field under simultaneous conjugation by a non-singular matrix. It is tame if the pairwise non-isomorphic indecomposable modules in each dimension can be parametrized by a finite number of parameters.

Matrix reduction techniques have been successfully used to enrich the representation theory of algebras, notably in the proof of fundamental results such as *Drozd's tame and wild theorem* (which states that, over an algebraically closed field, any finite-dimensional algebra is either tame or wild, but not both, see [9]) and Crawley-Boevey's theorems on tame algebras (see [7] and [8]). These techniques were introduced by the Kiev School in the representation theory of algebras (see [10]), in an attempt to formalize and generalize matrix problems methods. Here we follow the formulation of this methodology described in [6], which uses the language of ditalgebras, and we use these lecture notes as a general reference for this work. We refer to Chapter XIX of [11] for background on tame and wild finite-dimensional algebras.

Throughout this paper, we have a fixed base field k. All our algebras are associative k-algebras with unit element, Λ-Mod denotes the category of (left) Λ-modules, and Λ-mod denotes the full subcategory of Λ-Mod formed

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by the finite-dimensional Λ-modules. Right Λ-modules are identified with left modules over the opposite algebra Λ^{op}. The functor $D = \mathrm{Hom}_k(-,k) : \Lambda\text{-Mod} \to \Lambda^{\mathrm{op}}\text{-Mod}$ restricts to a duality $D : \Lambda\text{-mod} \to \Lambda^{\mathrm{op}}\text{-mod}$ with $D^2 \cong \mathrm{Id}$.

Consider the following well known situation (see for instance [1, I.6] and, for the corresponding situation in the context of categories, [2] and [3]). Let Λ be a finite-dimensional algebra and take any idempotent e_0 of Λ. If we set $\Lambda_0 := e_0 \Lambda e_0$, we have the standard restriction functor $\rho : \Lambda\text{-Mod} \to \Lambda_0\text{-Mod}$, where $\rho(M) = e_0 M$ for any $M \in \Lambda\text{-Mod}$. It has a left adjoint functor $\mathrm{tens} = \Lambda e_0 \otimes_{\Lambda_0} -$ and a right adjoint functor $\mathrm{hom} = \mathrm{Hom}_{\Lambda_0}(e_0\Lambda, -)$.

The functors tens and hom are both full and faithful, and they are dual to each other. More precisely, the following square commutes up to isomorphism:

$$\begin{array}{ccc} \Lambda\text{-Mod} & \xrightarrow{D} & \Lambda^{\mathrm{op}}\text{-Mod} \\ {\scriptstyle \mathrm{tens}}\uparrow & & \uparrow{\scriptstyle \mathrm{hom}} \\ \Lambda_0\text{-Mod} & \xrightarrow{D_0} & \Lambda_0^{\mathrm{op}}\text{-Mod} \end{array}$$

where $D := \mathrm{Hom}_k(-,k)$ and D_0 is the corresponding functor for Λ_0. Indeed, if $M \in \Lambda_0\text{-Mod}$, we have a natural isomorphism

$$\begin{aligned} \mathrm{hom}\, D_0(M) &= \mathrm{Hom}_{\Lambda_0^{\mathrm{op}}}(e_0\Lambda^{\mathrm{op}}, \mathrm{Hom}_k(M,k)) \cong \mathrm{Hom}_k(M \otimes_{\Lambda_0^{\mathrm{op}}} e_0\Lambda^{\mathrm{op}}, k) \\ &\cong \mathrm{Hom}_k(\Lambda e_0 \otimes_{\Lambda_0} M, k) = D\,\mathrm{tens}(M) \end{aligned}$$

determined by the isomorphism $\Lambda e_0 \otimes_{\Lambda_0} M \cong M \otimes_{\Lambda_0^{\mathrm{op}}} e_0 \Lambda^{\mathrm{op}}$ of left Λ-modules, which is natural in M.

In this work, we will assume furthermore that Λ_0 is a convex algebra in Λ in the following sense. The notation in the following definitions will be kept throughout this paper.

Definition 1.1. Let Λ be a finite-dimensional basic algebra over the field k and assume that there is a semisimple subalgebra S of Λ such that Λ admits the S-S-bimodule decomposition $\Lambda = S \oplus \mathrm{rad}\,\Lambda$. Consider a decomposition $1 = \sum_{e \in E} e$ of the unit element as a sum of central primitive orthogonal idempotents of S and let E_0 be a non-empty subset of E. Then E_0 is called:

- *convex* if $e''\Lambda e'\Lambda e \neq 0$ with $e'', e \in E_0$ and $e' \in E$ implies $e' \in E_0$;
- *final* if $e'\Lambda e \neq 0$ with $e' \in E$ and $e \in E_0$ implies $e' \in E_0$;
- *cofinal* if $e'\Lambda e \neq 0$ with $e \in E$ and $e' \in E_0$ implies $e \in E_0$.

Notice that E_0 is convex whenever it is final or cofinal. Given a convex subset E_0 of E, we are interested in the algebra $\Lambda_0 := e_0\Lambda e_0$, where $e_0 := \sum_{e \in E_0} e$, and we want to establish some relations between the categories Λ-mod and

Λ_0-mod. Notice that Λ_0 is also a basic finite-dimensional algebra which splits over its radical: $\Lambda_0 = S_0 \oplus \operatorname{rad}\Lambda_0$, where $S_0 = e_0 S e_0$ and $\operatorname{rad}\Lambda_0 = e_0(\operatorname{rad}\Lambda)e_0$.

The algebra Λ_0 is called *convex in* Λ if E_0 is a convex subset of E; and Λ_0 is *final* (resp. *cofinal*) *in* Λ if E_0 is final (resp. cofinal) in E.

Given a convex algebra Λ_0 in Λ, the morphism $\psi : \Lambda \to \Lambda_0$ given by $\psi(\lambda) = e_0\lambda e_0$ for $\lambda \in \Lambda$ is a morphism of algebras. This yields natural structures of a Λ_0-Λ-bimodule and of a Λ-Λ_0-bimodule on Λ_0. Hence, we have the following two natural new types of "restriction functor".

Definition 1.2. Given a convex algebra Λ_0 in Λ, we have the functors

$$\operatorname{res} := \Lambda_0 \otimes_\Lambda - : \Lambda\text{-Mod} \to \Lambda_0\text{-Mod},$$
$$\operatorname{res}' := \operatorname{Hom}_\Lambda(\Lambda_0, -) : \Lambda\text{-Mod} \to \Lambda_0\text{-Mod}.$$

In Section 2, we will collect some basic properties of res. The corresponding basic properties of res′ are given in Section 7. Although res (resp. res′) coincides with the standard restriction functor ρ in case Λ_0 is a cofinal (resp. final) algebra in Λ, in general it does not.

As an application of our study of the extension/restriction interactions for modules over ditalgebras developed in Sections 3 and 4, we will prove in Section 6 the following result.

Theorem 1.3. *Assume that Λ is a basic finite-dimensional tame algebra over an algebraically closed field k, and consider a decomposition of the unit $1 = \sum_{e\in E} e$ as a sum of primitive orthogonal idempotents of Λ. Consider a convex subset E_0 of E and the associated convex algebra Λ_0. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}_0(d)$ of indecomposable Λ_0-modules such that, for any indecomposable Λ-module M with $\dim_k M \le d$ and such that M does not admit a minimal projective presentation with direct summands of the form Λe with $e \in E_0$, the module* $\operatorname{res}(M)$ *is isomorphic to a direct sum of modules in $\mathcal{I}_0(d)$.*

The passage from ditalgebras to algebras is discussed in Section 5. In the final Section 7, we present the dual formulation of our results for algebras.

2. Convex algebras and restrictions

Lemma 2.1. *Assume that the algebra Λ_0 is convex in Λ, and denote by $\mathcal{P}(\Lambda)$ and $\mathcal{P}(\Lambda_0)$ the categories of morphisms between projective Λ-modules and projective Λ_0-modules, respectively. Then the functor* res *preserves projectives, and hence induces a functor* $\operatorname{Res} : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda_0)$ *such that the following square commutes up to isomorphism:*

$$\begin{array}{ccc} \mathcal{P}(\Lambda) & \xrightarrow{\mathrm{Cok}} & \Lambda\text{-Mod} \\ {\scriptstyle \mathrm{Res}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathcal{P}(\Lambda_0) & \xrightarrow{\mathrm{Cok}_0} & \Lambda_0\text{-Mod} \end{array}$$

Here, Cok *and* Cok_0 *are the corresponding cokernel functors.*

Proof. First notice that the isomorphism $\Lambda_0 \otimes_\Lambda \Lambda \to \Lambda_0$ of Λ_0-Λ-bimodules restricts to isomorphisms $\Lambda_0 \otimes_\Lambda \Lambda e_i \to \Lambda_0 e_i$ of Λ_0-modules for any $e_i \in E$. Here, $\Lambda_0 e_i = 0$ whenever $e_i \notin E_0$. Thus, the functor res preserves projectives, because it preserves direct sums. Then, given an object $\phi : P_1 \to P_0$ in $\mathcal{P}(\Lambda)$, we can consider the object $\mathrm{Res}(\phi) := 1_{\Lambda_0} \otimes \phi : \Lambda_0 \otimes_\Lambda P_1 \to \Lambda_0 \otimes_\Lambda P_0$ in $\mathcal{P}(\Lambda_0)$. Given a morphism $(u, v) : \phi \to \phi'$ in $\mathcal{P}(\Lambda)$, the rule $\mathrm{Res}(u, v) = (\mathrm{res}\, u, \mathrm{res}\, v)$ clearly defines a functor. Since res is right exact, for any $\phi \in \mathcal{P}(\Lambda)$ there is an isomorphism $\eta_\phi : \mathrm{Cok}_0 \mathrm{Res}\, \phi \to \mathrm{res}\, \mathrm{Cok}\, \phi$. It is natural in the variable ϕ. ∎

Write $J := \mathrm{rad}\, \Lambda$. Then, as usual, we denote by $\mathcal{P}^1(\Lambda)$ the full subcategory of $\mathcal{P}(\Lambda)$ whose objects are the morphisms $\alpha : P \to Q$ with image contained in JQ.

LEMMA 2.2. *If Λ_0 is a convex algebra in Λ, we have* $\mathrm{Res}(\mathcal{P}^1(\Lambda)) \subseteq \mathcal{P}^1(\Lambda_0)$*, and therefore* res *preserves projective covers.*

Proof. This follows from the observation that any morphism $\phi : M \to N$ in Λ-Mod which factors through JN is mapped by res to a morphism $\mathrm{res}\, \phi : \mathrm{res}\, M \to \mathrm{res}\, N$ factoring through $J_0\, \mathrm{res}\, N$, where $J_0 = e_0 J e_0 = \mathrm{rad}\, \Lambda_0$. ∎

LEMMA 2.3. *If Λ_0 is a cofinal algebra in Λ, then* res *is isomorphic to the standard restriction functor $\rho : \Lambda$-Mod $\to \Lambda_0$-Mod.*

Proof. If Λ_0 is cofinal in Λ, we have $\Lambda_0 = e_0 \Lambda e_0 = e_0 \Lambda$, an equality of right Λ-modules. Hence, given $M \in \Lambda$-Mod, we have $\Lambda_0 \otimes_\Lambda M \cong e_0 \Lambda \otimes_\Lambda M \cong e_0 M$, a natural isomorphism in the variable M. ∎

REMARK 2.4. Given a convex algebra Λ_0 in the finite-dimensional algebra Λ, it is not always true that the functor res is isomorphic to the standard restriction functor $\rho : \Lambda$-Mod $\to \Lambda_0$-Mod. Indeed, res annihilates every indecomposable projective Λe_i with $e_i \in E \setminus E_0$.

The functor res does not preserve, in general, minimal projective presentations. For example, if Λ is the path algebra of the quiver $1 \to 2$ and Λ_0 is defined by the idempotent e_2 corresponding to the vertex 2, then the minimal projective presentation of the simple Λ-module S_1 corresponding to the vertex 1 is not preserved by res.

LEMMA 2.5. *Let Λ_0 be a convex algebra in Λ. Then the functor* $\mathrm{tens} = \Lambda e_0 \otimes_{\Lambda_0} - : \Lambda_0$-Mod $\to \Lambda$-Mod *preserves projectives and induces a functor*

Tens : $\mathcal{P}(\Lambda_0) \to \mathcal{P}(\Lambda)$ *such that the following diagram commutes up to isomorphism:*

$$\begin{array}{ccc} \mathcal{P}(\Lambda) & \xrightarrow{\text{Cok}} & \Lambda\text{-Mod} \\ {\scriptstyle \text{Tens}}\uparrow & & \uparrow{\scriptstyle \text{tens}} \\ \mathcal{P}(\Lambda_0) & \xrightarrow{\text{Cok}_0} & \Lambda_0\text{-Mod} \end{array}$$

Moreover,

$$\text{res}\,\text{tens} \cong 1_{\Lambda_0\text{-Mod}}$$

and so, given $M \in \Lambda\text{-Mod}$, *we have* $M \cong \text{tens}\,\text{res}(M)$ *if and only if* $M \cong \text{tens}(M')$ *for some* $M' \in \Lambda_0\text{-Mod}$.

