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SLANT SUBMANIFOLDS IN COSYMPLECTIC MANIFOLDS

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Abstract. We give some examples of slant submanifolds of cosymplectic manifolds. Also, we study some special slant submanifolds, called austere submanifolds, and establish a relation between minimal and anti-invariant submanifolds which is based on properties of the second fundamental form. Moreover, we give an example to illustrate our result.

1. Introduction. The notion of a slant submanifold of an almost Hermitian manifold was introduced by Chen [7]. Examples of slant submanifolds of $\mathbb{C}^2$ and $\mathbb{C}^4$ were given by Chen and Tazawa [12], while those of slant submanifolds of a Kähler manifold were given by Maeda, Ohnita and Udagawa [21]. On the other hand, A. Lotta [19] defined and studied slant submanifolds of an almost contact metric manifold. He also studied the intrinsic geometry of 3-dimensional non-anti-invariant slant submanifolds of K-contact manifolds [20]. Later, L. Cabrerizo and others investigated slant submanifolds of a Sasakian manifold and obtained many interesting results [2] and examples. Slant submanifolds of cosymplectic manifolds have been studied in [16].

Lotta [19] has proved that a non-anti-invariant slant submanifold of a contact metric manifold must be odd-dimensional. This motivated us to find examples of slant submanifolds of a cosymplectic manifold with dimension greater than or equal to 3. In this paper we give some examples of minimal and non-minimal slant submanifolds with dimension 3. We also obtain sufficient conditions for slant submanifolds to be either austere or minimal.

2. Preliminaries. Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ a vector field, η a 1-form and g the Riemannian metric on $\overline{M}$. These tensors satisfy [1]

$$(2.1)\qquad \begin{cases} \phi^2 X = -X + \eta(X)\xi,\ \phi\xi = 0,\ \eta(\xi) = 1, & \eta(\phi X) = 0; \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), & \eta(X) = g(X, \xi), \end{cases}$$

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for any $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$. A normal almost contact metric manifold is called a *cosymplectic manifold* if

$$(\overline{\nabla}_X \varphi)(Y) = 0, \qquad \overline{\nabla}_X \xi = 0 \tag{2.2}$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$.

Let M be an m-dimensional Riemannian manifold with induced metric g isometrically immersed in $\overline{M}$. We denote by TM the Lie algebra of vector fields in M and by $T^{\perp}M$ the set of all vector fields normal to M.

For any $X \in TM$ and $N \in T^{\perp}M$, we write

$$\phi X = PX + FX \quad \text{and} \quad \phi N = tN + fN \tag{2.3}$$

where PX (resp. FX) denotes the tangential (resp. normal) component of ϕX, and tN (resp. fN) denotes the tangential (resp. normal) component of ϕN.

From now on, we suppose that the structure vector field ξ is tangent to M. Hence, if we denote by D the orthogonal distribution to ξ in TM, we can consider the orthogonal decomposition $TM = D \oplus \{\xi\}$.

For each non-zero X tangent to M at x such that X is not proportional to ξ_x, we denote by $\theta(X)$ the *Wirtinger angle* of X, that is, the angle between ϕX and T_xM.

The submanifold M is called *slant* if $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and $X \in T_xM - \{\xi_x\}$ (see [19]). The Wirtinger angle θ of a slant immersion is called the *slant angle* of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle 0 and $\pi/2$, respectively. A slant immersion which is neither invariant nor anti-invariant is called *proper*.

Let ∇ be the Riemannian connection on M. Then the Gauss and Weingarten formulae are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.4}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.5}$$

for $X, Y \in TM$ and $N \in T^{\perp}M$, where h and A_N are the second fundamental forms related by

$$g(A_N X, Y) = g(h(X, Y), N) \tag{2.6}$$

and $\nabla^{\perp}$ is the connection in the normal bundle $T^{\perp}M$ of M.

The mean curvature vector H is defined by $H = \frac{1}{m}(\operatorname{trace} h)$. We say that M is *minimal* if H vanishes identically.

