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ON TAME DYNAMICAL SYSTEMS

BY

E. GLASNER (Tel Aviv)

Abstract. A dynamical version of the Bourgain–Fremlin–Talagrand dichotomy shows that the enveloping semigroup of a dynamical system is either very large and contains a topological copy of $\beta\mathbb{N}$, or it is a "tame" topological space whose topology is determined by the convergence of sequences. In the latter case we say that the dynamical system is tame. We show that (i) a metric distal minimal system is tame iff it is equicontinuous, (ii) for an abelian acting group a tame metric minimal system is PI (hence a weakly mixing minimal system is never tame), and (iii) a tame minimal cascade has zero topological entropy. We also show that for minimal distal-but-not-equicontinuous systems the canonical map from the enveloping operator semigroup onto the Ellis semigroup is never an isomorphism. This answers a long standing open question. We give a complete characterization of minimal systems whose enveloping semigroup is metrizable. In particular it follows that for an abelian acting group such a system is equicontinuous.

Introduction. The *enveloping* (or *Ellis*) semigroup of a dynamical system was introduced by R. Ellis in [12]. It proved to be an indispensable tool in the abstract theory of topological dynamical systems (see e.g. Ellis [13]). However explicit computations of enveloping semigroups are quite rare. Some examples are to be found in Namioka [31] (1984), Milnes [29] (1986) and [30] (1989), Glasner [16] (1976) and [20] (1993), Berg, Gove & Hadad [4] (1998), Budak, Işik, Milnes & Pym [9] (2001), and Glasner & Megrelishvili [23] (2004). Rarely is the enveloping semigroup metrizable (a notable exception is the case of weakly almost periodic metric systems; see Downarowicz [10] (1998) and Glasner [22] (2003), Theorem 1.48).

In an interesting paper [28], A. Köhler pointed out the relevance of a theorem of Bourgain, Fremlin & Talagrand [7] to the study of enveloping semigroups. She calls a dynamical system, (X,ϕ), where X is a compact Hausdorff space and $\phi : X \to X$ a continuous map, *regular* if for every function $f \in C(X)$ the sequence $\{f \circ \phi^n : n \in \mathbb{N}\}$ does not contain an ℓ^1 subsequence (the sequence $\{f_n\}_{n=1}^{\infty}$ is an ℓ^1 *sequence* if there are strictly

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positive constants a and b such that

$$a\sum_{k=1}^{n}|c_k| \le \Big\|\sum_{k=1}^{n} c_k f_k\Big\| \le b\sum_{k=1}^{n}|c_k|$$

for all $n \in \mathbb{N}$ and $c_1,\ldots,c_n \in \mathbb{C}$). Since the word "regular" is already overused in topological dynamics I will call such systems *tame*. It turns out that for a metric system (X,ϕ) this is the same as the condition that $E(X,\phi)$, the enveloping semigroup of (X,ϕ), be a Rosenthal compact (see [23]).

In the above mentioned paper Köhler also considers another useful notion, that of the *enveloping operator semigroup*. For a Banach space K and a bounded linear operator $T: K \to K$ this is defined as

$$\mathcal{E}(T) = \mathrm{cls}_{w^*}\{T^n : n \in \mathbb{N}\}.$$

Köhler shows that when (X,ϕ) is a dynamical system, $K = C(X)$, and $T: C(X)^* \to C(X)^*$ is the operator induced by ϕ on the dual space $C(X)^*$, then there is always a surjective homomorphism of dynamical systems

$$\Phi : \mathcal{E}(T) \to E(X,\phi).$$

If we view $M(X)$, the compact space of probability measures on X equipped with the weak* topology, as a subset of $C(X)^*$ with $\mathrm{span}(M(X)) = C(X)^*$, we see that this map Φ is nothing else than the restriction of an element of $\mathcal{E}(T)$ to the subspace of Dirac measures $\{\delta_x : x \in X\}$. Theorem 5.3 of [28] says that for a tame metric dynamical system (X,ϕ), the map Φ is an isomorphism of the enveloping operator semigroup onto the Ellis semigroup. (We will re-prove this theorem in Section 1 as Theorem 1.5.) In this paper I will call a dynamical system (X,ϕ) for which Φ is an isomorphism an *injective system*.

In [28] there are several other cases where systems are shown to be injective and the author raises the question whether this is always the case. As she points out, this question was posed earlier by J. S. Pym (see [32]). In [25], S. Immervoll gives an example of a dynamical system which is not injective. His example is of the form (X,H) where $X = [0,1]$ is the unit interval and H is an uncountable semigroup of continuous maps from X to itself. This leaves the question open for $\mathbb{Z}$- (or $\mathbb{N}$-) systems, for group actions and for minimal systems.