Proof. The functor tens preserves projectives. Indeed, a typical projective Λ_0-module is a direct sum of Λ_0-modules of the form $\Lambda_0 e_i$ for some idempotent e_i of E_0. But $\Lambda e_0 \otimes_{\Lambda_0} \Lambda_0 e_i \cong \Lambda e_i$ and $\Lambda e_0 \otimes_{\Lambda_0} -$ preserves direct sums. Thus, $\Lambda e_0 \otimes_{\Lambda_0} -$ induces a functor

$$\text{Tens} : \mathcal{P}(\Lambda_0) \to \mathcal{P}(\Lambda)$$

such that $\text{Tens}(\phi) = 1 \otimes \phi$ for any object $\phi : P \to Q$ of $\mathcal{P}(\Lambda_0)$, and $\text{Tens}(u, v) = (1 \otimes u, 1 \otimes v)$ for any morphism $(u, v) : \phi \to \phi'$ in $\mathcal{P}(\Lambda_0)$. From the fact that $\Lambda e_0 \otimes_{\Lambda_0} -$ is right exact, we get, for each $\phi \in \mathcal{P}(\Lambda_0)$, an isomorphism $\eta_\phi : \text{Cok}(1 \otimes \phi) \to \Lambda e_0 \otimes_{\Lambda_0} \text{Cok}_0\,\phi$. It is easy to verify that $\eta : \text{Cok}\,\text{Tens} \to \text{tens}\,\text{Cok}_0$ is a natural isomorphism.

Now, notice that $\Lambda_0 \otimes_\Lambda \Lambda e_0 \cong \Lambda_0$, hence, for $M \in \Lambda_0\text{-Mod}$, we have the isomorphisms of Λ_0-modules $\Lambda_0 \otimes_\Lambda \Lambda e_0 \otimes_{\Lambda_0} M \cong \Lambda_0 \otimes_{\Lambda_0} M \cong M$, which are natural in the variable M. ∎

LEMMA 2.6. *Given a convex algebra* Λ_0 *in* Λ *and* $M \in \Lambda\text{-Mod}$, *we have* $M \cong \text{tens}(\text{res}(M))$ *if and only if the projectives in the minimal projective presentation of* M *are direct sums of modules of the form* Λe_i *with* $e_i \in E_0$.

Proof. In general, for arbitrary algebras $\Lambda_0 = e_0 \Lambda e_0$ with e_0 any idempotent of Λ, we know from the argument in the proof of [1, I.6.8] that a Λ-module $M \in \Lambda\text{-Mod}$ is of the form $M \cong \text{tens}(N)$ for some $N \in \Lambda_0\text{-Mod}$ if and only if there is an exact sequence $P_1 \to P_0 \to M \to 0$, where P_1 and P_0 are direct sums of summands of Λe_0. Then, for a convex algebra Λ_0 in Λ, having in mind 2.5, the fact that minimal presentations of M arise as direct summands in any projective presentation of M, and the uniqueness of decompositions in finite-dimensional indecomposables, we can easily derive our statement. ∎

3. Subditalgebras and reduction functors. Let us recall from [6] the notion of a proper subditalgebra.

Definition 3.1. Let $\mathcal{A} = (T, \delta)$ be any ditalgebra with layer (R, W). Assume we have R-R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = W_1' \oplus W_1''$. Consider the subalgebra T' of T generated by R and $W' = W_0' \oplus W_1'$. Then T' is freely generated by R and W' (see [6, 1.3]). Let us write $A' := [T']_0$, which is freely generated by the pair (R, W_0'), and assume furthermore that $\delta(W_0') \subseteq A'W_1'A'$ and $\delta(W_1') \subseteq A'W_1'A'W_1'A'$. Then the differential δ on T restricts to a differential δ' on the t-algebra T' and we obtain a new ditalgebra $\mathcal{A}' = (T', \delta')$ with layer (R, W'). A layered ditalgebra $\mathcal{A}'$ is called a *proper subditalgebra of* $\mathcal{A}$ if it is obtained from an R-R-bimodule decomposition of W as just described.

The inclusion $r : T' \to T$ yields a morphism of ditalgebras $r : \mathcal{A}' \to \mathcal{A}$, and hence a *restriction functor*

$$R^{\mathcal{A}}_{\mathcal{A}'} := F_r : \mathcal{A}\text{-Mod} \to \mathcal{A}'\text{-Mod}.$$

The projection $\pi : A = [T]_0 \to [T']_0 = A'$ yields an *extension functor*

$$E^{\mathcal{A}}_{\mathcal{A}'} := F_\pi : A'\text{-Mod} \to A\text{-Mod}.$$

Definition 3.2. Let $\mathcal{A} = (T, \delta)$ be a ditalgebra with layer (R, W). Then an algebra B is called a *proper subalgebra of* $\mathcal{A}$ if $B = [T']_0$ for some proper subditalgebra $\mathcal{A}' = (T', \delta')$ of $\mathcal{A}$ associated to R-R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = W_1' \oplus W_1''$, where $W_1' = 0$.

Remark 3.3. With the notation of the previous definitions, notice that we can identify the category B-Mod with $\mathcal{A}'$-Mod, and the algebra $\mathrm{End}_B(X)$ with $\mathrm{End}_{\mathcal{A}'}(X)$ for any $\mathcal{A}'$-module X. Assume that X is an admissible B-module (that is, an admissible $\mathcal{A}'$-module X, as in [6, 12.4]). Thus, we have a splitting $\mathrm{End}_B(X)^{\mathrm{op}} = S \oplus P$ and, in this case, the construction $\mathcal{A} \mapsto \mathcal{A}^X$, described in [6, 12.7–12.9], has the following simple form: $W^X = W_0^X \oplus W_1^X$, where $W_0^X = X^* \otimes_B BW_0''B \otimes_B X$ and $W_1^X = (X^* \otimes_B BW_1B \otimes_B X) \oplus P^*$. Then, by definition, $\mathcal{A}^X = (T^X, \delta^X)$, where $T^X = T_S(W^X)$ and the differential δ^X is determined, for $w \in BW_0''B \cup BW_1B$, $\nu \in X^*$ and $x \in X$, by the formula

$$\delta^X(\nu \otimes w \otimes x) = \lambda(\nu) \otimes w \otimes x + \sigma_{\nu,x}(\delta(w)) + (-1)^{\deg w+1}\nu \otimes w \otimes \rho(x),$$

where $\lambda : X^* \to P^* \otimes_S X^*$ and $\rho : X \to X \otimes_S P^*$ are the morphisms defined in [6, 11.10] and $\sigma_{\nu,x} : T \to T^X$ is the linear map defined in [6, 12.8]. Moreover, for $\gamma \in P^*$, by definition, $\delta^X(\gamma) = \mu(\gamma)$, where $\mu : P^* \to P^* \otimes_S P^*$ is the comultiplication morphism, as in [6, 11.7]. The ditalgebra $\mathcal{A}^X$ has layer (S, W^X) and there is an associated functor (see [6, 12.10])

$$F^X : \mathcal{A}^X\text{-Mod} \to \mathcal{A}\text{-Mod}.$$

Remark 3.4. Suppose that $\mathcal{A}'$ is a proper subditalgebra of the layered ditalgebra $\mathcal{A}$ and that B is a proper subalgebra of $\mathcal{A}'$. Then B is a proper subalgebra of $\mathcal{A}$.

Proof. Assume that $\mathcal{A} = (T, \delta)$ has layer (R, W). Suppose that $\mathcal{A}' = (T', \delta')$ is the proper subditalgebra of $\mathcal{A}$ associated to R-R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = W_1' \oplus W_1''$. In particular, δ' is just the restriction of δ to T'. Since B is a proper subalgebra of $\mathcal{A}'$, it is associated to R-R-bimodule decompositions $W_0' = V_0' \oplus V_0''$ and $W_1' = V_1' \oplus V_1''$ with $V_1' = 0$. Then B is the proper subalgebra of $\mathcal{A}$ associated to the R-R-bimodule decompositions $W_0 = V_0' \oplus (V_0'' \oplus W_0'')$ and $W_1 = V_1' \oplus (V_1'' \oplus W_1'')$, where $V_1' = 0$. ∎

LEMMA 3.5. *Assume that $\mathcal{A}'$ is a proper subditalgebra of the layered ditalgebra $\mathcal{A}$ and that B is a proper subalgebra of the layered ditalgebra $\mathcal{A}'$ (hence of $\mathcal{A}$ too). Therefore, according to the above remarks, for any admissible B-module X, we can consider the associated functors*

$$\mathcal{A}^X\text{-Mod} \xrightarrow{F^X} \mathcal{A}\text{-Mod} \quad \textit{and} \quad \mathcal{A}'^X\text{-Mod} \xrightarrow{F'^X} \mathcal{A}'\text{-Mod}$$

In this case, $\mathcal{A}'^X$ is a proper subditalgebra of $\mathcal{A}^X$ and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{A}^X\text{-Mod} & \xrightarrow{F^X} & \mathcal{A}\text{-Mod} \\ {\scriptstyle R^{\mathcal{A}^X}_{\mathcal{A}'^X}}\downarrow & & \downarrow{\scriptstyle R^{\mathcal{A}}_{\mathcal{A}'}} \\ \mathcal{A}'^X\text{-Mod} & \xrightarrow{F'^X} & \mathcal{A}'\text{-Mod} \end{array}$$

where $R^{\mathcal{A}^X}_{\mathcal{A}'^X}$ and $R^{\mathcal{A}}_{\mathcal{A}'}$ denote the corresponding restriction functors. Moreover, for any $M \in A'^X$-Mod, we have $F^X E^{A^X}_{A'^X}(M) = E^{A}_{A'} F'^X(M)$.

Proof. Here, $A = [T]_0$, $A' = [T']_0$, $A^X = [T^X]_0$ and $A'^X = [T'^X]_0$. We use the notation introduced in the previous remarks. Then

$$\mathcal{A}^X = (T_S(W_0^X \oplus W_1^X), \delta^X) \quad \text{and} \quad \mathcal{A}'^X = (T_S(W_0'^X \oplus W_1'^X), \delta'^X).$$

Thus, $\mathcal{A}^X$ has layer

$$(S, [X^* \otimes_B B(V_0'' \oplus W_0'')B \otimes_B X] \oplus [X^* \otimes_B B(V_1'' \oplus W_1'')B \otimes_B X] \oplus P^*),$$

while $\mathcal{A}'^X$ has layer

$$(S, [X^* \otimes_B BV_0''B \otimes_B X] \oplus [X^* \otimes_B BV_1''B \otimes_B X] \oplus P^*).$$

We want to see that δ'^X is the restriction of δ^X. For this, take $\nu \in X^*$, $w \in V_0'' \cup V_1''$ and $x \in X$, and let us show that $\delta'^X(\nu \otimes w \otimes x) = \delta^X(\nu \otimes w \otimes x)$. It is clear that the linear map $\sigma_{\nu,x} : T \to T^X$ defined in [6, 12.8] restricts to the corresponding linear map $\sigma'_{\nu,x} : T' \to T'^X$. Since $\mathcal{A}'$ is a proper subditalgebra of $\mathcal{A}$, we also know that $\delta'(w) = \delta(w)$. Thus, the expressions

$$\delta^X(\nu \otimes w \otimes x) = \lambda(\nu) \otimes w \otimes x + \sigma_{\nu,x}(\delta(w)) + (-1)^{\deg w+1} \nu \otimes w \otimes \rho(x)$$

and

$$\delta'^X(\nu \otimes w \otimes x) = \lambda(\nu) \otimes w \otimes x + \sigma'_{\nu,x}(\delta'(w)) + (-1)^{\deg w+1} \nu \otimes w \otimes \rho(x)$$

coincide. Finally, $\delta'^X(\gamma) = \mu(\gamma) = \delta^X(\gamma)$ for $\gamma \in P^*$. Therefore, $\mathcal{A}'^X$ is a proper subditalgebra of $\mathcal{A}^X$.

Now we show that $R^{\mathcal{A}}_{\mathcal{A}'} F^X = F'^X R^{\mathcal{A}^X}_{\mathcal{A}'^X}$. Take $M \in \mathcal{A}^X$-Mod and recall, from [6, 12.10], that $F^X(M)$ has underlying B-module $X \otimes_S M$ and the action of A on $F^X(M)$ is determined by the formula

$$w \cdot (x \otimes m) = \sum_{i \in I} x_i \otimes (\nu_i \otimes w \otimes x) * m,$$

where $(x_i, \nu_i)_{i\in I}$ is a fixed dual basis of X_S and $*$ denotes the left action of T^X on M, $w \in BV_0''B \cup BW_0''B$, $x \in X$ and $m \in M$. Then $R^{\mathcal{A}}_{\mathcal{A}'} F^X(M)$ has underlying B-module $X \otimes_S M$ where A' acts via the same formula given above for $w \in BV_0''B$. Now, the result of the action of a typical generator $\nu\otimes w\otimes x$ of $W_0'^X$ on $m \in R^{\mathcal{A}^X}_{\mathcal{A}'^X}(M)$ is again $(\nu\otimes w\otimes x)*m$. Thus, $F'^X R^{\mathcal{A}^X}_{\mathcal{A}'^X}(M)$ has underlying B-module $X \otimes_S M$ and action $\cdot'$ given by

$$w \cdot' (x \otimes m) = \sum_{i \in I} x_i \otimes (\nu_i \otimes w \otimes x) * m = w \cdot (x \otimes m).$$

Hence $R^{\mathcal{A}}_{\mathcal{A}'} F^X(M) = F'^X R^{\mathcal{A}^X}_{\mathcal{A}'^X}(M)$. Given $f = (f^0, f^1) \in \mathrm{Hom}_{\mathcal{A}^X}(M, N)$, we find that $(F^X(f))^0[x \otimes m] = x \otimes f^0(m) + \sum_{j\in J} xp_j \otimes f^1(\gamma_j)[m]$ and $(F^X(f))^1(w)[x \otimes m] = \sum_{i\in I} x_i \otimes f^1(\nu_i \otimes w \otimes x)[m]$, where $x \in X$, $m \in M$ and $w \in W_1$. Here, $(p_j, \gamma_j)_{j\in J}$ is a fixed dual basis of P_S.