A submanifold is said to be *austere* if the set of eigenvalues of A_N is invariant under multiplication by -1.

If P is the endomorphism defined by (2.3), then

$$g(PX,Y)+g(X,PY)=0. \tag{2.7}$$

Thus P^2, denoted by Q, is self-adjoint.

We define the covariant derivatives of Q, P and F by

$$(\nabla_X Q)Y=\nabla_X(QY)-Q(\nabla_X Y), \tag{2.8}$$

$$(\nabla_X P)Y=\nabla_X(PY)-P(\nabla_X Y), \tag{2.9}$$

$$(\nabla_X F)Y=\nabla_X^{\perp}(FY)-F(\nabla_X Y), \tag{2.10}$$

for any $X,Y\in TM$.

For 3-dimensional proper slant submanifolds of a cosymplectic manifold, we first prove:

LEMMA 2.1. *Let M be a 3-dimensional proper slant submanifold of a cosymplectic manifold. Then*

$$(\nabla_X P)Y=0 \quad \textit{for any } X,Y\in TM. \tag{2.11}$$

Proof. Let $p\in M$ and $\{e_1,e_2\}$ be an orthonormal frame on M defined in a neighbourhood U of p (cf. [20, Lemma 2.1, p. 40]). Put $\xi|_U=e_3$, and let ω_i^j be the structural 1-forms defined by

$$\nabla_X e_i=\sum_{j=1}^{3}\omega_i^j(X)e_j$$

for each vector field X tangent to M. By (2.2), we have

$$(\nabla_X P)e_3=\nabla_X Pe_3-P(\nabla_X e_3)=0.$$

Similarly, we get

$$(\nabla_X P)e_1=(\cos\theta)\omega_2^3(X)e_3, \qquad (\nabla_X P)e_2=-(\cos\theta)\omega_1^3(X)e_3.$$

On the other hand, writing

$$Y=\eta(Y)e_3+g(Y,e_1)e_1+g(Y,e_2)e_2$$

for all $Y\in TM$ and using the above formulae we obtain $(\nabla_X P)Y=0$, where we have used $\omega_2^3(X)=\omega_1^3(X)=0$. ∎

Now, using (2.11), we have

$$(\nabla_X Q)Y=0. \tag{2.12}$$

On the other hand, Gauss and Weingarten formulae together with (2.2) and (2.3) imply

$$(\nabla_X P)Y=A_{FY}X+th(X,Y), \tag{2.13}$$

$$\nabla_X^{\perp}(FY)-F(\nabla_X Y)=(\nabla_X F)Y=fh(X,Y)-h(X,PY), \tag{2.14}$$

for any $X, Y \in TM$. It is easy to see that (2.11) holds if and only if

$$A_{FY}X = A_{FX}Y, \tag{2.15}$$

where we have used (2.13). A similar calculation using (2.14) shows that

$$(\nabla_X F)Y = 0 \quad \text{if and only if} \quad A_N PY = -A_{fN}Y \tag{2.16}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$.

We state the following results for later use.

THEOREM A ([2]). *Let M be a submanifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = -\lambda(I - \eta \otimes \xi). \tag{2.17}$$

Furthermore, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

COROLLARY A ([2]). *Let M be a slant submanifold of an almost contact metric manifold $\overline{M}$ with slant angle θ. Then*

$$g(PX, PY) = (\cos^2\theta)\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{2.18}$$

$$g(FX, FY) = (\sin^2\theta)\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{2.19}$$

LEMMA A ([19]). *Let M be a slant submanifold of an almost contact metric manifold $\overline{M}$ with slant angle θ. Then, at each point x of M, $Q|_D$ has only one eigenvalue $\lambda_1 = \cos^2\theta$.*

Let M be a proper slant submanifold M with slant angle θ. For a unit tangent vector field e_1 on M perpendicular to ξ, we put

$$e_2 = (\sec\theta)Pe_1, \quad e_3 = \xi, \quad e_4 = (\csc\theta)Fe_1, \quad e_5 = (\csc\theta)Fe_2.$$