In the present work the setup is that of a compact (mostly metrizable) Γ dynamical system (X,Γ) where Γ is an arbitrary topological group. In [23] we have shown that metrizable weakly almost periodic (WAP) systems and more generally metrizable hereditarily almost equicontinuous (HAE) systems are tame. However, most of the results presented here are concerned with the case where (X,Γ) is a *minimal* dynamical system. In the first section it is shown that a tame dynamical system is injective. This, in conjunction with a theorem of Ellis and an old work of mine on affine dynamical systems ([19]), is used to deduce that a metric distal minimal system is injective iff it is equicontinuous. It therefore follows that every metric minimal distal-but-not-equicontinuous system serves as a counterexample to the question of Pym and Köhler. It is also shown that a tame minimal cascade (X, T) has zero topological entropy.

In the second section I show that for abelian Γ a metric minimal tame system is PI (proximal-isometric), hence in particular a minimal weakly mixing Γ-system is never tame. In the third section I consider the case when (X, Γ) is minimal and $E = E(X, \Gamma)$ is metrizable. Under these assumptions it is shown that there is a unique minimal ideal I in E, that the group K of automorphisms of the system (I, Γ) is compact, and that the quotient dynamical system $(I/K, \Gamma)$ is proximal. If we also assume that Γ is abelian then (X, Γ) is equicontinuous. In the last section I consider the question how big $E(X, \Gamma)$ can be in X^X.

The reader is referred to the sources [13], [16], [34], [8], [2], [35] and [21], on the abstract theory of topological dynamics and the structure theory of minimal dynamical systems including the notion of PI systems.

The questions treated in this paper arose during the work on another one, [23], written jointly with Michael Megrelishvili. I owe him much for fruitful discussions on these subjects. I am also indebted to Benjy Weiss for helpful conversations; in particular the content of Section 4 was the subject of a conversation over lunch several years ago.

1. Tame systems are injective. Recall that a topological space K is called a *Rosenthal compact* [24] if it is homeomorphic to a pointwise compact subset of the space $B_1(X)$ of functions of the first Baire class on a Polish space X. All metric compact spaces are Rosenthal. An example of a separable non-metrizable Rosenthal compact is the *Helly compact* of all (not only strictly) increasing self-maps of $[0, 1]$ in the pointwise topology. Another is the "two arrows" space of Aleksandrov and Urysohn (see Engelking [15]).

A topological space K is a *Fréchet space* if for every $A \subset K$ and every $x \in \overline{A}$ there exists a sequence $x_n \in A$ with $\lim_{n\to\infty} x_n = x$ (see [15]). A topological space K is *angelic* if every relatively countably compact subset $A \subset K$ has the properties: (i) A is relatively compact and (ii) for every $x \in \overline{A}$ there exists a sequence $x_n \in A$ with $\lim_{n\to\infty} x_n = x$. Thus a compact space is angelic iff it is Fréchet. Clearly, $\beta\mathbb{N}$, the Stone–Čech compactification of the natural numbers, cannot be embedded into a Fréchet space.

The following theorem is due to Bourgain, Fremlin and Talagrand [7, Theorem 3F], generalizing a result of Rosenthal. The second assertion (BFT dichotomy) is presented as in the book of Todorčević [33] (see Proposition 1 of Section 13).

1.1. THEOREM.

1. *Every Rosenthal compact space K is angelic.*
2. (BFT dichotomy) *Let X be a Polish space and let $\{f_n\}_{n=1}^\infty \subset C(X)$ be a sequence of real-valued functions which is* pointwise bounded (*i.e. for each $x \in X$ the sequence $\{f_n(x)\}_{n=1}^\infty$ is bounded in $\mathbb{R}$*). *Let K be the pointwise closure of $\{f_n\}_{n=1}^\infty$ in $\mathbb{R}^X$. Then either $K \subset B_1(X)$ (i.e. K is Rosenthal compact) or K contains a homeomorphic copy of $\beta\mathbb{N}$.*

The following dynamical BFT dichotomy is derived in [23].

1.2. THEOREM (A dynamical BFT dichotomy). *Let (X, Γ) be a metric dynamical system and let $E = E(X, \Gamma)$ be its enveloping semigroup. We have the following alternative. Either*

1. *E is a separable Rosenthal compact*, *hence* $\operatorname{card} E \le 2^{\aleph_0}$; *or*
2. *the compact space E contains a homeomorphic copy of $\beta\mathbb{N}$*, *hence* $\operatorname{card} E = 2^{2^{\aleph_0}}$.

1.3. DEFINITION. We will say that an enveloping semigroup $E(X, \Gamma)$ is *tame* if it is separable and Fréchet. A dynamical system (X, Γ) is *tame* when $E(X, \Gamma)$ is tame.

In these terms Theorem 1.2 can be rephrased as saying that *a metric dynamical system (X, Γ) is either tame or $E(X, \Gamma)$ contains a topological copy of $\beta\mathbb{N}$*. When (X, Γ) is a metrizable system the group Γ is embedded in the Polish group $\mathrm{Homeo}(X)$ of homeomorphisms of X equipped with the topology of uniform convergence. From this fact it is easy to deduce that the enveloping semigroup $E(X, \Gamma)$ is separable. If moreover (X, Γ) is tame then $E = E(X, \Gamma)$ is Fréchet and every element $p \in E$ is the limit of a sequence of elements of Γ, $p = \lim_{n\to\infty} \gamma_n$.