Now, $[R^{\mathcal{A}}_{\mathcal{A}'} F^X(f)]^0[x \otimes m]$ and $[R^{\mathcal{A}}_{\mathcal{A}'} F^X(f)]^1(w)[x \otimes m]$ have the same recipe as $(F^X(f))^0[x \otimes m]$ and $(F^X(f))^1(w)[x \otimes m]$ above when evaluated at any $w \in W_1'$. Also, $[F'^X R^{\mathcal{A}^X}_{\mathcal{A}'^X}(f)]^0[x \otimes m]$ and $[F'^X R^{\mathcal{A}^X}_{\mathcal{A}'^X}(f)]^1(w)[x \otimes m]$ have the same recipes. Thus, $R^{\mathcal{A}}_{\mathcal{A}'} F^X(f) = F'^X R^{\mathcal{A}^X}_{\mathcal{A}'^X}(f)$ and the square in the statement of the lemma commutes.

Finally, take $M \in A'^X$-Mod; we will see that $F^X E^{A^X}_{A'^X}(M) = E^{A}_{A'} F'^X(M)$.

Recall that $E^{A}_{A'} = F_\pi : A'$-Mod $\to A$-Mod is induced by the projection morphism of algebras $\pi : A \to A'$. Thus, for $N \in A'$-Mod, the A-module $E^{A}_{A'}(N)$ has underlying R-module N and the action of A on $n \in N$ is determined by $w * n = wn$ if $w \in W_0'$, and $w * n = 0$ if $w \in W_0''$.

Now, $F^X E^{A^X}_{A'^X}(M)$ has underlying B-module $X \otimes_S M$ and the action of $w \in BV_0''B \cup BW_0''B$ on $X \otimes_S M$ (recall that A is freely generated by B and $BV_0''B + BW_0''B$) is given by

$$w \cdot (x \otimes m) = \sum_{i \in I} x_i \otimes (\nu_i \otimes w \otimes x) * m,$$

where $*$ is the action of W_0^X on $E^{A^X}_{A'^X}(M)$. Thus,

$$w \cdot (x \otimes m) = \sum_{i \in I} x_i \otimes (\nu_i \otimes w \otimes x) \circledast m \quad \text{ if } w \in BV_0''B,$$

and $w \cdot (x \otimes m) = 0$ if $w \in BW_0''B$, where $\circledast$ denotes the action of A'^X on m. Moreover, $F'^X(M)$ has underlying B-module $X \otimes_S M$ and the action of $w \in BV_0''B$ on $X \otimes_S M$ is given by

$$w \odot (x \otimes m) = \sum_{i \in I} x_i \otimes (\nu_i \otimes w \otimes x) \circledast m.$$

Next, the action of $BV_0''B \cup BW_0''B$ on $E^{\mathcal{A}}_{\mathcal{A}'}F'^X(M)$ is given by

$$w \odot (x \otimes m) = \sum_{i \in I} x_i \otimes (\nu_i \otimes w \otimes x) \circledast m \quad \text{ if } w \in BV_0''B,$$

and $w \odot (x \otimes m) = 0$ if $w \in BW_0''B$. Hence, the action $\cdot$ coincides with $\odot$ and we are done. ∎

Lemma 3.6. *Assume that $\mathcal{A}' = (T', \delta')$ is a proper subditalgebra of the layered ditalgebra $\mathcal{A} = (T, \delta)$. With the notation of* 3.1, *assume that the ditalgebra $\mathcal{A}'^a$ is obtained from $\mathcal{A}'$ by absorption of the bimodule V_0', as in* [6, 8.20], *where $W_0' = V_0' \oplus V_0''$ is a given R-R-bimodule decomposition and $\delta(V_0') = 0$. Consider also the ditalgebra $\mathcal{A}^a$ obtained from $\mathcal{A}$ by absorption of the same bimodule V_0'. Then $\mathcal{A}'^a$ is a proper subditalgebra of $\mathcal{A}^a$ and there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{A}^a\text{-Mod} & \xrightarrow{F^a} & \mathcal{A}\text{-Mod} \\ \Big\downarrow R^{\mathcal{A}^a}_{\mathcal{A}'^a} & & \Big\downarrow R^{\mathcal{A}}_{\mathcal{A}'} \\ \mathcal{A}'^a\text{-Mod} & \xrightarrow{F'^a} & \mathcal{A}'\text{-Mod} \end{array}$$

where F^a and F'^a denote the associated reduction functors. Moreover, for any $M \in A'^a$-Mod, we have $F^a E^{\mathcal{A}^a}_{\mathcal{A}'^a}(M) = E^{\mathcal{A}}_{\mathcal{A}'}F'^a(M)$.

Proof. We are considering the R-R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = W_1' \oplus W_1''$, which define $\mathcal{A}'$ and its layer (R, W'). Thus, $W_0 = V_0' \oplus V_0'' \oplus W_0''$ and $\mathcal{A}^a$ has layer (R^a, W^a), where R^a is the subalgebra of T freely generated by R and V_0', and we have $W_0^a = R^a(V_0'' \oplus W_0'')R^a$ and $W_1^a = R^aW_1R^a$. Likewise, $\mathcal{A}'^a$ has layer (R^a, W'^a), where $W_0'^a = R^aV_0''R^a$ and $W_1'^a = R^aW_1'R^a$. Then $W_0^a = W_0'^a \oplus R^aW_0''R^a$ and $W_1^a = W_1'^a \oplus R^aW_1''R^a$. By definition, $\mathcal{A}^a = (T^a, \delta^a) = (T, \delta)$ and $\mathcal{A}'^a = (T'^a, \delta'^a) = (T', \delta')$. Therefore, δ'^a is the restriction of δ^a, and $\mathcal{A}'^a$ is a proper subditalgebra of $\mathcal{A}^a$. Here, the equality $R^{\mathcal{A}}_{\mathcal{A}'}F^a = F'^aR^{\mathcal{A}^a}_{\mathcal{A}'^a}$ is clear because all these functors are identity functors. The projection algebra morphism $A^a = [T^a]_0 \to [T'^a]_0 = A'^a$ coincides with the projection morphism $A = [T]_0 \to [T']_0 = A'$. Thus, $E^{\mathcal{A}}_{\mathcal{A}'} = E^{\mathcal{A}^a}_{\mathcal{A}'^a}$ and the last formula of the lemma holds trivially. ∎

Lemma 3.7. *Assume that $\mathcal{A}'$ is a proper subditalgebra of the layered ditalgebra $\mathcal{A}$. Assume that the ditalgebras $\mathcal{A}'^d$ and $\mathcal{A}^d$ are obtained from*

$\mathcal{A}'$ and $\mathcal{A}$, respectively, by deletion of the same idempotent (as in [6, 8.17]). Then $\mathcal{A}'^d$ is a proper subditalgebra of $\mathcal{A}^d$ and there is a commutative diagram

$$\begin{array}{ccc} \mathcal{A}^d\text{-Mod} & \xrightarrow{F^d} & \mathcal{A}\text{-Mod} \\ \Big\downarrow{\scriptstyle R^{\mathcal{A}^d}_{\mathcal{A}'^d}} & & \Big\downarrow{\scriptstyle R^{\mathcal{A}}_{\mathcal{A}'}} \\ \mathcal{A}'^d\text{-Mod} & \xrightarrow{F'^d} & \mathcal{A}'\text{-Mod} \end{array}$$

where F^d and F'^d denote the associated reduction functors. Moreover, for any $M \in A'^d$-Mod, we have $F^d E^{A^d}_{A'^d}(M) = E^A_{A'} F'^d(M)$.

Proof. Adopt the notation of 3.1 and let e be the idempotent in question. Recall that if $\mathcal{A}$ has layer (R, W), then $\mathcal{A}^d$ has layer $(eRe, eW_0e \oplus eW_1e)$. Likewise, if $\mathcal{A}'$ has layer (R, W'), then $\mathcal{A}'^d$ has layer $(eRe, eW'_0e \oplus eW'_1e)$. We have projection morphisms of ditalgebras $\eta : \mathcal{A} \to \mathcal{A}^d$ and $\eta' : \mathcal{A}' \to \mathcal{A}'^d$. Moreover, if we consider the inclusion morphisms $r : \mathcal{A}' \to \mathcal{A}$ and $r^d : \mathcal{A}'^d \to \mathcal{A}^d$, we have the equality $\eta r = r^d \eta'$. Hence, $R^{\mathcal{A}}_{\mathcal{A}'} F^d = F_r F_\eta = F_{\eta'} F_{r^d} = F'^d R^{\mathcal{A}^d}_{\mathcal{A}'^d}$. We can also consider the morphisms of algebras $\eta_0 : A \to A^d$ and $\eta'_0 : A' \to A'^d$ obtained by restriction from η and η', respectively, and the canonical projections of algebras $\pi : A \to A'$ and $\pi^d : A^d \to A'^d$ which satisfy the equality $\eta'_0 \pi = \pi^d \eta_0$. Considering the induced functors between the categories of modules over the corresponding algebras, we obtain $F^d E^{A^d}_{A'^d}(M) = E^A_{A'} F'^d(M)$ for any $M \in A'^d$-Mod. ∎

Lemma 3.8. *Assume that $\mathcal{A}'$ is a proper subditalgebra of the layered ditalgebra $\mathcal{A}$. Assume that the ditalgebras $\mathcal{A}'^r$ and $\mathcal{A}^r$ are obtained from $\mathcal{A}'$ and $\mathcal{A}$, respectively, by regularization of the same bimodule (as in [6, 8.19]). Then $\mathcal{A}'^r$ is a proper subditalgebra of $\mathcal{A}^r$ and there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{A}^r\text{-Mod} & \xrightarrow{F^r} & \mathcal{A}\text{-Mod} \\ \Big\downarrow{\scriptstyle R^{\mathcal{A}^r}_{\mathcal{A}'^r}} & & \Big\downarrow{\scriptstyle R^{\mathcal{A}}_{\mathcal{A}'}} \\ \mathcal{A}'^r\text{-Mod} & \xrightarrow{F'^r} & \mathcal{A}'\text{-Mod} \end{array}$$

where F^r and F'^r denote the associated reduction functors. Moreover, for any $M \in A'^r$-Mod, we have $F^r E^{A^r}_{A'^r}(M) = E^A_{A'} F'^r(M)$.

Proof. Adopt the notation of 3.1 and denote by V'_0 the bimodule in question. Thus, $W_0 = W'_0 \oplus W''_0$ and $W_1 = W'_1 \oplus W''_1$ are the R-R-bimodule decompositions which define $\mathcal{A}'$. Moreover, we also have R-R-bimodule decompositions $W'_0 = V'_0 \oplus V''_0$ and $W'_1 = \delta'(V'_0) \oplus V''_1$. Recall that $\mathcal{A}$ has layer (R, W) and $\mathcal{A}^r$ has layer $(R, (V''_0 \oplus W''_0) \oplus (V''_1 \oplus W''_1))$. Likewise, $\mathcal{A}'$ has layer (R, W') and $\mathcal{A}'^r$ has layer $(R, V''_0 \oplus V''_1)$. Since δ^r and δ'^r are induced by δ and δ', respectively, and δ' is the restriction of δ, it follows that δ'^r is the restriction of δ^r and $\mathcal{A}'^r$ is a proper subditalgebra of $\mathcal{A}^r$. The canonical projection morphisms of ditalgebras $\eta : \mathcal{A} \to \mathcal{A}^r$ and $\eta' : \mathcal{A}' \to \mathcal{A}'^r$, and the inclusion morphisms $s : \mathcal{A}' \to \mathcal{A}$ and $s^r : \mathcal{A}'^r \to \mathcal{A}^r$, satisfy the equality $\eta s = s^r \eta'$. Hence, $R^{\mathcal{A}}_{\mathcal{A}'} F^r = F_s F_\eta = F_{\eta'} F_{s^r} = F'^r R^{\mathcal{A}^r}_{\mathcal{A}'^r}$. We can also consider the morphisms of algebras $\eta_0 : A \to A^r$ and $\eta'_0 : A' \to A'^r$ obtained by restriction from η and η', respectively, and the canonical projections of algebras $\pi : A \to A'$ and $\pi^r : A^r \to A'^r$, which satisfy the equality $\eta'_0 \pi = \pi^r \eta_0$. Considering the induced functors between the categories of modules over the corresponding algebras, we obtain $F^r E^{A^r}_{A'^r}(M) = E^{A}_{A'} F'^r(M)$ for any $M \in A'^r$-Mod. ■

Proposition 3.9. *Assume that $\mathcal{A}'$ is a proper subditalgebra of the layered ditalgebra $\mathcal{A}$ and that B is a proper subalgebra of the layered ditalgebra $\mathcal{A}'$ (hence of $\mathcal{A}$ too). From 3.5, for any admissible B-module X, $\mathcal{A}'^X$ is a proper subditalgebra of $\mathcal{A}^X$ and we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{A}^X\text{-Mod} & \xrightarrow{F^X} & \mathcal{A}\text{-Mod} \\ {\scriptstyle R^{\mathcal{A}^X}_{\mathcal{A}'^X}}\downarrow & & \downarrow{\scriptstyle R^{\mathcal{A}}_{\mathcal{A}'}} \\ \mathcal{A}'^X\text{-Mod} & \xrightarrow{F'^X} & \mathcal{A}'\text{-Mod} \end{array}$$

Assume that $\mathcal{A}$ is a Roiter ditalgebra and that $\mathcal{A}'$ admits a triangular layer. Then, for any $M \in \mathcal{A}$-Mod with $R^{\mathcal{A}}_{\mathcal{A}'}(M) \cong F'^X(N')$ for some $N' \in \mathcal{A}'^X$-Mod, there is $N \in \mathcal{A}^X$-Mod such that $F^X(N) \cong M$. If X is complete, then also $R^{\mathcal{A}^X}_{\mathcal{A}'^X}(N) \cong N'$.