Then $e_1 = -(\sec\theta)Pe_2$ and by (2.2) and (2.3), e_1, e_2, $\xi = e_3$, e_4, e_5 form an orthonormal frame such that e_1, e_2, ξ are tangent to M and e_3, e_4 are normal to M. We call such an orthonormal frame an *adapted slant frame*. We also have

$$te_4 = -(\sin\theta)e_1, \quad te_5 = -(\sin\theta)e_2, \quad fe_4 = -(\cos\theta)e_5, \quad fe_5 = (\cos\theta)e_4.$$

If we put $h^r_{ij} = g(h(e_i, e_j), e_r)$, $i, j = 1, 2, 3$, $r = 4, 5$, then from [16, Lemma 3.1] we have

$$h^4_{12} = h^5_{11}, \quad h^4_{22} = h^5_{12}, \tag{2.20}$$

$$h^4_{13} = h^4_{32} = h^4_{33} = h^5_{13} = h^5_{23} = h^5_{33} = 0. \tag{2.21}$$

If $\dim \overline{M} = \overline{m}$, a local field of orthonormal frames $\{e_1, \dots, e_m, e_{m+1}, \dots, e_{\overline{m}}\}$ can be chosen such that, when restricted to M, the vectors $e_1, \dots, e_m$ are tangent to M and hence $e_{m+1}, \dots, e_{\overline{m}}$ are normal to M. Then, for any vector field X tangent to M, we can write

$$\overline{\nabla}_X e_i = \sum_{j=1}^{m} \omega_i^j(X) e_j + \sum_{k=m+1}^{\overline{m}} \omega_i^k(X) e_k, \tag{2.22}$$

$$\overline{\nabla}_X e_r = \sum_{j=1}^{m} \omega_r^j(X) e_j + \sum_{k=m+1}^{\overline{m}} \omega_r^k(X) e_k, \tag{2.23}$$

for $i \in \{1, \ldots, m\}$ and $r \in \{m+1, \ldots, \overline{m}\}$, where ω_i^j, ω_i^k, ω_r^j and ω_r^k are the connection forms of M in $\overline{M}$.

3. Examples of slant submanifolds. In the present section, we introduce a method to find examples of slant submanifolds of $\mathbb{R}^{2m+1}$ with almost contact metric structure $(\varphi_0, \xi, \eta, g)$, which satisfy

$$(\overline{\nabla}_X \varphi_0)(Y) = 0, \qquad \overline{\nabla}_X \xi = 0$$

for $X, Y \in T\mathbb{R}^{2m+1}$.

The cosymplectic structure on $T\mathbb{R}^{2m+1}$ is given by

$$\eta = dz, \qquad \xi = \partial/\partial z, \tag{3.1}$$

$$g = \eta \otimes \eta + \sum_{i=1}^{m} (dx^i \otimes dx^i + dy^i \otimes dy^i) \tag{3.2}$$

and

$$\varphi_0\biggl(\sum_{i=1}^{m} \Bigl(X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i} \Bigr) + Z \frac{\partial}{\partial z} \biggr) = \sum_{i=1}^{m} \Bigl(Y_i \frac{\partial}{\partial x^i} - X_i \frac{\partial}{\partial y^i} \Bigr) \tag{3.3}$$

where (x^i, y^i, z), $i = 1, \ldots, m$, are the cartesian coordinates on $\mathbb{R}^{2m+1}$. The following theorem yields examples of slant submanifolds in $\mathbb{R}^5(\varphi_0, \xi, \eta, g)$.

THEOREM 3.1. *Let*

$$x(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v), f_4(u,v))$$

define a slant surface S in $\mathbb{C}^2$ with its usual Kählerian structure, such that $\partial/\partial u$ and $\partial/\partial v$ are non-zero and perpendicular. Then

$$y(u,v,t) = (f_1(u,v), f_2(u,v), f_3(u,v), f_4(u,v), t)$$

defines a three-dimensional slant submanifold M in $\mathbb{R}^5(\varphi_0, \xi, \eta, g)$ with the same slant angle such that, if we put $e_1 = \partial/\partial u$, $e_2 = \partial/\partial v$, then (e_1, e_2, ξ) is an orthogonal basis of the tangent bundle of the submanifold.