Examples of tame dynamical systems include metric minimal equicontinuous systems, almost periodic (WAP) systems (E. Akin, J. Auslander and K. Berg [1]), and hereditarily nonsensitive (HNS) systems (Glasner and Megrelishvili [23]).

The cardinality distinction between the two cases entails the first part of the following proposition.

1.4. PROPOSITION.

1. *For metric dynamical systems tameness is preserved by taking*
   (a) *subsystems*,
   (b) *countable self-products*, *and*
   (c) *factors.*
2. *Every metric dynamical system (X, Γ) admits a unique maximal tame factor.*

Proof. As pointed out, the first statement follows from cardinality arguments (note that $E(X,\Gamma) = E(X^\kappa,\Gamma)$ for any cardinal number κ). To prove the second, use Zorn's lemma, the first part of the theorem, and the fact that a chain of factors of a metric system is necessarily countable to find a maximal tame factor. Then use the first part again to deduce that such a maximal factor is unique. ∎

As was mentioned in the Introduction the following theorem is due to Köhler; our proof, though, is different (see also [22, Lemma 1.49]).

1.5. Theorem. *Let (X,Γ) be a metric tame dynamical system. Let $M(X)$ denote the compact convex set of probability measures on X (with the weak* topology). Then each element $p \in E(X,\Gamma)$ defines an element $p_* \in E(M(X),\Gamma)$ and the map $p \mapsto p_*$ is both a dynamical system and a semigroup isomorphism of $E(X,\Gamma)$ onto $E(M(X),\Gamma)$.*

Proof. Since $E(X,\Gamma)$ is Fréchet we have for every $p \in E$ a sequence γ_i of elements of Γ converging to p. Now for every $f \in C(X)$ and every probability measure $\nu \in M(X)$ we get, by the Riesz representation theorem and Lebesgue's dominated convergence theorem,

$$\gamma_i\nu(f) = \nu(f \circ \gamma_i) \to \nu(f \circ p) =: p_*\nu(f).$$

Since the Baire class 1 function $f \circ p$ is well defined and does not depend upon the choice of the convergent sequence $\gamma_i \to p$, this defines the map $p \mapsto p_*$ uniquely. It is easy to see that this map is an isomorphism of dynamical systems, whence a semigroup isomorphism. Finally, as Γ is dense in both enveloping semigroups, it follows that this isomorphism is onto. ∎

1.6. Definition. We will say that the dynamical system (X,Γ) is *injective* if the natural map $E(M(X),\Gamma) \to E(X,\Gamma)$ is an isomorphism.

In these terms the previous theorem can be restated as follows. *A tame dynamical system is injective.* Our next theorem, which relies on [17], answers a question of J. S. Pym and A. Köhler (see also S. Immervoll [25]).

1.7. Theorem. *A minimal distal metric dynamical system is injective iff it is equicontinuous.*

Proof. It is well known that when (X,Γ) is equicontinuous, $E = E(X,\Gamma)$ is a compact topological group and in that case it is easy to see that (X,Γ) is injective. By a theorem of Ellis (see e.g. [13]), a system (X,Γ) is distal iff $E(X,\Gamma)$ is a group. Thus, if (X,Γ) is distal, metric and injective then $E(X,\Gamma) = E(M(X),\Gamma)$ is a group and it follows that the dynamical system $(M(X),\Gamma)$ is also distal. By Theorem 1.1 of [17], the system (X,Γ) is equicontinuous. ∎

1.8. Corollary. *A minimal distal metric system is tame iff it is equicontinuous.*

Proof. A metric minimal equicontinuous system is isomorphic to its own enveloping semigroup. For the other direction observe that if (X,Γ) is tame then by Theorem 1.5 it is injective, and hence, by Theorem 1.7, it is equicontinuous. ∎

By way of illustration consider, given an irrational number $\alpha \in \mathbb{R}$, the minimal distal dynamical $\mathbb{Z}$-system on the two-torus $(\mathbb{T}^2, T)$ given by

$$T(x,y) = (x+\alpha, y+x) \ (\mathrm{mod}\, 1).$$

Since this system is not equicontinuous Theorem 1.7 and Corollary 1.8 show that it is neither tame nor injective.

The fact that tame systems are injective also yields the result that metric tame minimal systems have zero topological entropy. For this we need the following (simplified version of a) theorem of Blanchard, Glasner, Kolyada and Maass [5, Theorem 2.3]. Recall that a pair of points $\{x,y\} \subseteq X$ is said to be a *Li–Yorke pair* if simultaneously

$$\limsup_{n\to\infty} d(T^n x, T^n y) = \delta > 0 \quad \text{and} \quad \liminf_{n\to\infty} d(T^n x, T^n y) = 0.$$

In particular a Li–Yorke pair is proximal. A set $S \subseteq X$ is called *scrambled* if any pair of distinct points $\{x,y\} \subseteq S$ is a Li–Yorke pair. A dynamical system (X,T) is called *chaotic in the sense of Li and Yorke* if X contains an uncountable scrambled set.