Proof. From [6, 16.1], we know that for any S-module N' such that there is $L \in \mathcal{A}$-Mod with underlying B-module structure equal to the canonical B-module $X \otimes_S N'$, there is a unique $N \in \mathcal{A}^X$-Mod with underlying S-module N' such that $F^X(N) = L$. We will deduce the proposition from this fact.

Assume that $M \in \mathcal{A}$-Mod is such that $R^{\mathcal{A}}_{\mathcal{A}'}(M) \cong F'^X(N')$ for some $N' \in \mathcal{A}'^X$-Mod. Consider an isomorphism $f = (f^0, f^1) : R^{\mathcal{A}}_{\mathcal{A}'}(M) \to F'^X(N')$. We know that $\mathcal{A}$ is a Roiter ditalgebra and that $\mathcal{A}'$ admits a triangular layer. From [6, 12.3], $\mathcal{A}'$ is a Roiter ditalgebra and $f^0 : M \to X \otimes_S N'$ is an isomorphism of B-modules (recall that $\delta'(B) = 0$). Thus, we can copy the A-module structure of M onto the B-module $X \otimes_S N'$ with the help of the morphism f^0 of B-modules, and obtain a new A-module L. Hence, $a\cdot(x\otimes n) = f^0(a(f^0)^{-1}(x\otimes n))$ for any $a \in A$, $x \in X$ and $n \in N'$. Therefore, $a\cdot(x\otimes n) = ax\otimes n$ for $a \in B$, which means that the underlying B-module of L is just $X \otimes_S N'$. From the fact stated above, there is a unique $N \in \mathcal{A}^X$-Mod such that $F^X(N) = L \cong M$.

Finally, if X is a complete admissible B-module, we know from [6, 13.5] that F'^X is full and faithful. Thus F'^X reflects isomorphisms and, from

$F^X(N) \cong M$, we get $F'^X(N') \cong R^{\mathcal{A}}_{\mathcal{A}'}(M) \cong R^{\mathcal{A}}_{\mathcal{A}'}F^X(N) \cong F'^X R^{\mathcal{A}^X}_{\mathcal{A}'^X}(N)$ and we can derive our last claim. ∎

Lemma 3.10. *Assume that $\mathcal{A}'$ is an initial subditalgebra of the triangular ditalgebra $\mathcal{A}$, as in* [6, 14.8]. *From* [6, 14.9], *we know that $\mathcal{A}'$ is triangular. Then the following statements hold.*

(1) *Suppose that $\mathcal{A}'^z$ and $\mathcal{A}^z$ are obtained from $\mathcal{A}'$ and $\mathcal{A}$ for $z \in \{a, d, r\}$ as in* 3.6–3.8, *respectively. Then $\mathcal{A}'^z$ is an initial subditalgebra of the triangular ditalgebra $\mathcal{A}^z$.*
(2) *Assume that B is an initial subalgebra of the triangular ditalgebra $\mathcal{A}'$. Suppose that X is a triangular admissible B-module* (*see* [6, 14.6], *having in mind that we are looking at a splitting* $\mathrm{End}_B(X)^{\mathrm{op}} = S \oplus P$). *Then $\mathcal{A}'^X$ is an initial subditalgebra of the triangular ditalgebra $\mathcal{A}^X$.*

Proof. This follows in all cases by inspection of the bimodule filtrations of the layer. The bimodule filtrations of the layer of $\mathcal{A}^a$ are described in [6, 8.20], and the corresponding filtrations for $\mathcal{A}^d$ and $\mathcal{A}^r$ can be derived from those described in [6, 8.12]. In the remaining case, we have to look carefully at the description of the bimodule filtrations of the layer of $\mathcal{A}^X$ given in [6, 14.10]. Here, if we assume that $\mathcal{A}$ has layer (R, W), that $\mathcal{A}'$ has layer (R, W'), and that B is identified with the initial subditalgebra $\mathcal{A}''$ of $\mathcal{A}'$ and has layer (R, V'), then the triangular filtration of W_0 has the form

$$0 = W_0^0 \subseteq W_0^1 \subseteq \cdots \subseteq W_0^{\ell_0''} = V_0' \subseteq \cdots \subseteq W_0^{\ell_0'} = W_0' \subseteq \cdots \subseteq W_0^{\ell_0} = W_0.$$

Thus, the triangular filtration $\{[W_0'^X]_m\}_m$ of the bimodule $W_0'^X$ is initial in the triangular filtration $\{[W_0^X]_n\}_n$ of W_0^X, with $[W_0'^X]_m = [W_0^X]_m$ for all $m \leq 2\ell_X(\ell_0' - \ell_0'' + 1)$. The situation for triangular filtrations in degree one is similar. ∎

4. Main result for ditalgebras. In this section, the ground field k is assumed to be algebraically closed. We shall prove the following theorem for modules over a seminested tame ditalgebra with an initial subditalgebra (see [6, 23.5]).

Theorem 4.1. *Assume that $\mathcal{A}'$ is an initial subditalgebra of the seminested tame ditalgebra $\mathcal{A}$ over the algebraically closed field k. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}(d)$ of indecomposable $\mathcal{A}'$-modules such that, for any indecomposable $\mathcal{A}$-module M with* $\dim_k M \leq d$ *and $M \not\cong E^{\mathcal{A}}_{\mathcal{A}'}(N)$ in* $\mathcal{A}$-Mod *for any $N \in \mathcal{A}'$*-Mod, *the module $R^{\mathcal{A}}_{\mathcal{A}'}(M)$ is isomorphic in* $\mathcal{A}'$-Mod *to a direct sum of modules in $\mathcal{I}(d)$.*

Recall that, given a seminested ditalgebra $\mathcal{A}$ and a fixed vertex v of $\mathcal{A}$, a module $N \in \mathcal{A}$-Mod is called *concentrated at v* if $\operatorname{supp} N = \{v\}$ and $\alpha N = 0$

for any solid arrow α of $\mathcal{A}$. We recall from [6, 28.8] the following theorem (which was stated in [9] and proved in detail in [5]).

THEOREM 4.2. *Assume $\mathcal{A}$ is a seminested tame ditalgebra over the algebraically closed field k. Assume that $d \in \mathbb{N}$ and v is a marked vertex of $\mathcal{A}$, say with marked loop z. Then there is a finite subset $\mathcal{S}(d,v)$ of k such that for any indecomposable $M \in \mathcal{A}$-*Mod *with* $\dim_k M \leq d$ *and such that $M_v \neq 0$ and* $\operatorname{spec} M(z) \not\subseteq \mathcal{S}(d,v)$, *there is $N \in \mathcal{A}$-*mod *concentrated at v with $N \cong M$.*

We can derive the following consequence, which will play a fundamental role in the proof of our main result.

THEOREM 4.3. *Assume that $\mathcal{A}'$ is a proper minimal subditalgebra of the tame seminested ditalgebra $\mathcal{A}$ over the algebraically closed field k. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}(d)$ of indecomposable $\mathcal{A}'$-modules such that, for any indecomposable $M \in \mathcal{A}$-*Mod *with* $\dim_k M \leq d$ *and* $M \ncong E^{\mathcal{A}}_{\mathcal{A}'}(N)$ *in* $\mathcal{A}$-Mod *for any* $N \in \mathcal{A}'$-Mod, *the module* $R^{\mathcal{A}}_{\mathcal{A}'}(M)$ *is isomorphic in* $\mathcal{A}'$-Mod *to a direct sum of modules in* $\mathcal{I}(d)$.

Proof. Consider all the marked vertices $v_1, \ldots, v_t$ of $\mathcal{A}'$. Given $d \in \mathbb{N}$, we can apply 4.2 to each of these marked vertices $v_1, \ldots, v_t$ of $\mathcal{A}$ and obtain the corresponding sets of scalars $\mathcal{S}(d, v_i)$ for $i \in [1,t]$. For each $i \in [1,t]$, consider the family $\mathcal{I}(d,v_i) := \{J_n(\lambda, v_i) \mid n \leq d \text{ and } \lambda \in \mathcal{S}(d,v_i)\}$ of $\mathcal{A}'$-modules. Consider also the non-marked points $v_{t+1}, \ldots, v_n$ of $\mathcal{A}'$ and the corresponding one-dimensional $\mathcal{A}'$-modules $S_{t+1}, \ldots, S_n$. Then we have the finite family of indecomposable $\mathcal{A}'$-modules $\mathcal{I}(d) := (\bigcup_{i=1}^t \mathcal{I}(d,v_i)) \cup \{S_{t+1}, \ldots, S_n\}$. If $M \in \mathcal{A}$-Mod is indecomposable with $\dim_k M \leq d$ and not isomorphic to any $\mathcal{A}$-module concentrated at any vertex v_i, then $R^{\mathcal{A}}_{\mathcal{A}'}(M)$ is isomorphic to a direct sum of $\mathcal{A}'$-modules in the family $\mathcal{I}(d)$. It remains to notice that $M \ncong E^{\mathcal{A}}_{\mathcal{A}'}(N)$ for any $N \in \mathcal{A}'$-Mod implies that M is not isomorphic to any $\mathcal{A}$-module concentrated at any v_i. Indeed, if $M \cong M'$ with M' concentrated at some v_i, then $M \cong M' \cong E^{\mathcal{A}}_{\mathcal{A}'} R^{\mathcal{A}}_{\mathcal{A}'}(M')$. ∎

REMARK 4.4. Given a seminested ditalgebra $\mathcal{A}$ over our algebraically closed field k, we shall consider the five basic operations $\mathcal{A} \mapsto \mathcal{A}^z$, where $z \in \{d,a,r,e,u\}$, called *deletion of idempotents* as in [6, 23.14], *regularization of a solid arrow* as in [6, 23.15], *absorption of a loop* as in [6, 23.16], *reduction of an edge* as in [6, 23.18] and *unravelling of a loop* as in [6, 23.23], and their corresponding reduction functors $F^z : \mathcal{A}^z\text{-Mod} \to \mathcal{A}\text{-Mod}$.

Assume that $\mathcal{A}'$ is an initial subditalgebra of a seminested ditalgebra $\mathcal{A}$. Then $\mathcal{A}'$ is a seminested ditalgebra too. Thus, if we can perform a basic operation $\mathcal{A}' \mapsto \mathcal{A}'^z$ for $z \in \{d,a,r,e,u\}$, we can also perform the basic operation $\mathcal{A} \mapsto \mathcal{A}^z$, where we respectively delete the same idempotent, absorb the same loop, regularize the same arrow, reduce the same edge or unravel the same loop as before. In this case, we shall say that $\mathcal{A}'^z$ and $\mathcal{A}^z$ are *simultaneously obtained from* $\mathcal{A}'$ *and* $\mathcal{A}$ *by a basic operation of type* z.

The only delicate point in the last observation occurs in the case of the edge reduction $\mathcal{A}' \mapsto \mathcal{A}'^e$, where we reduce an edge, say α, of $\mathcal{A}'$, which requires, in order that $\mathcal{A}'^e$ is indeed a seminested ditalgebra, that the proper subalgebra B of $\mathcal{A}'$ which supports the edge α is an initial subalgebra of $\mathcal{A}'$. Here, since $\mathcal{A}'$ is an initial subditalgebra of $\mathcal{A}$, we see that B is also an initial subalgebra of $\mathcal{A}$, and we can perform the operation $\mathcal{A} \mapsto \mathcal{A}^e$ within the context of seminested ditalgebras.

LEMMA 4.5. *Suppose that* $\mathcal{A}'$ *is an initial subditalgebra of the seminested ditalgebra* $\mathcal{A}$. *Assume that the ditalgebras* $\mathcal{A}'^z$ *and* $\mathcal{A}^z$ *are simultaneously obtained from the seminested ditalgebras* $\mathcal{A}'$ *and* $\mathcal{A}$, *respectively, by one of the five basic operations* $z \in \{d, a, r, e, u\}$. *Consider the corresponding reduction functors*

$$\mathcal{A}^z\text{-Mod} \xrightarrow{F^z} \mathcal{A}\text{-Mod} \quad \textit{and} \quad \mathcal{A}'^z\text{-Mod} \xrightarrow{F'^z} \mathcal{A}'\text{-Mod}.$$

Then, for any $M \in \mathcal{A}\text{-Mod}$ *with* $R^{\mathcal{A}}_{\mathcal{A}'}(M) \cong F'^z(N')$ *for some* $N' \in \mathcal{A}'^z\text{-Mod}$, *there is* $N \in \mathcal{A}^z\text{-Mod}$ *such that* $F^z(N) \cong M$ *and* $R^{\mathcal{A}^z}_{\mathcal{A}'^z}(N) \cong N'$.