Proof. By means of the basis (e_1, e_2, ξ), it is easy to show that M is a three-dimensional submanifold of $\mathbb{R}^5$. To prove that M is slant, we write

$$X = \lambda_1 e_1 + \lambda_2 e_2 + \eta(X)\xi \qquad \text{for } X \in \chi(M).$$

Then

$$\sqrt{|X|^2 - \eta^2(X)} = \sqrt{\lambda_1^2 + \lambda_2^2}. \tag{3.4}$$

Now, since (e_1, e_2, ξ) is an orthogonal basis of $\chi(M)$, using (2.3) we obtain

$$|PX|^2 = \frac{g^2(\varphi_0 X, e_1)}{g(e_1, e_1)} + \frac{g^2(\varphi_0 X, e_2)}{g(e_2, e_2)}. \tag{3.5}$$

We may consider a vector field $X_0 \in TS$ such that $X_0 = \lambda_1 e_1 + \lambda_2 e_2$ and denoting by J the usual almost complex structure of $\mathbb{C}^2$, we find that

$$g(\varphi_0 X, e_1) = g(JX_0, e_1) \quad \text{and} \quad g(\varphi_0 X, e_2) = g(JX_0, e_2).$$

If $P_0 X_0$ is the tangent projection of JX_0 and θ is the slant angle of S, then from (3.4) and (3.5), we get

$$\frac{|PX|}{\sqrt{|X|^2 - \eta^2(X)}} = \frac{|P_0 X_0|}{X_0} = \cos\theta. \tag{3.6}$$

Hence, M is a slant submanifold with the same slant angle θ. ∎

By applying the examples given in [7] and the above theorem, we have the following examples of slant submanifolds of cosymplectic manifolds in $\mathbb{R}^5(\varphi_0, \xi, \eta, g)$:

EXAMPLE 3.1. For any $\theta \in [0, \pi/2]$,

$$x(u, v, t) = (u\cos\theta, u\sin\theta, v, 0, t)$$

defines a three-dimensional minimal slant submanifold M with slant angle θ.

We may choose an orthonormal basis (e_1, e_2, ξ) of $\chi(M)$ such that

$$e_1 = \cos\theta\,\frac{\partial}{\partial x^1} + \sin\theta\,\frac{\partial}{\partial x^2}, \qquad e_2 = \frac{\partial}{\partial y^1}, \qquad e_3 = \xi = \frac{\partial}{\partial z}.$$

Moreover, the vector fields

$$e_1^* = -\sin\theta\,\frac{\partial}{\partial x^1} + \cos\theta\,\frac{\partial}{\partial x^2}, \qquad e_2^* = \frac{\partial}{\partial y^2}$$

form an orthonormal basis for $T^{\perp}M$. Since $\overline{\nabla}_{e_i} e_i = 0$, we have $h(e_1, e_1) = 0$, $h(e_2, e_2) = 0$, $h(e_3, e_3) = 0$ and the submanifold is minimal.

EXAMPLE 3.2. For any positive constant k,

$$x(u, v, t) = (e^{ku}\cos u\cos v, e^{ku}\sin u\cos v, e^{ku}\cos u\sin v, e^{ku}\sin u\sin v, t)$$

defines a three-dimensional non-minimal slant submanifold M with the slant angle

$$\theta = \cos^{-1}\left(\frac{k}{\sqrt{1 + k^2}}\right).$$

In this case we may choose an orthonormal basis (e_1, e_2, ξ) of $\chi(M)$ such that

$$e_1 = \frac{e^{-ku}}{\sqrt{1+k^2}}\frac{\partial}{\partial u}, \qquad e_2 = e^{-ku}\frac{\partial}{\partial v}, \qquad e_3 = \xi = \frac{\partial}{\partial z}.$$

Also, at the points of the submanifold, we have

$$(x^1)^2 + (x^2)^2 + (y^1)^2 + (y^2)^2 = e^{2ku}.$$

Then, by a straightforward computation, we get $|H| = e^{-ku}/3\sqrt{1+k^2}$.