1.9. Theorem. *Let (X,T) be a topological dynamical system such that $h_{\mathrm{top}}(X,T) > 0$. Let μ be a T-ergodic probability measure with* $\mathrm{supp}(\mu) = X$ *and $h_\mu(X,T) > 0$. Then there exists a topologically transitive subsystem $(W, T\times T)$ with $W \subseteq X\times X$ such that for every open $U \subseteq X$ there exists a Cantor scrambled set $K \subseteq U$ with $K\times K \setminus \Delta_X \subseteq W_{\mathrm{tr}}$, where W_{tr} is the set of transitive points in W. Thus a dynamical system with positive topological entropy is chaotic in the sense of Li and Yorke.*

We note that the set W in Theorem 1.9 has the following special form. There exists a measure-theoretical weakly mixing factor map $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, T)$ with a corresponding measure disintegration

$$\mu = \int_Y \mu_y \, d\nu(y)$$

having the property that μ_y is nonatomic for ν-a.e. y. The subsystem W is then given as $W = \mathrm{supp}(\lambda)$, where

$$\lambda = \mu \underset{\nu}{\times} \mu = \int_Y \mu_y \times \mu_y \, d\nu(y).$$

Consequently, if $X_0 \subset X$ is any μ-measurable set with $\mu(X_0) = 1$ then with no loss of generality we can assume that for ν-almost every y the measure μ_y satisfies the condition $\mu_y(X_0) = 1$. It then follows that the Cantor set in Theorem 1.9 can be chosen to be a subset of X_0.

1.10. THEOREM. *A minimal metric tame $\mathbb{Z}$ dynamical system (X, T) has zero topological entropy.*

Proof. By the variational principle it suffices to show that $h_\mu(T) = 0$ for every T-invariant probability measure μ on X. Let μ be such a measure. By Theorem 1.5, (X, T) is injective and therefore $v_*(\mu) = \mu$ for any minimal idempotent $v \in E = E(X, T)$. Since $v_*\mu(f) = \mu(f \circ v)$ for every $f \in C(X)$ it follows that $\mu(vX) = 1$. (Note that vX is an analytic set hence universally measurable.) Now if $h_\mu(T) > 0$ then by Theorem 1.9 with $X_0 = vX$, there is a Cantor set $K \subset X_0$ such that for any distinct points x, x' in K the pair (x, x') is proximal. However, since pairs $(x, x') \in vX \times vX$ with $x \neq x'$ are *almost periodic* (i.e. have minimal orbit closure in $X \times X$) they are never proximal and we conclude that $h_\mu(T) = 0$. ∎

1.11. REMARK. In the proof of Theorem 1.10, with slight modifications, one can use instead of the results in [5] a theorem of Blanchard, Host and Ruette [6] on the abundance of asymptotic pairs in a system (X, T) with positive topological entropy.

2. Minimal tame systems are PI. As we have seen, when (X, Γ) is a metrizable tame system the enveloping semigroup $E(X, \Gamma)$ is a separable Fréchet space. Therefore each element $p \in E$ is a limit of a sequence of elements of Γ, $p = \lim_{n\to\infty} \gamma_n$. It follows that the subset $C(p)$ of continuity points of each $p \in E$ is a dense G_δ subset of X. More generally, if $Y \subset X$ is any closed subset then the set $C_Y(p)$ of continuity points of the map $p\restriction Y : Y \to X$ is a dense G_δ subset of Y. For an idempotent $v = v^2 \in E$ we write C_v for $C_{\overline{vX}}(v)$.

2.1. LEMMA. *Let (X, Γ) be a metrizable tame dynamical system, and $E = E(X, \Gamma)$ its enveloping semigroup.*

1. *For every $p \in E$ the set $C(p) \subset X$ is a dense G_δ subset of X.*
2. *For every minimal idempotent $v \in E$, we have $C_v \subset vX$.*
3. *When Γ is commutative we have $C(v) \subset vX$.*

Proof. 1. See the remark above.

2. Given $x \in C_v$ choose a sequence $x_n \in vX$ with $\lim_{n\to\infty} x_n = x$. We then have $vx = \lim_{n\to\infty} vx_n = \lim_{n\to\infty} x_n = x$, hence $C_v \subset vX$.

3. When Γ is commutative we have $\gamma p = p\gamma$ for every $\gamma \in \Gamma$ and $p \in E$. In particular the subset vX is Γ-invariant, hence dense in X. Thus $\overline{vX} = X$, hence $C(v) = C_v \subset vX$ by part 2. ∎

We next proceed to the main theorem of this section.