Proof. For $z \in \{u, e\}$, this was proved in 3.9. For $z \in \{r, a\}$ it follows from the fact that F^z is an equivalence. For $z = d$, denote by e the idempotent such that $1 - e$ is to be eliminated. Then $M \in \mathcal{A}\text{-Mod}$ with $R^{\mathcal{A}}_{\mathcal{A}'}(M) \cong F'^d(N')$ for some $N' \in \mathcal{A}'^d\text{-Mod}$ implies that $eM = eR^{\mathcal{A}}_{\mathcal{A}'}(M) = R^{\mathcal{A}}_{\mathcal{A}'}(M) = M$. Hence, $M \cong F^d(N)$ for some $N \in \mathcal{A}^d\text{-Mod}$. ∎

Proof of Theorem 4.1. Since $\mathcal{A}$ is seminested and $\mathcal{A}'$ is an initial subditalgebra of $\mathcal{A}$, we infer that $\mathcal{A}'$ is also a seminested ditalgebra. From Drozd's theorem, any seminested ditalgebra $\mathcal{A}$ is tame if and only if it is not wild. From [6, 22.13], since $\mathcal{A}$ is a tame seminested ditalgebra, so is $\mathcal{A}'$. Fix any $d \in \mathbb{N}$. From [6, 28.22], there is a finite sequence of basic operations

$$\mathcal{A}' \mapsto \mathcal{A}'^{z_1} \mapsto \mathcal{A}'^{z_1 z_2} \mapsto \cdots \mapsto \mathcal{A}'^{z_1 \cdots z_t},$$

where $z_1, \ldots, z_t \in \{d, a, r, e, u\}$ and $\mathcal{A}'^{z_1 \cdots z_t}$ is a minimal ditalgebra. Moreover, if we consider the associated reduction functors

$$F'^{z_i} : \mathcal{A}'^{z_1 \cdots z_{i-1} z_i}\text{-Mod} \to \mathcal{A}'^{z_1 \cdots z_{i-1}}\text{-Mod}$$

for $i \in [1, t]$, then the composition functor

$$F' := F'^{z_1} F'^{z_2} \cdots F'^{z_t} : \mathcal{A}'^{z_1 \cdots z_t}\text{-Mod} \to \mathcal{A}'\text{-Mod}$$

has the property that, for any $M' \in \mathcal{A}'\text{-Mod}$ with $\dim_k M' \leq d$, there is some $N' \in \mathcal{A}'^{z_1 \cdots z_t}\text{-Mod}$ with $F'(N') \cong M'$.

From 3.10 and 4.4, we can consider simultaneously the finite sequence of basic operations

$$\mathcal{A} \mapsto \mathcal{A}^{z_1} \mapsto \mathcal{A}^{z_1 z_2} \mapsto \cdots \mapsto \mathcal{A}^{z_1 \cdots z_t},$$

and the associated reduction functors

$$F^{z_i} : \mathcal{A}^{z_1 \cdots z_{i-1} z_i}\text{-Mod} \to \mathcal{A}^{z_1 \cdots z_{i-1}}\text{-Mod},$$

where, for each $i \in [1,t]$, the ditalgebra $\mathcal{A}'^{z_1 \cdots z_i} = (T'^{z_1 \cdots z_i}, \delta'^{z_1 \cdots z_i})$ is an initial subditalgebra of the seminested ditalgebra $\mathcal{A}^{z_1 \cdots z_i} = (T^{z_1 \cdots z_i}, \delta^{z_1 \cdots z_i})$ for $i \in [1,t]$. We shall also consider the composition functor

$$F := F^{z_1} \cdots F^{z_t} : \mathcal{A}^{z_1 \cdots z_t}\text{-Mod} \to \mathcal{A}\text{-Mod}.$$

As before, we use the notation $A'^{z_1 \cdots z_i} = [T'^{z_1 \cdots z_i}]_0$ and $A^{z_1 \cdots z_i} = [T^{z_1 \cdots z_i}]_0$ for $i \in [1,t]$. We introduce the short notation for the extension functors

$$E_i := E^{A^{z_1 \cdots z_i}}_{A'^{z_1 \cdots z_i}} : A'^{z_1 \cdots z_i}\text{-Mod} \to A^{z_1 \cdots z_i}\text{-Mod},$$

and for the restriction functors

$$R_i := R^{\mathcal{A}^{z_1 \cdots z_i}}_{\mathcal{A}'^{z_1 \cdots z_i}} : \mathcal{A}^{z_1 \cdots z_i}\text{-Mod} \to \mathcal{A}'^{z_1 \cdots z_i}\text{-Mod},$$

for $i \in [1,t]$. Set

$$R_0 := R^{\mathcal{A}}_{\mathcal{A}'} : \mathcal{A}\text{-Mod} \to \mathcal{A}'\text{-Mod} \quad \text{and} \quad E_0 := E^{A}_{A'} : A'\text{-Mod} \to A\text{-Mod}.$$

Then, from the previous section applied to the basic reductions (which are particular cases of those considered before), we have:

1. $F'^{z_i} R_i = R_{i-1} F^{z_i}$ for $i \in [1,t]$;
2. $F^{z_i} E_i(N') = E_{i-1} F'^{z_i}(N')$ for $N' \in A'^{z_1 \cdots z_i}$-Mod and $i \in [1,t]$.

Therefore, $R_0 F = F' R_t$ and $F E_t(N') = E_0 F'(N')$ for any $N' \in \mathcal{A}'^{z_1 \cdots z_t}$-Mod.

Since $\mathcal{A}$ is a tame ditalgebra, so is $\mathcal{A}^{z_1 \cdots z_t}$ (see [6, 22.8] and [6, 22.10]). From 4.3, there is a finite family $\mathcal{I}_t(d)$ of indecomposable $\mathcal{A}'^{z_1 \cdots z_t}$-modules such that, for any indecomposable $\mathcal{A}^{z_1 \cdots z_t}$-module M' with $\dim_k M' \leq d$ and $M' \not\cong E_t(N'')$, and for any $N'' \in \mathcal{A}'^{z_1 \cdots z_t}$-Mod, the module $R_t(M')$ is isomorphic to a direct sum of indecomposables in $\mathcal{I}_t(d)$.

Consider the finite family $\mathcal{I}(d)$ of indecomposable $\mathcal{A}'$-modules of the form $F'(N')$ for some $N' \in \mathcal{I}_t(d)$. Take an indecomposable $M \in \mathcal{A}$-Mod with $\dim_k M \leq d$ and $M \not\cong E_0(M')$ for any $M' \in \mathcal{A}'$-Mod. Since $\dim_k R_0(M) = \dim_k M \leq d$, there is an $\mathcal{A}'^{z_1 \cdots z_t}$-module N' with $F'(N') \cong R_0(M)$. From 4.5, there is $N \in \mathcal{A}^{z_1 \cdots z_t}$-Mod such that $F(N) \cong M$ and $R_t(N) \cong N'$. Since M is indecomposable, so is N. Assume that $N \cong E_t(N'')$ for some $N'' \in \mathcal{A}'^{z_1 \cdots z_t}$-Mod; then $M \cong F(N) \cong F E_t(N'') = E_0 F'(N'')$. This contradicts the hypothesis on M, thus $N \not\cong E_t(N'')$ for any $N'' \in \mathcal{A}'^{z_1 \cdots z_t}$-Mod. But $\dim_k N \leq \dim_k F(N) = \dim_k M \leq d$ (see [6, 28.2]). Therefore, $R_t(N) \cong \bigoplus_{i=1}^{\ell} N'_i$, with $N'_i \in \mathcal{I}_t(d)$. It follows that $R_0(M) \cong R_0 F(N) = F' R_t(N) \cong \bigoplus_{i=1}^{\ell} F'(N'_i)$ with $F'(N'_i) \in \mathcal{I}(d)$. This finishes the proof of the theorem. ∎

5. Convex algebras and Drozd's ditalgebras

DEFINITION 5.1. Let $\mathcal{A}$ be a seminested ditalgebra with layer (R, W) and a set $\mathcal{P}$ of points. Then a proper subditalgebra $\mathcal{A}'$ of $\mathcal{A}$, say associated to the R-R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = W_1' \oplus W_1''$, is called *convex* if there is a subset $\mathcal{P}_0$ of $\mathcal{P}$ such that

$$eW_0'e = W_0' \quad \text{and} \quad eW_1'e = W_1', \quad \text{where} \quad e = \sum_{x\in\mathcal{P}_0} e_x.$$

REMARK 5.2. Assume that $\mathcal{A}'$ is a convex subditalgebra of the seminested ditalgebra $\mathcal{A}$. Suppose that $\mathcal{A}$ has layer (R, W) and a set $\mathcal{P}$ of points, and that the convex subditalgebra $\mathcal{A}'$ is associated to the R-R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = W_1' \oplus W_1''$, and to the subset $\mathcal{P}_0$ of $\mathcal{P}$. Consider the central orthogonal idempotents

$$e := \sum_{x\in\mathcal{P}_0} e_x \quad \text{and} \quad f := 1 - e = \sum_{x\in\mathcal{P}\setminus\mathcal{P}_0} e_x$$

of R. By assumption, the ditalgebra $\mathcal{A}'$ has layer (R, W'), and we have the decomposition of R-R-bimodules $W' = W_0' \oplus W_1'$ with $W_0' = eW_0'e$ and $W_1' = eW_1'e$. Then $R \cong R_e \times R_f$, where $R_e := eRe$ and $R_f = fRf$. Moreover, we have isomorphisms of R-R-bimodules: $W_0' \cong W_0^e \times 0$, where W_0^e denotes the R_e-R_e-bimodule obtained from W_0' by restriction and 0 is the trivial R_f-R_f-bimodule; and $W_1' \cong W_1^e \times 0$, where W_1^e denotes the R_e-R_e-bimodule obtained from W_1' by restriction and 0 is the trivial R_f-R_f-bimodule. Then we have an isomorphism of graded t-algebras $T_R(W') \cong T_{R_e}(W^e) \times T_{R_f}(0)$, where $W^e = W_0^e \oplus W_1^e$ (see [6, 10.1]). We already have the differential δ' of $\mathcal{A}'$, defined on the t-algebra $T' \cong T_R(W')$ by restriction of the differential δ of $\mathcal{A}$. For $i \in \{0, 1\}$, notice that whenever the R-bimodule W_i is freely generated by the set $\mathbb{B}_i$ of arrows, the R-bimodule $W_i' = eW_ie$ is freely generated by the subset $\mathbb{B}_i'$ of $\mathbb{B}_i$ formed by the arrows starting and ending at points of $\mathcal{P}_0$. Thus, $\mathcal{A}'$ is a seminested ditalgebra. Moreover, the R_e-bimodule W_i^e is freely generated by the same set $\mathbb{B}_i'$ of arrows. Then we can also consider the differential δ^e defined on each arrow α of the t-algebra $T^e := T_{R_e}(W^e)$ by the same formal expression for $\delta'(\alpha)$. Thus, we can consider the seminested ditalgebra $\mathcal{A}^e = (T^e, \delta^e)$, with points $\mathcal{P}^e = \mathcal{P}_0$ and with the same arrows as $\mathcal{A}'$. If we consider the minimal ditalgebra $\mathcal{A}^f = (T_{R_f}(0), 0)$, then it is now clear that $\mathcal{A}'$ is a product of ditalgebras, $\mathcal{A}' \cong \mathcal{A}^e \times \mathcal{A}^f$, as in [6, 10.2].

LEMMA 5.3. *Let Λ be a basic finite-dimensional algebra over the algebraically closed field k and let Λ_0 be a convex algebra in Λ. Consider the Drozd ditalgebra $\mathcal{D} = \mathcal{D}^\Lambda$ of Λ (as in* [6, 23.25]*). Then there is a convex subditalgebra $\mathcal{D}'$ of $\mathcal{D}$ and a functor $\Xi' : \mathcal{D}'\text{-Mod} \to \mathcal{P}^1(\Lambda_0)$ such that the following square commutes up to isomorphism:*

$$\begin{array}{ccc} \mathcal{D}\text{-Mod} & \xrightarrow{\Xi_\Lambda} & \mathcal{P}^1(\Lambda) \\ {\scriptstyle R^{\mathcal{D}}_{\mathcal{D}'}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}} \\ \mathcal{D}'\text{-Mod} & \xrightarrow{\Xi'} & \mathcal{P}^1(\Lambda_0) \end{array}$$

Here, Ξ_Λ denotes the usual equivalence of [6, 19.8].

Proof. By assumption, there is a semisimple subalgebra S of Λ such that Λ admits the S-S-bimodule decomposition $\Lambda = S \oplus P$, where $P = \operatorname{rad}\Lambda$. Consider a decomposition $1 = \sum_{i\in I} e_i$ of the unit element as a sum of central primitive orthogonal idempotents of S. Consider the set $E := \{e_i \mid i \in I\}$ of idempotents and the convex subset $E_0 := \{e_i \mid i \in I_0\}$ of E such that $\Lambda_0 = e_0\Lambda e_0$, with $e_0 = \sum_{i\in I_0} e_i$.