EXAMPLE 3.3. For any positive constant k,

$$x(u, v, t) = (u, k\cos v, v, k\sin v, t)$$

defines a three-dimensional non-minimal slant submanifold M with the slant angle

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{1+k^2}}\right).$$

Moreover, the following statements are equivalent: (i) $k = 0$, (ii) M is invariant, (iii) M is minimal. In this case orthonormal basis (e_1, e_2, ξ) of $\chi(M)$ is given by

$$e_1 = \frac{\partial}{\partial x^1}, \qquad e_2 = \frac{1}{\sqrt{1+k^2}}\left(-y^2\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^1} + x^2\frac{\partial}{\partial y^2}\right), \qquad e_3 = \xi = \frac{\partial}{\partial z}.$$

Moreover, by applying the vector fields $e_1^* = x^2\partial/\partial x^2 + y^2\partial/\partial y^2$ of $T^\perp M$ and some computation, we see that the mean curvature vector is

$$\overrightarrow{H} = -\frac{k}{3(1+k^2)}\, e_1^*.$$

EXAMPLE 3.4. For any non-zero constants a and b,

$$x(u, v, t) = (a\cos u, b\cos v, a\sin u, b\sin v, t)$$

gives a compact totally real submanifold M with $\overline{\nabla} h = 0$. In this case, we may take the orthonormal basis (e_1, e_2, ξ) of $\chi(M)$ as

$$e_1 = -\frac{y^1}{a}\frac{\partial}{\partial x^1} + \frac{x^1}{a}\frac{\partial}{\partial y^1}, \qquad e_2 = -\frac{y^2}{b}\frac{\partial}{\partial x^2} + \frac{x^2}{b}\frac{\partial}{\partial y^2}, \qquad e_3 = \xi = \frac{\partial}{\partial z}.$$

Moreover, the vector fields

$$e_1^* = -\frac{x^1}{a}\frac{\partial}{\partial x^1} - \frac{y^1}{a}\frac{\partial}{\partial y^1}, \qquad e_2^* = -\frac{x^2}{b}\frac{\partial}{\partial x^2} - \frac{y^2}{b}\frac{\partial}{\partial y^2}$$

generate the normal space $T^\perp M$.

4. Slant submanifolds and second fundamental forms. In this section, we study some properties of slant submanifolds related to the second fundamental form. We have:

PROPOSITION 4.1. *Any totally umbilical slant submanifold M of a cosymplectic manifold is totally geodesic.*

Proof. Since M is totally umbilical, we get $h(X,Y)=g(X,Y)H$ for all $X,Y\in\chi(M)$. From (2.2), we have $h(\xi,\xi)=0$, and consequently $H=0$. Hence $h(X,Y)=0$ for all $X,Y\in\chi(M)$ and the submanifold is totally geodesic.

From the above proposition it can be deduced that a totally umbilical submanifold is totally geodesic if and only if it is minimal.

Now, we consider another type of minimal submanifolds, namely austere submanifolds. We have the following:

THEOREM 4.2. *Let M be a proper slant submanifold of a cosymplectic manifold $\overline{M}$. If $(\nabla_X F)Y=0$ for all $X,Y\in\chi(M)$, then M is an austere submanifold.*

Proof. Since $(\nabla_X F)Y=0$, from (2.14) we have

$$fh(X,Y)=h(X,PY)\quad\text{for any } X,Y\in\chi(M). \tag{4.1}$$

It is easy to show that $(M,(\sec\theta)P,\xi,\eta,g)$ is an almost contact metric manifold, and we consider a local orthonormal basis

$$\{e_1,(\sec\theta)Pe_1,\dots,e_m,(\sec\theta)Pe_m,\xi\} \tag{4.2}$$

on M. Moreover, from (4.1) and (2.17), we get

$$h((\sec\theta)Pe_i,(\sec\theta)Pe_j)=-h(e_i,e_i)\quad\text{for any } i,j=1,\dots,m. \tag{4.3}$$