2.2. Definition. Let (X, Γ) be a dynamical system. We say that a closed Γ-invariant set $W \subset X \times X$ is an *M-set* if it satisfies the conditions:

1. The system (W, Γ) is topologically transitive.
2. The almost periodic points are dense in W.

A theorem of Bronshteĭn asserts that a metric system (X, Γ) is PI iff every M-set in $X \times X$ is minimal ([8], see also [21]).

2.3. Theorem. *Let Γ be a commutative group. Then any metric tame minimal system (X, Γ) is PI.*

Proof. We will prove that the Bronshteĭn condition holds, i.e. that every M-set in $X \times X$ is minimal. So let $W \subset X \times X$ be an M-set. Let $v = v^2$ be some minimal idempotent in $E(X, \Gamma)$. By Lemma 2.1.3 the set $C(v)$ of continuity points of the map $v : X \to X$ is a dense G_δ subset of X and moreover $C(v) \subset vX$. Let U be a relatively open subset of W; then there exists a minimal subset $M \subset W$ with $M \cap U \neq \emptyset$. Let $\pi_i : M \to X$, $i = 1, 2$, denote the projection maps. Because M is minimal we have $\pi_i(M) = X$ and the map π_i is semi-open, i.e. $\mathrm{int}(\pi_i(\overline{V})) \neq \emptyset$ for every nonempty open subset V of M (see e.g. [18, Lemma 1.5]; these observations are due to Auslander and Markley). It follows that the sets $\pi_i^{-1}(C(v))$, $i = 1, 2$, are dense G_δ subsets of M and therefore so is the set

$$(C(v) \times C(v)) \cap M = \pi_1^{-1}(C(v)) \cap \pi_2^{-1}(C(v)).$$

In particular $U \cap (C(v) \times C(v)) \neq \emptyset$ and we conclude that $W_0 = (C(v) \times C(v)) \cap W$ is a dense G_δ subset of W.

Let W_{tr} be the dense G_δ subset of transitive points in W and observe that $W_0 \cap W_{\mathrm{tr}} \neq \emptyset$. If (x, x') is a point in $W_0 \cap W_{\mathrm{tr}}$, then $\overline{\Gamma(x, x')} = W$, and since $(x, x') \in vX \times vX$ it follows that W is minimal. ∎

2.4. Corollary. *Let Γ be a commutative group and (X, Γ) a minimal weakly mixing metric tame dynamical system. Then (X, Γ) is trivial.*

Proof. A minimal system which is weakly mixing and PI is necessarily trivial.

A direct proof of Corollary 2.4 that does not require the PI theory is as follows. Fix a minimal idempotent $u \in E$ and let $C(u) \subset X$ be the dense G_δ subset of continuity points of u. Fix some $x \in X$; then, by a theorem of Weiss, $P[x]$, the proximal cell of x, is also a dense G_δ subset of X (see [22, Theorem 1.13]). Set $A = C(u) \cap P[x]$; then for $y \in A$ there is a sequence $\gamma_j \in \Gamma$ such that $\lim_{j\to\infty} \gamma_j x = \lim_{j\to\infty} \gamma_j y = x$. By the continuity of u at x we have

$$ux = u \lim_{j\to\infty} \gamma_j x = \lim_{j\to\infty} \gamma_j ux = u \lim_{j\to\infty} \gamma_j y = \lim_{j\to\infty} \gamma_j uy,$$

so that $(ux, uy) \in P$. This implies $ux = uy$ and we conclude that $ux = uy$ for every $y \in A$. For an arbitrary element $\gamma \in \Gamma$, the set $\gamma^{-1}A \cap A$ is a residual subset of X and for each y in this set we get $ux = u\gamma y = \gamma uy = uy$. Since Γ is commutative and (Γ, X) is minimal we conclude that $\gamma z = z$ for every $z \in X$. Thus Γ acts trivially on X and the minimality of (Γ, X) implies that X is a one-point space. ∎

In [23] there is an example of a minimal tame dynamical cascade (i.e. a $\mathbb{Z}$-system) on the Cantor set with an enveloping semigroup which is not metrizable (in fact E in this example is homeomorphic to the "two arrows" space). This system has the structure of an almost 1-1 (hence proximal) extension of an irrational rotation on the circle $\mathbb{T}$. Another such example is in R. Ellis [14] where the enveloping semigroup of the $\mathrm{SL}(2,\mathbb{R})$ action on the projective line $\mathbb{P}$ is shown to be tame but not metrizable. Here the system $(\mathbb{P}, \mathrm{SL}(2,\mathbb{R}))$ is proximal. In view of these examples, Corollary 1.8, Theorem 1.10, Corollary 2.4, and Theorem 3.1 below, it is reasonable to raise the following question.

2.5. Problem. Is it true that every minimal metrizable tame system (X, Γ) with an abelian acting group is a proximal extension of an equicontinuous system? (Or, for the general acting group, is X proximally equivalent to a factor of an isometric extension of a proximal system?)