Let us recall, from [6, 23.25], the description of the bigraph of the nested ditalgebra $\mathcal{D}$. We consider a special dual basis $(p_j, \gamma_{p_j})_{j\in J}$ of the right S-module P (as constructed in [6, 23.11]). Thus, $\{p_j\}_{j\in J}$ and $\{\gamma_{p_j}\}_{j\in J}$ are vector space bases for P and P^*, respectively. Consider also the structural constants $c^t_{i,j}$ of the product of Λ restricted to P. Hence, $p_sp_r = \sum_t c^t_{s,r}p_t$ for any basic elements p_r and p_s of P. Then $R = R^\Lambda$ is a trivial algebra, with canonical decomposition $1 = (\sum_{i\in I} e'_i) + (\sum_{i\in I} e''_i)$, where $e'_i = \left(\begin{smallmatrix} e_i & 0 \\ 0 & 0\end{smallmatrix}\right)$ and $e''_i = \left(\begin{smallmatrix} 0 & 0 \\ 0 & e_i\end{smallmatrix}\right)$. Thus, the bigraph of $\mathcal{D}$ has $2|I|$ points associated to these idempotents, which we denote by the same symbols. For each basic element $p \in e_jPe_i$, we have the basic element $\gamma_p \in e_iP^*e_j$ such that $\gamma_p(q) = \delta_{p,q}$ (the Kronecker delta of the basic elements $p, q \in P$). Every such basic element p determines: a solid arrow $\alpha_p := \left(\begin{smallmatrix} 0 & 0 \\ \gamma_p & 0\end{smallmatrix}\right)$ of $\mathcal{D}$ from e'_j to e''_i; a dotted arrow $v'_p := \left(\begin{smallmatrix} \gamma_p & 0 \\ 0 & 0\end{smallmatrix}\right)$ of $\mathcal{D}$ from e'_j to e'_i; and a dotted arrow $v''_p := \left(\begin{smallmatrix} 0 & 0 \\ 0 & \gamma_p\end{smallmatrix}\right)$ of $\mathcal{D}$ from e''_j to e''_i. These are all the arrows of $\mathcal{D}$. The values of the differential δ^Λ of $\mathcal{D}$ on these arrows are given by

$$\delta^\Lambda(\alpha_p) = \sum_{r,s,t} c^t_{s,r}\delta_{p,p_t}v''_{p_r}\alpha_{p_s} - \sum_{r,s,t} c^t_{s,r}\delta_{p,p_t}\alpha_{p_r}v'_{p_s},$$
$$\delta^\Lambda(v'_p) = \sum_{r,s,t} c^t_{s,r}\delta_{p,p_t}v'_{p_r}v'_{p_s}, \qquad \delta^\Lambda(v''_p) = \sum_{r,s,t} c^t_{s,r}\delta_{p,p_t}v''_{p_r}v''_{p_s}.$$

Equivalently,

$$\delta^\Lambda(\alpha_{p_t}) = \sum_{r,s} c^t_{s,r}v''_{p_r}\alpha_{p_s} - \sum_{r,s} c^t_{s,r}\alpha_{p_r}v'_{p_s},$$
$$\delta^\Lambda(v'_{p_t}) = \sum_{r,s} c^t_{s,r}v'_{p_r}v'_{p_s}, \qquad \delta^\Lambda(v''_{p_t}) = \sum_{r,s} c^t_{s,r}v''_{p_r}v''_{p_s}.$$

Now, consider the convex proper subditalgebra $\mathcal{D}'$ of $\mathcal{D}$ determined by the set of idempotents $E^\bullet_0 := \{e'_i \mid i \in I_0\} \cup \{e''_i \mid i \in I_0\}$. Then consider the idempotent $e := \sum_{i\in I_0} e'_i + \sum_{i\in I_0} e''_i$ of $R = R^\Lambda$, and the R-R-subbimodules

$W_0' := eW_0e$ of $W_0 = W_0^\Lambda$ and $W_1' := eW_1e$ of $W_1 = W_1^\Lambda$. If we consider the idempotent $f := 1-e$ of R, we have the R-R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = W_1' \oplus W_1''$, where $W_0'' := fW_0f \oplus eW_0f \oplus fW_0e$ and $W_1'' := fW_1f \oplus eW_1f \oplus fW_1e$. In order to show that $\mathcal{D}'$ is the proper subditalgebra associated to these bimodule decompositions, we just have to check that $\delta(W_0') \subseteq D'W_1'D'$ and $\delta(W_1') \subseteq D'W_1'D'W_1'D'$, where D' denotes the subalgebra of $D = [T]_0$ generated by R and W_0'. If α_{p_t} is a typical solid arrow of $\mathcal{D}'$, which is a typical solid arrow of $\mathcal{D}$ between idempotents of $E_0^\bullet$, thus $p_t \in ePe$, we want to see that $\delta^\Lambda(\alpha_{p_t}) = \sum_{r,s} c_{s,r}^t v_{p_r}'' \alpha_{p_s} - \sum_{r,s} c_{s,r}^t \alpha_{p_r} v_{p_s}' \in D'W_1'D'$. Indeed, $c_{s,r}^t \neq 0$ means that the basic element p_t appears with non-zero coefficient in the expression of the product $p_s p_r$ in terms of basic elements of P. From the convexity of E_0, since $p_s p_r \neq 0$, we know that p_s and p_r, which start and end at idempotents in E_0, have to connect at an idempotent of E_0 too (recall that each basic element p_r is directed, as in [6, 23.1]). Thus, v_{p_r}'' is a dashed arrow of W_1' and α_{p_s} is a solid arrow of W_0'. Similarly, α_{p_r} is a solid arrow of W_0' and v_{p_s}' is a dashed arrow of W_1'. The fact that $\delta^\Lambda(v_{p_t}') = \sum_{r,s} c_{s,r}^t v_{p_r}' v_{p_s}'$ and $\delta^\Lambda(v_{p_t}'') = \sum_{r,s} c_{s,r}^t v_{p_r}'' v_{p_s}''$ live in $D'W_1'D'W_1'D'$ is verified similarly. This shows that $\mathcal{D}'$ is indeed a convex subditalgebra of $\mathcal{D}$.

Now, let us construct the functor $\Xi' : \mathcal{D}'\text{-Mod} \to \mathcal{P}^1(\Lambda_0)$. According to 5.2, there is an isomorphism of ditalgebras $\mathcal{D}' \cong \mathcal{D}^e \times \mathcal{D}^f$. As a consequence, for instance from [6, 16.3], we have an equivalence

$$\mathcal{D}^e\text{-Mod} \times \mathcal{D}^f\text{-Mod} \to \mathcal{D}'\text{-Mod},$$

and hence a projection functor $H : \mathcal{D}'\text{-Mod} \to \mathcal{D}^e\text{-Mod}$. Given $M \in \mathcal{D}'\text{-Mod}$, we have $H(M) = eM$, and given $g \in \mathrm{Hom}_{\mathcal{D}'}(M,N)$, we have $H(g) = (H(g)^0, H(g)^1)$ with $H(g)^0(em) = eg^0(m)$ and $H(g)^1(v)(em) = g^1(v)(em)$ for $v \in W_1^e = eW_1e$ and $m \in eM$.

Moreover, if we consider the Drozd nested ditalgebra $\mathcal{D}^{\Lambda_0}$ of the algebra Λ_0, there is a very natural isomorphism of nested ditalgebras $\mathcal{D}^e \cong \mathcal{D}^{\Lambda_0}$ determined by the isomorphisms

$$R^e = eR^\Lambda e = e\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} e \cong \begin{pmatrix} S_0 & 0 \\ 0 & S_0 \end{pmatrix} = R^{\Lambda_0},$$

$$W_0^e = eW_0^\Lambda e = e\begin{pmatrix} 0 & 0 \\ P^* & 0 \end{pmatrix} e \cong \begin{pmatrix} 0 & 0 \\ P_0^* & 0 \end{pmatrix} = (W_0)^{\Lambda_0},$$

$$W_1^e = eW_1^\Lambda e = e\begin{pmatrix} P^* & 0 \\ 0 & P^* \end{pmatrix} e \cong \begin{pmatrix} P_0^* & 0 \\ 0 & P_0^* \end{pmatrix} = (W_1)^{\Lambda_0}.$$

Here, the last two isomorphisms are determined by the canonical isomorphism of S_0-S_0-bimodules $e_0 P^* e_0 \cong P_0^*$, where the first dual is taken over the algebra S and the second over S_0. By construction, our special dual basis $(p_j,\gamma_{p_j})_{j\in J}$ of the S-S-bimodule P contains a special dual basis $(p_j,\gamma_{p_j})_{j\in J_0}$ of the S_0-S_0-bimodule $P_0=e_0Pe_0$. More precisely, $\{p_j \mid j\in J_0\}$ is a k-basis for P_0 and $\{\gamma_{p_j}\mid j\in J_0\}$ is a k-basis for $e_0P^*e_0$, which we shall identify with P_0^*. Then the given isomorphisms map each solid arrow α_p of $\mathcal{D}^e$ to the solid arrow α_p of $\mathcal{D}^{\Lambda_0}$, and similarly for dashed arrows. The non-zero structural constants of the product of basic elements p_r,p_s of e_0Pe_0 coincide with those of the same basic elements considered in P. This means that the differentials δ^e and δ^{Λ_0} coincide on the arrows. Thus, we have an isomorphism $\varphi:\mathcal{D}^{\Lambda_0}\to\mathcal{D}^e$ of nested ditalgebras, and therefore an isomorphism of categories $F_\varphi:\mathcal{D}^e\text{-Mod}\to\mathcal{D}^{\Lambda_0}\text{-Mod}$.

Now, we can define the functor Ξ' to be the composition

$$\mathcal{D}'\text{-Mod}\xrightarrow{H}\mathcal{D}^e\text{-Mod}\xrightarrow{F_\varphi}\mathcal{D}^{\Lambda_0}\text{-Mod}\xrightarrow{\Xi_{\Lambda_0}}\mathcal{P}^1(\Lambda_0).$$

It remains to show that the square of functors in the statement of our lemma commutes up to isomorphism. Recall that any $\mathcal{D}$-module M determines a triple (M_1,M_2,ψ_M), where $M_1,M_2\in S$-Mod and ψ_M is a morphism in $\operatorname{Hom}_{S\text{-}S}(P^*,\operatorname{Hom}_k(M_1,M_2))$, and conversely. By definition, $\Xi_\Lambda(M):\Lambda\otimes_S M_1\to\Lambda\otimes_S M_2$ is the object in $\mathcal{P}^1(\Lambda)$ such that, for $\lambda\in\Lambda$ and $m_1\in M_1$, we have $\Xi_\Lambda(M)(\lambda\otimes m_1)=\sum_{j\in J}\lambda p_j\otimes\psi_M(\gamma_{p_j})[m_1]$. Thus, $\operatorname{Res}\Xi_\Lambda(M)=1_{\Lambda_0}\otimes\Xi_\Lambda(M):\Lambda_0\otimes_\Lambda\Lambda\otimes_S M_1\to\Lambda_0\otimes_\Lambda\Lambda\otimes_S M_2$.

For $m\in M_1$, $\lambda\in\Lambda$ and $\lambda_0\in\Lambda_0$, we have

$$\lambda_0\otimes\lambda\otimes m_1=\lambda_0e_0\lambda e_0\otimes 1\otimes m_1=\lambda_0\otimes e_0\lambda e_0\otimes m_1=\lambda_0\otimes e_0\lambda e_0\otimes e_0m_1.$$

Then

$$\begin{aligned}\operatorname{Res}\Xi_\Lambda(M)(\lambda_0\otimes\lambda\otimes m_1)&=\operatorname{Res}\Xi_\Lambda(M)(\lambda_0\otimes e_0\lambda e_0\otimes e_0m_1)\\&=\lambda_0\otimes\sum_{j\in J}e_0\lambda e_0p_j\otimes\psi_M(\gamma_{p_j})[e_0m_1]\\&=\sum_{j\in J_0}\lambda_0\otimes e_0\lambda e_0p_j\otimes\psi_M(\gamma_{p_j})[e_0m_1],\end{aligned}$$

where the non-zero terms $\lambda_0\otimes e_0\lambda e_0p_j\otimes e_0\psi_M(\gamma_{p_j})[m_1]$ of the sum over J correspond to indices $j\in J$ with $e_0p_je_0\neq 0$, which means indices $j\in J_0$.

Let us examine the other composition. The $\mathcal{D}^{\Lambda_0}$-module $F_\varphi HR^{\mathcal{D}}_{\mathcal{D}'}(M)=eM$ has associated triple $(e_0M_1,e_0M_2,\psi_{eM})$, where

$$\begin{aligned}\psi_{eM}\in{}&\operatorname{Hom}_{S_0\text{-}S_0}(P_0^*,\operatorname{Hom}_k(e_0M_1,e_0M_2))\\&\cong\operatorname{Hom}_{S_0\text{-}S_0}(e_0P^*e_0,\operatorname{Hom}_k(M_1,M_2))\end{aligned}$$

is the restriction of ψ_M. Then $\Xi_{\Lambda_0}F_\varphi HR^{\mathcal{D}}_{\mathcal{D}'}(M):\Lambda_0\otimes_{S_0}e_0M_1\to\Lambda_0\otimes_{S_0}e_0M_2$ acts as $\lambda_0\otimes e_0m_1\mapsto\sum_{j\in J_0}\lambda_0p_j\otimes\psi_{eM}(\gamma_{p_j})[e_0m_1]$ and we obtain the following isomorphism in $\mathcal{P}^1(\Lambda_0)$:

$$\begin{array}{ccc} \Lambda_0 \otimes_\Lambda \Lambda \otimes_S M_1 & \xrightarrow{\operatorname{Res} \Xi_\Lambda(M)} & \Lambda_0 \otimes_\Lambda \Lambda \otimes_S M_2 \\ \downarrow \cong & & \downarrow \cong \\ \Lambda_0 \otimes_{S_0} e_0 M_1 & \xrightarrow{\Xi_{\Lambda_0} F_\varphi H R^{\mathcal{D}}_{\mathcal{D}'}(M)} & \Lambda_0 \otimes_{S_0} e_0 M_2 \end{array}$$

We have exhibited an isomorphism $\eta_M : \operatorname{Res} \Xi_\Lambda(M) \to \Xi_{\Lambda_0} F_\varphi H R^{\mathcal{D}}_{\mathcal{D}'}(M)$. It is not hard to see that it is natural in M. ∎

LEMMA 5.4. *Given a convex subditalgebra $\mathcal{A}'$ of a seminested ditalgebra $\mathcal{A}$, we can modify the triangular filtrations of $\mathcal{A}$, obtaining a different seminested ditalgebra $\overline{\mathcal{A}}$ with the same underlying layered ditalgebra $\mathcal{A}$, such that $\mathcal{A}'$ is an initial convex subditalgebra of $\overline{\mathcal{A}}$. Thus, $\mathcal{A}$ and $\overline{\mathcal{A}}$ coincide as ditalgebras and share the same layer (and the same basis of their layer), but the heights of their arrows are different. We have $\mathcal{A}$-*Mod $= \overline{\mathcal{A}}$-Mod*; as we shall see later, sometimes it is possible and convenient to replace $\mathcal{A}$ by $\overline{\mathcal{A}}$.*