On the other hand, we write $\widetilde{X}=X-\eta(X)\xi$ and $X_*=(\sec\theta)PX$. Now, we shall show that if μ is a non-zero eigenvalue of A_N for any $N\in T^{\perp}M$, then $-\mu$ is also an eigenvalue of A_N for some non-zero vector $X_*=(\sec\theta)PX$ associated with $X\in\chi(M)$, i.e. $A_NX_*=-\mu X_*$.

From (4.2), we can write

$$\widetilde{X}=\sum_{i=1}^{m/2}\lambda_ie_i+\sum_{i=1}^{m/2}\mu_ie_{i*}. \tag{4.4}$$

Then

$$A_N\widetilde{X}=\sum_{i=1}^{m/2}\lambda_iA_Ne_i+\sum_{i=1}^{m/2}\mu_iA_Ne_{i*}. \tag{4.5}$$

Now, from (2.2) and (2.6), we get

$$A_Ne_i=\sum_{j=1}^{m/2}g(h(e_i,e_j),N)e_j+\sum_{j=1}^{m/2}g(h(e_i,e_{j*}),N)e_{j*}. \tag{4.6}$$

From (4.3), we get

$$A_N e_{i*} = \sum_{j=1}^{m/2} g(h(e_{i*}, e_j), N)e_j - \sum_{j=1}^{m/2} g(h(e_i, e_j), N)e_{j*}. \tag{4.7}$$

Applying P to (4.4), multiplying by $\sec\theta$ and using (2.17), we get

$$X_* = \sum_{i=1}^{m/2} \lambda_i e_{i*} - \sum_{i=1}^{m/2} \mu_i e_i. \tag{4.8}$$

Moreover, using $h(e_{i*}, e_j) = h(e_i, e_{j*})$, we get $A_N X_* = -\mu X_*$, which proves the result.

Now, we establish a relation between 3-dimensional minimal slant submanifolds and anti-invariant submanifolds of cosymplectic manifolds.

We have the following:

LEMMA 4.3. *Let M be a 3-dimensional proper slant submanifold of a 5-dimensional cosymplectic manifold $\overline{M}$ with slant angle θ. If $\{e_1, e_2, e_3 = \xi, e_4, e_5\}$ is an adapted slant basis, then*

$$\omega_4^5 - \omega_1^2 = -(\cot\theta)((\operatorname{trace} h^4)\omega^1 + (\operatorname{trace} h^5)\omega^2), \tag{4.9}$$

where ω^1, ω^2 are the dual forms of e_1, e_2.

Proof. Putting $X = Y = e_1$ in (2.14), we have

$$\nabla^{\perp}_{e_1} e_4 = \csc\theta\{F(\nabla_{e_1} e_1) + fh(e_1, e_1) - h(e_1, Pe_1)\}. \tag{4.10}$$

Using (2.22) and applying F, we get

$$F(\nabla_{e_1} e_1) = (\sin\theta)\omega_1^2(e_1)e_5. \tag{4.11}$$

On the other hand,

$$fh(e_1, e_1) = h_{11}^4 fe_4 + h_{11}^5 fe_5 = (\cos\theta)\{-h_{11}^4 e_5 + h_{11}^5 e_4\}, \tag{4.12}$$

$$h(e_1, Pe_1) = (\cos\theta)h(e_1, e_2) = (\cos\theta)\{h_{12}^4 e_4 + h_{12}^5 e_5\}. \tag{4.13}$$

Substituting (4.11)–(4.13) in (4.10), we find

$$\nabla^{\perp}_{e_1} e_4 = \omega_1^2(e_1)e_5 + (\cot\theta)(-h_{11}^4 e_5 + h_{11}^5 e_4 - h_{12}^4 e_4 - h_{12}^5 e_5)$$