3. Metrizable enveloping semigroups. In this section we consider the case of a minimal dynamical system for which $E = E(X, \Gamma)$ is metrizable. Of course then E is tame and if $I \subset E$ is a minimal (left) ideal in E then the dynamical system (I, Γ) is metric with $E(I, \Gamma) \cong E(X, \Gamma)$ so that it is also tame.

3.1. Theorem. *Let (X, Γ) be a minimal dynamical system such that $E = E(X, \Gamma)$ is metrizable. Then*:

1. *There is a unique minimal ideal $I \subset E = E(X, \Gamma) \cong E(I, \Gamma)$.*
2. *The Polish group $G_U = \mathrm{Aut}(I, \Gamma)$, of automorphisms of the system (I, Γ) equipped with the topology of uniform convergence, is compact.*
3. *The quotient dynamical system $(I/K, \Gamma)$ is proximal.*
4. *The quotient map $\pi : I \to I/K$ is a K-extension.*
5. *If in addition (X, Γ) is incontractible then $I = K$ and Γ acts on K by translations via a continuous homomorphism $J : \Gamma \to K$ with $J(\Gamma)$ dense in K. In particular (I, Γ), and hence also (X, Γ), is equicontinuous.*
6. *If Γ is commutative then $X = I = K$ and $K = \mathrm{cls}\, J(\Gamma)$ is also commutative.*

Proof. We split the proof into several steps.

I. If $I \subset E$ is a minimal left ideal then (X,Γ) is a factor of the dynamical system (I,Γ) and the enveloping semigroup $E(I,\Gamma)$ is isomorphic to $E(X,\Gamma)$, where each $p \in E$ is identified with the map $L_p : E \to E$, $q \mapsto pq$.

II. Let $u = u^2$ be a fixed idempotent in I, and as usual define $G = uI \subset I$. Then to each $\alpha \in G$ corresponds an automorphism $\widehat{\alpha} : I \to I$ defined by $\widehat{\alpha}(p) = p\alpha$, $\forall p \in I$. The map $G \to G_U$, $\alpha \mapsto \widehat{\alpha}$, is a surjective algebraic isomorphism. The inverse map $G_U \to G \subset I$ is given by $\widehat{\alpha} \mapsto \widehat{\alpha}(u) = u\alpha = \alpha$. Thus G_U acts on I by right multiplication. In what follows we will identify $\widehat{\alpha}$ with α.

III. By Lemma 2.1, for each idempotent $v = v^2 \in I$, the restricted map $v : \overline{vI} \to vI$, $q \mapsto vq$, has a dense G_δ subset $C_v \subset \overline{vI}$ of continuity points. Again by Lemma 2.1, $C_v \subset vI$. Since clearly $C_vG \subset C_v$, we get $C_v = vG = vI$.

IV. If $p \in vI$ then also $p : \overline{vI} \to vI$ and thus its set of continuity points C_p is also a dense G_δ subset of $\overline{vI}$. Therefore $C_p \cap C_v = C_p \cap vI \neq \emptyset$, and since $C_pG \subset C_p$ we conclude that $C_p \supset vI$.

V. The G-dynamical system $(\overline{I,G})$ admits a minimal subset M, and it is clearly of the form $M = \overline{vI} = \overline{vG}$ for some $v = v^2 \in I$. By minimality we have $M = \overline{wI}$ for any other idempotent $w = w^2 \in \overline{vI}$. Since, by step III, $C_v = vI$ and $C_w = wI$ are residual subsets of M, their intersection is nonempty and the structure of I as a disjoint union of groups implies that $v = w$, hence $wI = vI = M$. Thus $v : I \to I$, $p \mapsto vp$, has a closed range vI and the right action of G on $M = vI = vG$ is algebraically transitive. (The right action of G on I, hence also on vI, is free.) Moreover, from step IV we see that every $\alpha \in G$ acts continuously on vI on the *left*; that is $p_n \to p$, for $p_n, p \in vI$, implies $\alpha p_n \to \alpha p$. Thus in the compact group vG, with the topology inherited from I, both left and right multiplications are continuous. By a theorem of Ellis ([11]) it follows that vG is a compact topological group. Being a closed subset of I it is also Polish.

VI. Now the map $v : G_U \to vG$, $\alpha \mapsto v\alpha$, is clearly a continuous surjective 1-1 homomorphism of Polish topological groups and a theorem of Banach ([3]) implies that it is a topological isomorphism (see also [19, Lemma 3]). We therefore conclude that G_U is a compact subgroup of $\mathrm{Aut}(I,\Gamma)$.