Proof. We use the notation of 5.1 and consider the R-bimodule filtrations

$$0 = W_t^0 \subseteq W_t^1 \subseteq \cdots \subseteq W_t^i \subseteq \cdots \subseteq W_t^{\ell_t-1} \subseteq W_t^{\ell_t} = W_t,$$

with $t \in \{0,1\}$, given by the triangularity of $\mathcal{A}$ (see [6, 5.1]). Now, consider the ditalgebra $\overline{\mathcal{A}} = (T,\delta)$ with the same layer (R,W) as $\mathcal{A} = (T,\delta)$, but with new R-bimodule filtrations of length $2\ell_0$ for W_0 and of length $2\ell_1$ for W_1, given, for $t \in \{0,1\}$, by

$$\overline{W}_t^i = eW_t^i e \quad \text{for } i \in [0,\ell_t],$$
$$\overline{W}_t^{\ell_t+i} = eW_te \oplus C_t^i, \text{ where } C_t^i = eW_t^i f \oplus fW_t^i f \oplus fW_t^i e, \text{ for } i \in [1,\ell_t];$$

here f denotes the idempotent $1-e$ of R. It remains to show that these are triangular filtrations of the layer, as in [6, 5.1]. Denote by $\overline{A}_i$ the subalgebra of A generated by R and $\overline{W}_0^i$ for $i \in [0,2\ell_0]$. We want to show that

$$\delta(\overline{W}_0^{i+1}) \subseteq \overline{A}_i W_1 \overline{A}_i \qquad \text{for } i \in [0,2\ell_0-1],$$
$$\delta(\overline{W}_1^{i+1}) \subseteq A\overline{W}_1^i A \overline{W}_1^i A \quad \text{for all } i \in [0,2\ell_1-1].$$

Denote by A_i the subalgebra of A generated by R and W_0^i for $i \in [0,\ell_0]$. Then, for $i \in [0,\ell_0-1]$, we have $\delta(W_0') \subseteq A'W_1'A' \subseteq \overline{A}_{\ell_0+i}W_1\overline{A}_{\ell_0+i}$ and $\delta(C_0^{i+1}) \subseteq A_iW_1A_i \subseteq \overline{A}_{\ell_0+i}W_1\overline{A}_{\ell_0+i}$, therefore $\delta(\overline{W}_0^{\ell_0+i+1}) \subseteq \overline{A}_{\ell_0+i}W_1\overline{A}_{\ell_0+i}$.

For $i \in [0,\ell_1-1]$, we have $\delta(W_1') \subseteq A'W_1'A'W_1'A' \subseteq A\overline{W}_1^{\ell_1+i}A\overline{W}_1^{\ell_1+i}A$ and $\delta(C_1^{i+1}) \subseteq AW_1^iAW_1^iA \subseteq A\overline{W}_1^{\ell_1+i}A\overline{W}_1^{\ell_1+i}A$. Therefore, we also have $\delta(\overline{W}_1^{\ell_1+i+1}) \subseteq A\overline{W}_1^{\ell_1+i}A\overline{W}_1^{\ell_1+i}A$.

For $i \in [0,\ell_0-1]$, we have $\delta(\overline{W}_0^{i+1}) \subseteq eA_iW_1A_ie \cap A'W_1'A'$, but there is an R-bimodule decomposition $eA_iW_1A_ie = e\overline{A}_iW_1'\overline{A}_ie \oplus H_0$ with $H_0 \cap A'W_1'A' = 0$. Hence, $\delta(\overline{W}_0^{i+1}) \subseteq \overline{A}_iW_1'\overline{A}_i$.

For $i \in [0, \ell_1 - 1]$, we have $\delta(\overline{W}_1^{i+1}) \subseteq eAW_1^iAW_1^iAe \cap A'W_1'A'W_1'A'$, but there is an R-bimodule decomposition $eAW_1^iAW_1^iAe = eA'\overline{W}_1^iA'\overline{W}_1^iA'e \oplus H_1$ with $H_1 \cap A'W_1'A'W_1'A' = 0$. Hence, $\delta(\overline{W}_1^{i+1}) \subseteq A'\overline{W}_1^iA'\overline{W}_1^iA'$. ■

6. Main result for algebras

THEOREM 6.1. *Assume that Λ is a basic finite-dimensional tame algebra over an algebraically closed field k. Suppose that Λ_0 is a convex algebra in Λ. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}_0(d)$ of indecomposable Λ_0-modules such that, for any indecomposable Λ-module M with $\dim_k M \leq d$ and $M \ncong \operatorname{tens}(\operatorname{res}(M))$, the module $\operatorname{res}(M)$ is isomorphic to a direct sum of modules in $\mathcal{I}_0(d)$.*

Proof. Fix $d \in \mathbb{N}$. The functor Res considered in 2.1 restricts to a functor $\operatorname{Res} : \mathcal{P}^1(\Lambda) \to \mathcal{P}^1(\Lambda_0)$ and the following diagram commutes up to isomorphism:

$$\begin{array}{ccccc} \overline{\mathcal{D}}\text{-Mod} & \xrightarrow{\Xi_\Lambda} & \mathcal{P}^1(\Lambda) & \xrightarrow{\text{Cok}} & \Lambda\text{-Mod} \\ {\scriptstyle R^{\overline{\mathcal{D}}}_{\mathcal{D}'}}\downarrow & & {\scriptstyle \text{Res}}\downarrow & & \downarrow{\scriptstyle \text{res}} \\ \mathcal{D}'\text{-Mod} & \xrightarrow{\Xi'} & \mathcal{P}^1(\Lambda_0) & \xrightarrow{\text{Cok}_0} & \Lambda_0\text{-Mod} \end{array}$$

where $\overline{\mathcal{D}}$ was defined in 5.4 and its initial subditalgebra $\mathcal{D}'$ was constructed in 5.3. Since Λ is tame, from [6, 27.14], so is its Drozd ditalgebra $\mathcal{D}$, and so is $\overline{\mathcal{D}}$ too (recall that $\mathcal{D}$-Mod $= \overline{\mathcal{D}}$-Mod). Then we can apply 4.1 to the number $d' := (1 + \dim_k \Lambda)d \in \mathbb{N}$ to obtain a finite family $\mathcal{I}'(d')$ of indecomposable $\mathcal{D}'$-modules such that for any indecomposable $\overline{\mathcal{D}}$-module H with $\dim_k H \leq d'$ and $H \ncong E^{\overline{\mathcal{D}}}_{\mathcal{D}'}(H')$, and any $H' \in \mathcal{D}'$-Mod, the module $R^{\overline{\mathcal{D}}}_{\mathcal{D}'}(H)$ is isomorphic to a direct sum of indecomposables in $\mathcal{I}'(d')$. Having in mind the construction of $\mathcal{D}'$ and Ξ' in the proof of 5.3, hence the fact that $\mathcal{D}'$-Mod is equivalent to the product category $\mathcal{D}^e$-Mod $\times$ $\mathcal{D}^f$-Mod, we can consider the subfamily $\mathcal{I}''(d')$ of $\mathcal{I}'(d')$ obtained by excluding all the indecomposables from $\mathcal{D}^f$-Mod, as well as all the indecomposables $N' \in \mathcal{D}^e$-Mod such that $\Xi_{\Lambda_0}(N')$ has the form $Q \to 0$. Then $\mathcal{I}(d) := \operatorname{Cok}_0 \Xi'\mathcal{I}''(d')$ is a finite family of indecomposable Λ_0-modules.

Take any indecomposable Λ-module M with $\dim_k M \leq d$ and $M \ncong \operatorname{tens}(\operatorname{res}(M))$ and let us show that $\operatorname{res}(M)$ is isomorphic to a direct sum of Λ_0-modules in $\mathcal{I}(d)$. Consider a minimal projective presentation $Q' \to Q \to M \to 0$ of M. Then, there is an $N \in \mathcal{D}$-Mod $= \overline{\mathcal{D}}$-Mod such that $\Xi_\Lambda(N) \cong (Q' \to Q)$ and $\operatorname{Cok} \Xi_\Lambda(N) \cong M$. Since M is indecomposable, so is N.

Now, from [6, 22.19 and 27.13], if P denotes the radical of Λ,

$$\dim_k N = \ell_\Lambda(Q/PQ) + \ell_\Lambda(Q'/PQ') \leq \dim_k M \cdot (1 + \dim_k \Lambda) \leq d'.$$

Suppose $N \cong E^{\overline{D}}_{D'}(N')$ for some $N' \in \mathcal{D}'$-Mod. As $\mathcal{D}' \cong \mathcal{D}^e \times \mathcal{D}^f$, we can consider the projection morphisms $\pi^e : \mathcal{D}' \to \mathcal{D}^e$ and $\pi^f : \mathcal{D}' \to \mathcal{D}^f$. The induced functors $F^e : \mathcal{D}^e$-Mod $\to \mathcal{D}'$-Mod and $F^f : \mathcal{D}^f$-Mod $\to \mathcal{D}'$-Mod determine an equivalence of categories

$$\mathcal{D}^e\text{-Mod} \times \mathcal{D}^f\text{-Mod} \xrightarrow{F^e \oplus F^f} \mathcal{D}'\text{-Mod}$$

(see [6, 10.3]). There is an isomorphism $N' \cong F^e(N^e) \oplus F^f(N^f)$ in D'-Mod, for some $N^e \in \mathcal{D}^e$-Mod and $N^f \in \mathcal{D}^f$-Mod, which is preserved by the functor $E^{\overline{D}}_{D'}$. Then $N \cong E^{\overline{D}}_{D'}(N') \cong E^{\overline{D}}_{D'}F^e(N^e) \oplus E^{\overline{D}}_{D'}F^f(N^f)$, and since N is indecomposable, we have $N^e = 0$ or $N^f = 0$. If $N^f \neq 0$, then $N^e = 0$ and N^f is indecomposable. In order to justify this last statement, assume N^f decomposes non-trivially; then it does so in D^f-Mod, hence $F^d(N^f)$ has a non-trivial decomposition in D'-Mod, which is preserved by $E^{\overline{D}}_{D'}$, contradicting again the indecomposability of N. Since $\mathcal{D}$ has no marked points, N^f is a one-dimensional module of $\mathcal{D}^f$, thus $F^f(N^f)$ is a one-dimensional module corresponding to a point of $\mathcal{D}'$ not in $\mathcal{D}^e$. Then its extension $N \cong E^{\overline{D}}_{D'}F^f(N^f)$ is again such a one-dimensional $\mathcal{D}$-module, corresponding to a point not in $E_0^{\bullet}$. Its image under Ξ_Λ has the form $\Lambda \otimes_S N_1 \to \Lambda \otimes_S N_2$, where either $N_1 = 0$ or $N_2 = 0$. If $\Lambda \otimes_S N_2 = 0$, then $M \cong \operatorname{Cok} \Xi_\Lambda(N) = 0$, a contradiction. Thus, $\Lambda \otimes_S N_2 \neq 0$, and $M \cong \operatorname{Cok} \Xi_\Lambda(N) \cong \Lambda \otimes_S N_2 \cong \Lambda e_i$ with $e_i \in E \setminus E_0$, thus $\operatorname{res} M = 0$. Therefore, we can assume that $N^f = 0$, and hence $N \cong E^{\overline{D}}_{D'}F^e(N^e)$.

We claim that, for any $N^e \in \mathcal{D}^e$-Mod,

$$\Xi_\Lambda E^{\overline{D}}_{D'}F^e(N^e) \cong \operatorname{Tens} \Xi_{\Lambda_0}(N^e).$$

To verify this claim, notice first that $e_0(E^{\overline{D}}_{D'}F^e(N^e))_1 = (E^{\overline{D}}_{D'}F^e(N^e))_1$, so we have isomorphisms

$$\Lambda e_0 \otimes_{\Lambda_0} \Lambda_0 \otimes_{S_0} N_i^e \xrightarrow{\eta_i} \Lambda \otimes_S (E^{\overline{D}}_{D'}F^e(N^e))_i$$

for $i \in \{1, 2\}$, given by $\eta_i(\lambda \otimes \lambda_0 \otimes n) = \lambda\lambda_0 \otimes n$, where $\lambda \in \Lambda e_0$, $\lambda_0 \in \Lambda_0$ and $n \in N_i^e$. Here, (N_1^e, N_2^e, ψ^e) and $((E^{\overline{D}}_{D'}F^e(N^e))_1, (E^{\overline{D}}_{D'}F^e(N^e))_2, \psi)$ are the triples corresponding to the $\mathcal{D}^e$-module N^e and the $\mathcal{D}$-module $E^{\overline{D}}_{D'}F^e(N^e)$, respectively. As before, we can identify $e_0 P^* e_0$ with P_0^*, and $\mathcal{D}^e$ with $\mathcal{D}^{\Lambda_0}$. Observe that $\gamma_{p_j} \in P^* \setminus P_0^*$ implies that α_{p_j} is an arrow of $\mathcal{D}$ not in $\mathcal{D}'$; then, for $n \in (E^{\overline{D}}_{D'}F^e(N^e))_1$, we have $\psi(\gamma_{p_j})[n] = \left(\begin{smallmatrix} 0 & 0 \\ \gamma_{p_j} & 0 \end{smallmatrix}\right)n = \alpha_{p_j} n = 0$; while, for $\gamma_{p_j} \in P_0^*$, $\psi(\gamma_{p_j})[n] = \left(\begin{smallmatrix} 0 & 0 \\ \gamma_{p_j} & 0 \end{smallmatrix}\right)n = \psi^e(\gamma_{p_j})[n]$. Then the following diagram commutes:

$$\begin{array}{ccc}
\Lambda e_0 \otimes_{\Lambda_0} \Lambda_0 \otimes_{S_0} N_1^e & \xrightarrow{1\otimes \Xi_{\Lambda_0}(N^e)} & \Lambda e_0 \otimes_{\Lambda_0} \Lambda_0 \otimes_{S_0} N_2^e \\
\eta_1 \downarrow & & \downarrow \eta_2 \\
\Lambda \otimes_S (E_{D'}^{\overline{D}} F^e(N^e))_1 & \xrightarrow{\Xi_\Lambda E_{D'}^{\overline{D}} F^e(N^e)} & \Lambda \otimes_S (E_{D'}^{\overline{D}} F^e(N^e))_2
\end{array}$$