From equations (2.20) and (2.21), we have

$$\nabla^{\perp}_{e_1} e_4 = \omega_1^2(e_1)e_5 - (\cot\theta)(\operatorname{trace} h^4)e_5,$$

and from (2.23) we get

$$\omega_4^5(e_1) - \omega_1^2(e_1) = -(\cot\theta)(\operatorname{trace} h^4). \tag{4.14}$$

Similarly,

$$\omega_4^5(e_2) - \omega_1^2(e_2) = -(\cot\theta)(\operatorname{trace} h^4), \tag{4.15}$$

$$\omega_4^5(e_3) - \omega_1^2(e_3) = 0. \tag{4.16}$$

Now, since $\{e_1, e_2, e_3 = \xi\}$ is a local orthonormal basis of the tangent space of M, dual to $\{\omega^1, \omega^2, \eta\}$, equation (4.9) follows from (4.14)–(4.16).

We now prove:

THEOREM 4.4. *Let M be a 3-dimensional proper slant submanifold of a 5-dimensional cosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ with slant angle θ. Suppose that there exists on $\overline{M}$ an almost contact structure $\overline{\varphi}$ such that $(\overline{M}, \overline{\varphi}, \xi, \eta, g)$ is an almost contact metric manifold satisfying*

$$g((\overline{\nabla}_X \overline{\varphi})Y, Z) = 0 \tag{4.17}$$

for any X, Y, Z normal to the structure vector field. If M is an anti-invariant submanifold with respect to the structure $(\overline{\varphi}, \xi, \eta, g)$, then M is a minimal submanifold of $\overline{M}$.

Proof. Let $\{e_1, e_2, e_3 = \xi, e_4, e_5\}$ be an adapted slant basis of the cosymplectic manifold $(\overline{M}, \varphi, \xi, \eta, g)$ and $\{e_4, e_5\}$ be a local orthonormal frame of $T^{\perp}M$. Since M is an anti-invariant submanifold in $(\overline{M}, \overline{\varphi}, \xi, \eta, g)$, it follows that $\{\overline{\varphi}e_1, \overline{\varphi}e_2\}$ is another local orthonormal basis of $T^{\perp}M$. Consequently, there exists a function ψ on M such that

$$\begin{cases} e_4 = (\cos\psi)\overline{\varphi}e_1 + (\sin\psi)\overline{\varphi}e_2, \\ e_4 = (-\sin\psi)\overline{\varphi}e_1 + (\cos\psi)\overline{\varphi}e_2. \end{cases} \tag{4.18}$$

Consider $\widetilde{X} \in D$; then

$$\omega_4^5(\widetilde{X}) = g(\overline{\nabla}_{\widetilde{X}} e_4, e_5)$$

and further using (4.17) and (4.18), we get

$$\omega_4^5(\widetilde{X}) - \omega_1^2(\widetilde{X}) = \widetilde{X}\psi = d\psi(\widetilde{X}). \tag{4.19}$$

Now, consider any $X \in \chi(M)$, i.e. $X = \widetilde{X} + \eta(X)\xi$. We find, by using (4.17) and (4.19), that

$$\omega_4^5(X) - \omega_1^2(X) = \omega_4^5(\widetilde{X}) - \omega_1^2(\widetilde{X}) + \eta(X)(\omega_4^5(\xi) - \omega_1^2(\xi)) = d\psi(\widetilde{X}).$$

But

$$d\psi(\widetilde{X}) = d\psi(X - \eta(X)\xi) = d\psi(X) - \eta(X)\xi(\psi).$$

Therefore

$$\omega_4^5 - \omega_1^2 = d\psi - \xi(\psi)\eta.$$

Using (4.9), we get

$$d\psi - \xi(\psi)\eta = -(\cot\theta)((\operatorname{trace} h^4)\omega^1 + (\operatorname{trace} h^5)\omega^2). \tag{4.20}$$