Now, letting $K = G_U$ makes all the assertions of the theorem follow readily. ∎

3.2. Remark. Let Γ be a topological group and $J : \Gamma \to K$ a continuous homomorphism, where K is a compact metrizable topological group and $J(\Gamma)$ is dense in K. In addition let H be a closed subgroup of K for which $\bigcap_{k \in K} kHk^{-1} = \{e\}$. Then the dynamical system $(X,\Gamma) = (K/H,\Gamma)$, where $\gamma(kH) = J(\gamma)kH$ ($\gamma \in \Gamma$, $k \in K$), is a minimal dynamical system with

$E(X,\Gamma) = K$. In fact, these are the only examples I know of minimal systems with metrizable enveloping semigroup.

3.3. Problem. Is there a nontrivial minimal proximal system with a metrizable enveloping semigroup?

4. When is $E(X,\Gamma)$ all of X^X? We say that the system (X,Γ) is n-*complete* if for every point $(x_1,\dots,x_n) \in X^n$ with distinct components the orbit $\Gamma(x_1,\dots,x_n)$ is dense in X^n. It is called *complete* when it is n-complete for every $n \in \mathbb{N}$.

4.1. Theorem. *Let (X,Γ) be a dynamical system. Then $E(X,\Gamma) = X^X$ if and only if (X,Γ) is complete.*

Proof. Suppose $E(X,\Gamma) = X^X$ and let $(x_1,\dots,x_n),(x'_1,\dots,x'_n) \in X^n$. Then there exists an element $p \in E$ with $px_i = x'_i$, $i = 1,\dots,n$, hence $\overline{\Gamma(x_1,\dots,x_n)} = X^n$ and (X,Γ) is complete.

Conversely, if (X,Γ) is complete then clearly every element of X^X can be approximated by an element from Γ. As E is closed this concludes the proof. ∎

4.2. Corollary. *Let X be a topological space which is n-homogeneous for every $n \in \mathbb{N}$ (i.e. the group* $\mathrm{Homeo}(X)$ *acts n-transitively on X for every n). Then for any dense subgroup $\Gamma \subset$* $\mathrm{Homeo}(X)$ *the dynamical system (X,Γ) is complete*, *hence $E(X,\Gamma) = X^X$. For example this is the case for the Cantor set C, for any sphere S^n, $n \geq 2$, and for the Hilbert cube Q.*

If ϕ is a nontrivial continuous automorphism of a system (X,Γ) then $\phi p = p\phi$ for every $p \in E = E(X,\Gamma)$. Thus when the group $\mathrm{Aut}(X,\Gamma)$ is nontrivial then $E \subset \{p \in X^X : \phi p = p\phi, \forall \phi \in \mathrm{Aut}(X,\Gamma)\}$. In particular, when Γ is commutative,

$$E \subset \{p \in X^X : p\gamma = \gamma p, \forall \gamma \in \Gamma\}.$$

Are there dynamical systems (X,Γ) for which this inclusion is an equality? A $\mathbb{Z}$-dynamical system (X,T) is said to have 2*-fold topological minimal self-joinings* ([26], [27]) if it satisfies the following condition: For every pair $(x,x') \in X \times X$ with $x' \notin \{T^n x : n \in \mathbb{Z}\}$, the orbit $\{T^n(x,x') : n \in \mathbb{Z}\}$ is dense in $X \times X$. If it satisfies the analogous condition for every point $(x_1,\dots,x_n) \in X^n$ whose coordinates belong to n distinct orbits, then (X,Γ) has *n-fold topological minimal self-joinings.* As in the proof of Theorem 4.1 it is easy to see that

$$E(X,T) = \{p \in X^X : pT = Tp\}$$

iff (X,Γ) has n-fold topological minimal self-joinings for all $n \geq 1$. Now, in [27], J. King shows that no nontrivial map has 4-fold topological minimal self-joinings. We thus get the following.

4.3. THEOREM. *There exists no infinite minimal cascade* (X, T) *with*

$$E(X, T) = \{p \in X^X : pT = Tp\}.$$

4.4. REMARK. In [36], B. Weiss shows that every aperiodic ergodic zero entropy measure preserving system has a topological model which has two-fold topological minimal self-joinings (*doubly minimal* in his terminology).

REFERENCES

[1] E. Akin, J. Auslander and K. Berg, *Almost equicontinuity and the enveloping semigroup*, in: Topological Dynamics and Applications. A Volume in Honor of R. Ellis, Contemp. Math. 215, Amer. Math. Soc., Providence, RI, 1998, 75–81.

[2] J. Auslander, *Minimal Flows and their Extensions*, North-Holland Math. Stud. 153 (Notas Mat. 122), North-Holland, Amsterdam, 1988.

[3] S. Banach, *Théorie des opérations linéaires*, Chelsea, New York, 1963.

[4] K. Berg, D. Gove and K. Haddad, *Enveloping semigroups and mappings onto the two-shift*, Proc. Amer. Math. Soc. 126 (1998), 899–905.

[5] F. Blanchard, E. Glasner, S. Kolyada and A. Maass, *On Li–Yorke pairs*, J. Reine Angew. Math. 547 (2002), 51–68.