Indeed, for $\lambda \in \Lambda e_0$, $\lambda_0 \in \Lambda_0$ and $n \in N_1^e$, we have

$$\begin{aligned}
\eta_2(1 \otimes \Xi_{\Lambda_0}(N^e))[\lambda \otimes \lambda_0 \otimes n] &= \eta_2\Big(\lambda \otimes \sum_{j\in J_0} \lambda_0 p_j \otimes \psi^e(\gamma_{p_j})[n]\Big) \\
&= \sum_{j\in J_0} \lambda\lambda_0 p_j \otimes \psi^e(\gamma_{p_j})[n] \\
&= \sum_{j\in J} \lambda\lambda_0 p_j \otimes \psi(\gamma_{p_j})[n] \\
&= \Xi_\Lambda E_{D'}^{\overline{D}} F^e(N^e)[\lambda\lambda_0 \otimes n] \\
&= \Xi_\Lambda E_{D'}^{\overline{D}} F^e(N^e)\eta_1[\lambda \otimes \lambda_0 \otimes n].
\end{aligned}$$

Thus, $\Xi_\Lambda E_{D'}^{\overline{D}} F^e(N^e) \cong \operatorname{Tens} \Xi_{\Lambda_0}(N^e)$.

Apply this claim to our previously fixed N^e to obtain

$$\Xi_\Lambda(N) \cong \Xi_\Lambda E_{D'}^{\overline{D}} F^e(N^e) \cong \operatorname{Tens} \Xi_{\Lambda_0}(N^e).$$

Therefore, using 2.5, we obtain

$$M \cong \operatorname{Cok} \Xi_\Lambda(N) \cong \operatorname{Cok} \operatorname{Tens} \Xi_\Lambda(N^e) \cong \operatorname{tens} \operatorname{Cok} \Xi_\Lambda(N^e),$$

which is a contradiction (recall the last statement of 2.5). Hence, $N \not\cong E_{D'}^{\overline{D}}(N')$ for any $N' \in \mathcal{D}'$-Mod, and $R_{\mathcal{D}'}^{\overline{\mathcal{D}}}(N) \cong \bigoplus_i N_i$ for some indecomposable $\mathcal{D}'$-modules $N_i \in \mathcal{I}'(d')$. From 5.3, it follows that

$$\begin{aligned}
\operatorname{res}(M) &\cong \operatorname{res} \operatorname{Cok} \Xi_\Lambda(N) \cong \operatorname{Cok} \operatorname{Res} \Xi_\Lambda(N) \cong \operatorname{Cok} \Xi' R_{\mathcal{D}'}^{\overline{\mathcal{D}}}(N) \\
&\cong \bigoplus_i \operatorname{Cok} \Xi'(N_i),
\end{aligned}$$

a direct sum of modules in $\mathcal{I}(d)$, and we are done. ∎

Now, clearly, Theorem 1.3 follows from 6.1 and 2.6.

7. Dual results. The dual results concern, given a convex algebra Λ_0 in Λ, the restriction functor $\operatorname{res}' = \operatorname{Hom}_\Lambda(\Lambda_0, -) : \Lambda\text{-Mod} \to \Lambda_0\text{-Mod}$.

LEMMA 7.1. *Assume that the algebra Λ_0 is convex in Λ, and denote by $\mathcal{Q}(\Lambda)$ and $\mathcal{Q}(\Lambda_0)$ the categories of morphisms between injective Λ-modules and injective Λ_0-modules, respectively. The functor* res′ *preserves injectives, and hence induces a functor* $\operatorname{Res}' : \mathcal{Q}(\Lambda) \to \mathcal{Q}(\Lambda_0)$ *such that the following* *square commutes up to isomorphism:*

$$\begin{array}{ccc} \mathcal{Q}(\Lambda) & \xrightarrow{\text{Ker}} & \Lambda\text{-Mod} \\ {\scriptstyle \text{Res}'}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{res}'} \\ \mathcal{Q}(\Lambda_0) & \xrightarrow{\text{Ker}_0} & \Lambda_0\text{-Mod} \end{array}$$

Here, Ker *and* Ker_0 *are the corresponding kernel functors.*

Proof. Given an idempotent $e_i \in E$, we have the isomorphisms

$$\begin{aligned} \text{res}'(D(e_i\Lambda)) &= \text{Hom}_\Lambda(\Lambda_0, D(e_i\Lambda)) = \text{Hom}_\Lambda(\Lambda_0, \text{Hom}_k(e_i\Lambda, k)) \\ &\cong \text{Hom}_k(e_i\Lambda \otimes_\Lambda \Lambda_0, k) \cong \text{Hom}_k(e_i\Lambda_0, k) \cong D_0(e_i\Lambda_0), \end{aligned}$$

where the injective Λ_0-module $D_0(e_i\Lambda_0)$ is zero when $e_i \in E \setminus E_0$. This implies that the functor res′ preserves injectives. Indeed, any injective Λ-module Q has the form $Q \cong \bigoplus_{i\in I} D(e_i\Lambda)$ for some family $\{e_i\}_{i\in I}$ of idempotents of E, and so the inclusion morphism $\bigoplus_{i\in I} D(e_i\Lambda) \to \prod_{i\in I} D(e_i\Lambda)$ splits. Therefore, the induced monomorphism

$$\text{Hom}_\Lambda\Big(\Lambda_0, \bigoplus_{i\in I} D(e_i\Lambda)\Big) \to \text{Hom}_\Lambda\Big(\Lambda_0, \prod_{i\in I} D(e_i\Lambda)\Big) \cong \prod_{i\in I} \text{Hom}_\Lambda(\Lambda_0, D(e_i\Lambda)),$$

which has an injective codomain, also splits. It follows that the Λ_0-module $\text{res}'(Q) \cong \text{res}'(\bigoplus_{i\in I} D(e_i\Lambda))$ is injective.

Now, given an object $\phi : Q_1 \to Q_0$ in $\mathcal{Q}(\Lambda)$, we can consider the object $\text{Res}'(\phi) := \phi_* : \text{Hom}_\Lambda(\Lambda_0, Q_1) \to \text{Hom}_\Lambda(\Lambda_0, Q_0)$ in $\mathcal{Q}(\Lambda_0)$. Given a morphism $(u, v) : \phi \to \phi'$ in $\mathcal{Q}(\Lambda)$, the rule $\text{Res}'(u, v) = (\text{res}'\, u, \text{res}'\, v)$ clearly defines a functor. Since res′ is left exact, for any $\phi \in \mathcal{Q}(\Lambda)$ there is an isomorphism $\eta_\phi : \text{Ker}_0 \text{Res}'\, \phi \to \text{res}' \text{Ker}\, \phi$, natural in the variable ϕ. ■

LEMMA 7.2. *If* Λ_0 *is a final algebra in* Λ*, then* res′ *is isomorphic to the standard restriction functor* $\rho : \Lambda\text{-Mod} \to \Lambda_0\text{-Mod}$.

Proof. If Λ_0 is final in Λ, we have $\Lambda_0 = e_0\Lambda e_0 = \Lambda e_0$, an equality of left Λ-modules. Hence, given $M \in \Lambda\text{-Mod}$, we have $\text{Hom}_\Lambda(\Lambda_0, M) = \text{Hom}_\Lambda(\Lambda e_0, M) \cong e_0 M$, a natural isomorphism in the variable M. ■

LEMMA 7.3. *Let* Λ_0 *be a convex algebra in* Λ*. Consider the functor* $\text{hom} = \text{Hom}_{\Lambda_0}(e_0\Lambda, -) : \Lambda_0\text{-Mod} \to \Lambda\text{-Mod}$*. Then*

$$\text{res}'\,\text{hom} \cong 1_{\Lambda_0\text{-Mod}},$$

and hence, given $M \in \Lambda\text{-Mod}$*, we have* $M \cong \text{hom}\,\text{res}'(M)$ *if and only if* $M \cong \text{hom}(M')$ *for some* $M' \in \Lambda_0\text{-Mod}$.

Proof. Notice that $e_0\Lambda \otimes_\Lambda \Lambda_0 \cong \Lambda_0$. Hence, for $M \in \Lambda_0\text{-Mod}$, we have isomorphisms of Λ-modules $\text{res}'\,\text{hom}(M) = \text{Hom}_\Lambda(\Lambda_0, \text{Hom}_{\Lambda_0}(e_0\Lambda, M)) \cong \text{Hom}_{\Lambda_0}(e_0\Lambda \otimes_\Lambda \Lambda_0, M) \cong \text{Hom}_{\Lambda_0}(\Lambda_0, M) \cong M$, which are natural in M. ■

LEMMA 7.4. *Assume that Λ_0 is a convex algebra in Λ. Then the functors* res *and* res′ *are dual to each other in the sense that the following diagram commutes up to isomorphism:*

$$\begin{array}{ccc} \Lambda\text{-Mod} & \xrightarrow{D} & \Lambda^{\mathrm{op}}\text{-Mod} \\ {\scriptstyle \mathrm{res}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}'} \\ \Lambda_0\text{-Mod} & \xrightarrow{D_0} & \Lambda_0^{\mathrm{op}}\text{-Mod} \end{array}$$

where $D = \mathrm{Hom}_k(-,k)$ and D_0 is the corresponding functor for Λ_0.

Proof. If $M \in \Lambda$-Mod, we have a natural isomorphism

$$\begin{aligned} D_0 \operatorname{res}(M) &= \mathrm{Hom}_k(\Lambda_0 \otimes_\Lambda M, k) \cong \mathrm{Hom}_k(M \otimes_{\Lambda_0^{\mathrm{op}}} \Lambda_0^{\mathrm{op}}, k) \\ &\cong \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda_0^{\mathrm{op}}, \mathrm{Hom}_k(M,k)) = \operatorname{res}' D(M) \end{aligned}$$

determined by the isomorphism of left Λ_0-modules $\Lambda_0 \otimes_\Lambda M \cong M \otimes_{\Lambda_0^{\mathrm{op}}} \Lambda_0^{\mathrm{op}}$, which is natural in M. ∎

Now, we can state the following result dual to Theorem 6.1.

THEOREM 7.5. *Assume that Λ is a basic finite-dimensional tame algebra over an algebraically closed field k. Suppose that Λ_0 is a convex algebra in Λ. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}_0(d)$ of indecomposable Λ_0-modules such that for any indecomposable Λ-module M with $\dim_k M \le d$ and $M \not\cong \hom(\operatorname{res}'(M))$, the module $\operatorname{res}'(M)$ is isomorphic to a direct sum of modules in $\mathcal{I}_0(d)$.*

Proof. Apply first 6.1 to the algebra Λ_0^{op}, convex in Λ^{op}, to obtain a family $\mathcal{I}'(d)$ of indecomposable modules in Λ_0^{op}-mod such that for any indecomposable Λ^{op}-module N with $\dim_k N \le d$ and $N \not\cong \operatorname{tens}(\operatorname{res}(N))$, $\operatorname{res}(N)$ is isomorphic to a direct sum of modules in $\mathcal{I}_0'(d)$. Denote by $\mathcal{I}(d)$ the family of indecomposable Λ_0-modules of the form $D_0(L)$ for some L in $\mathcal{I}_0'(d)$. Take any indecomposable Λ-module M with $\dim_k M \le d$ and $M \not\cong \hom(\operatorname{res}'(M))$. If we had $D(M) \cong \operatorname{tens}(\operatorname{res}(D(M)))$, then, applying D, we obtain $M \cong D^2(M) \cong D \operatorname{tens}(\operatorname{res}(D(M))) \cong \hom D_0 \operatorname{res} D(M) \cong \hom \operatorname{res}' D^2(M) \cong \hom \operatorname{res}'(M)$, a contradiction. Hence, $\operatorname{res}(D(M))$ is a direct sum of modules in $\mathcal{I}_0'(d)$. It follows that $D_0 \operatorname{res} D(M) \cong \operatorname{res}' D^2(M) \cong \operatorname{res}'(M)$ is a direct sum of modules in $\mathcal{I}_0(d)$, as claimed. ∎

Finally, using the statement dual to 2.6, we get the following.

THEOREM 7.6. *Assume that Λ is a basic finite-dimensional tame algebra over an algebraically closed field k, and consider a decomposition $1 = \sum_{e \in E} e$ into a sum of primitive orthogonal idempotents of Λ. Consider a convex subset E_0 of E and the associated convex algebra Λ_0. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}_0(d)$ of indecomposable Λ_0-modules such that, for* *any indecomposable Λ-module M with* $\dim_k M \leq d$ *and such that M does not admit a minimal injective copresentation with direct summands of the form $D(e\Lambda)$ with $e \in E_0$, the module* $\mathrm{res}'(M)$ *is isomorphic to a direct sum of indecomposables in $\mathcal{I}_0(d)$.*

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