Also, from (4.17) and (4.18), we have

$$\begin{aligned} h_{11}^4 &= -g(\overline{\nabla}_{e_1} e_4, e_1) \\ &= (\cos\psi)g(h(e_1, e_1), \overline{\varphi}e_1) + (\sin\psi)g(h(e_1, e_2), \overline{\varphi}e_1). \end{aligned} \tag{4.21}$$

Again, from (4.18), we have

$$\begin{cases} \overline{\varphi}e_1 = (\cos\psi)e_4 - (\sin\psi)e_5, \\ \overline{\varphi}e_2 = (\sin\psi)e_4 + (\cos\psi)e_5. \end{cases} \tag{4.22}$$

Hence,

$$h^4_{11} = (\cos^2\psi)h^4_{11} - (\sin^2\psi)h^4_{22}.$$

Since $h^4_{33} = h^5_{33} = 0$, we get

$$(\sin^2\psi)(\operatorname{trace} h^4) = 0. \tag{4.23}$$

Similarly,

$$(\sin^2\psi)(\operatorname{trace} h^5) = 0. \tag{4.24}$$

Now, we set

$$U = \{x \in M : H(x) \neq 0\};$$

we will show that $U = \emptyset$. Indeed, if $x \in U$ then

$$\frac{1}{3}(\operatorname{trace} h) = \frac{1}{3}\{(\operatorname{trace} h^4)e_4 + (\operatorname{trace} h^5)e_5\} = H(x) \neq 0,$$

and hence

$$\operatorname{trace} h^4 \neq 0 \quad \text{or} \quad \operatorname{trace} h^5 \neq 0. \tag{4.25}$$

From (4.23) and (4.25), we conclude that $\psi \equiv 0 \pmod{\pi}$ in U. Thus, $d\psi = 0$ and $\xi(\psi) = 0$, and consequently, from (4.20), we have

$$(\cot\theta)((\operatorname{trace} h^4)\omega^1 + (\operatorname{trace} h^5)\omega^2) = 0.$$

Taking (4.25) into consideration, we get $\cot\theta = 0$, contrary to the fact that M is a proper slant submanifold. Hence $U = \emptyset$, and therefore M is minimal.

Finally, we consider an example: Let $\overline{\varphi}$ be the $(1,1)$-tensor field defined as follows:

$$\overline{\varphi}\left(\sum_{i=1}^{2}\left(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i} + Z\frac{\partial}{\partial z}\right)\right) = -X_2\frac{\partial}{\partial x^1} + X_1\frac{\partial}{\partial x^2} + Y_2\frac{\partial}{\partial y^1} - Y_1\frac{\partial}{\partial y^2}.$$

Then $\mathbb{R}^5(\overline{\varphi}, \xi, \eta, g)$ is an almost contact metric manifold. If we take the basis vectors as in Example 3.1, $e_1 = (\cos\theta)\partial/\partial x^1 + (\sin\theta)\partial/\partial x^2$, $e_2 = \partial/\partial y^1$ and $e_3 = \xi = \partial/\partial z$, then

$$\overline{\varphi}e_1 = -\sin\theta\frac{\partial}{\partial x^1} + \cos\theta\frac{\partial}{\partial x^2}$$

and

$$\begin{aligned} g(\overline{\varphi}e_1, e_2) &= \eta(\overline{\varphi}e_1)\eta(e_2) + dx^1(\overline{\varphi}e_1)dx^1(e_2) + dx^2(\overline{\varphi}e_1)dx^2(e_2) \\ &\quad + dy^1(\overline{\varphi}e_1)dy^1(e_2) + dy^2(\overline{\varphi}e_1)dy^2(e_2) \\ &= 0 = \sqrt{g(\overline{\varphi}e_1, \overline{\varphi}e_1)}\sqrt{g(e_2, e_2)}\cos\alpha, \end{aligned}$$

i.e. $\alpha = \pi/2$. Thus the submanifold is anti-invariant with respect to the structure $\overline{\varphi}$. Moreover, $\overline{\nabla}_{e_i} e_i = 0$, hence the submanifold is minimal.

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