[6] F. Blanchard, B. Host and S. Ruette, *Asymptotic pairs in positive-entropy systems*, Ergodic Theory Dynam. Systems 22 (2002), 671–686.

[7] J. Bourgain, D. H. Fremlin and M. Talagrand, *Pointwise compact sets of Baire-measurable functions*, Amer. J. Math. 100 (1978), 845–886.

[8] I. U. Bronšteĭn [I. U. Bronshteĭn], *A characteristic property of PD-extensions*, Bul. Akad. Ştiince RSS Moldoven. 1977, no. 3, 11–15 (in Russian).

[9] T. Budak, N. Işik, P. Milnes and J. Pym, *The action of a semisimple Lie group on its maximal compact subgroup*, Proc. Amer. Math. Soc. 129 (2001), 1525–1534.

[10] T. Downarowicz, *Weakly almost periodic flows and hidden eigenvalues*, in: Topological Dynamics and Applications. A Volume in Honor of R. Ellis, Contemp. Math. 215, Amer. Math. Soc., Providence, RI, 1998, 101–120.

[11] R. Ellis, *Locally compact transformation groups*, Duke Math. J. 24 (1957), 119–125.

[12] —, *A semigroup associated with a transformation group*, Trans. Amer. Math. Soc. 94 (1960), 272–281.

[13] —, *Lectures on Topological Dynamics*, W. A. Benjamin, New York, 1969.

[14] —, *The enveloping semigroup of projective flows*, Ergodic Theory Dynam. Systems 13 (1993), 635–660.

[15] R. Engelking, *General Topology*, Sigma Ser. in Pure Math. 6, Heldermann, Berlin, 1989.

[16] E. Glasner, *Proximal Flows*, Lecture Notes in Math. 517, Springer, Berlin, 1976.

[17] —, *Distal and semisimple affine flows*, Amer. J. Math. 109 (1987), 115–131.

[18] —, *A topological version of a theorem of Veech and almost simple flows*, Ergodic Theory Dynam. Systems 10 (1990), 463–482.

[19] —, *Regular PI metric flows are equicontinuous*, Proc. Amer. Math. Soc. 114 (1992), 269–277.

[20] —, *Minimal nil-transformations of class two*, Israel J. Math. 81 (1993), 31–51.

[21] —, *Structure theory as a tool in topological dynamics*, in: Descriptive Set Theory and Dynamical Systems, London Math. Soc. Lecture Note Ser. 277, Cambridge Univ. Press, Cambridge, 2000, 173–209.

[22] E. Glasner, *Ergodic Theory via Joinings*, Math. Surveys Monogr. 101, Amer. Math. Soc., Providence, RI, 2003.
[23] E. Glasner and M. Megrelishvili, *Hereditarily non-sensitive dynamical systems and linear representations*, Colloq. Math. 104 (2006), 223–283.
[24] G. Godefroy, *Compacts de Rosenthal*, Pacific J. Math. 91 (1980), 293–306.
[25] S. Immervoll, *A note on the enveloping semigroup of a flow*, Math. Proc. Roy. Irish Acad. Sect. A 99 (1999), 185–188.
[26] A. del Junco, *On minimal self-joinings in topological dynamics*, Ergodic Theory Dynam. Systems 7 (1987), 211–227.
[27] J. L. King, *A map with topological minimal self-joinings in the sense of del Junco*, ibid. 10 (1990), 745–761.
[28] A. Köhler, *Enveloping semigroups for flows*, Proc. Roy. Irish Acad. Sect. A 95 (1995), 179–191.
[29] P. Milnes, *Ellis groups and group extensions*, Houston J. Math. 12 (1986), 87–108.
[30] —, *Enveloping semigroups, distal flows and groups of Heisenberg type*, ibid. 15 (1989), 563–572.
[31] I. Namioka, *Ellis groups and compact right topological groups*, in: Conference in Modern Analysis and Probability (New Haven, CN, 1982), Contemp. Math. 26, Amer. Math. Soc., Providence, RI, 1984, 295–300.
[32] J. S. Pym, *Compact semigroups with one-sided continuity*, in: The Analytical and Topological Theory of Semigroups, de Gruyter Exp. Math. 1, de Gruyter, Berlin, 1990, 197–217.
[33] S. Todorčević, *Topics in Topology*, Lecture Notes in Math. 1652, Springer, Berlin, 1997.
[34] W. A. Veech, *Topological dynamics*, Bull. Amer. Math. Soc. 83 (1977), 775–830.
[35] J. de Vries, *Elements of Topological Dynamics*, Math. Appl. 257, Kluwer, Dordrecht, 1993.
[36] B. Weiss, *Multiple recurrence and doubly minimal systems*, in: Topological Dynamics and Applications (Minneapolis, MN, 1995), Contemp. Math. 215, Amer. Math. Soc., Providence, RI, 1998, 189–196.

Department of Mathematics
Tel Aviv University
Tel Aviv, Israel
E-mail: glasner@math.tau.ac.il

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