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DERIVED ENDO-DISCRETE ARTIN ALGEBRAS

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Abstract. Let Λ be an artin algebra. We prove that for each sequence $(h_i)_{i\in\mathbb{Z}}$ of non-negative integers there are only a finite number of isomorphism classes of indecomposables $X \in \mathcal{D}^{\mathrm{b}}(\Lambda)$, the bounded derived category of Λ, with $\operatorname{length}_{E(X)} H^i(X) = h_i$ for all $i \in \mathbb{Z}$ and $E(X)$ the endomorphism ring of X in $\mathcal{D}^{\mathrm{b}}(\Lambda)$ if and only if $\mathcal{D}^{\mathrm{b}}(\operatorname{Mod}\Lambda)$, the bounded derived category of the category $\operatorname{Mod}\Lambda$ of all left Λ-modules, has no generic objects in the sense of [4].

1. Introduction. Let Λ be an artin algebra over a commutative artinian ring k and $\mathcal{D}^{\mathrm{b}}(\Lambda)$ be its bounded derived category. We consider the category $\operatorname{Mod}\Lambda$ of left Λ-modules. We denote by $\operatorname{mod}\Lambda$, $\operatorname{Proj}\Lambda$ and $\operatorname{proj}\Lambda$ the full subcategories of $\operatorname{Mod}\Lambda$ consisting of the finitely generated, projective and finitely generated projective Λ-modules, respectively. By $\mathcal{D}^{\mathrm{b}}(\operatorname{Mod}\Lambda)$ we denote the bounded derived category of $\operatorname{Mod}\Lambda$; we recall that $\mathcal{D}^{\mathrm{b}}(\Lambda)$ is the bounded derived category of $\operatorname{mod}\Lambda$. If $X = (X^i, d_X^i)_{i\in\mathbb{Z}}$ is an object in $\mathcal{D}^{\mathrm{b}}(\Lambda)$, an invariant of it is given by its homology dimension $\mathbf{h}\mathrm{dim} = (h_i)_{i\in\mathbb{Z}}$ with $h_i = \operatorname{length}_k H^i(X)$.

A sequence $\mathbf{h} = (h_i)_{i\in\mathbb{Z}}$ of non-negative integers is called a *homology dimension* if $h_i = 0$ for all but finitely many $i \in \mathbb{Z}$. We recall that according to [5], $\mathcal{D}^{\mathrm{b}}(\Lambda)$ is called *discrete* and Λ *derived discrete* if there are only finitely many isoclasses of indecomposables $X \in \mathcal{D}^{\mathrm{b}}(\Lambda)$ with fixed homology dimension.

We recall that $X \in \mathcal{D}^{\mathrm{b}}(\operatorname{Mod}\Lambda)$ is called *endofinite* if for all $i \in \mathbb{Z}$, $H^i(X)$ has finite length as left $E(X) = \operatorname{End}_{\mathcal{D}^{\mathrm{b}}(\operatorname{Mod}\Lambda)}(X)$-module. In case X is endofinite its *homology endolength* is defined as

$$\mathbf{h}\mathrm{endol}(X) = (\operatorname{length}_{E(X)} H^i(X))_{i\in\mathbb{Z}}.$$

Observe that all objects in $\mathcal{D}^{\mathrm{b}}(\Lambda)$ are endofinite. The category $\mathcal{D}^{\mathrm{b}}(\Lambda)$ is called *endofinite discrete* and Λ *derived endo-discrete* if for each homology dimension $\mathbf{h}$ there are only a finite number of isomorphism classes of indecomposable objects X in $\mathcal{D}^{\mathrm{b}}(\Lambda)$ with $\mathbf{h}\mathrm{endol}(X) = \mathbf{h}$.

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We recall from [4] that $G \in \mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\,\Lambda)$ is called *generic* if G is not in $\mathcal{D}^{\mathrm{b}}(\Lambda)$, G is endofinite and indecomposable. If k is an algebraically closed field, then the generic objects in $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\,\Lambda)$ play an important role in the derived representation type of Λ (see Theorem 3.2 of [3]).

In this paper we prove the following.

THEOREM 1.1. *Let Λ be an artin algebra over k. Then*:

(i) *Λ is not derived endo-discrete if and only if $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\,\Lambda)$ has a generic object.*
(ii) *If k has infinite cardinality, then Λ is not derived discrete if and only if the category $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\,\Lambda)$ has a generic object.*

In [5] it has been proved that if k is an algebraically closed field, then Λ is derived discrete if and only if $\mathcal{D}^{\mathrm{b}}(\Lambda)_{\mathrm{prf}}$, the full subcategory of $\mathcal{D}^{\mathrm{b}}(\Lambda)$ whose objects are the perfect complexes, is discrete. In this paper we prove that the same result holds for artin algebras (see Proposition 2.5(i)).

For the proof of Theorem 1.1, in Section 2 we consider the category $\mathcal{A} = \mathrm{proj}\,\Lambda$ or $\mathcal{A} = \mathrm{Proj}\,\Lambda$ and $\mathbf{C}_m(\mathcal{A})$, which is the category of complexes $X = (X^i, d_X^i)$ over $\mathcal{A}$ with $X^i = 0$ for $i \notin \{1, \ldots, m\} = [1, m]$. We denote by $\mathbf{C}_m^1(\mathcal{A})$ the full subcategory of $\mathbf{C}_m(\mathcal{A})$ whose objects are the complexes $X = (X^i, d_X^i)$ such that $\mathrm{Im}\, d_X^{i-1} \subset \mathrm{rad}\, X^i$ for all $i \in \mathbb{Z}$.

In general if $\mathcal{C}$ is a k-category, a morphism $f : M \to N$ in $\mathcal{C}$ is called *radical* if for any split monomorphism $\sigma : X \to M$ and any split epimorphism $\pi : M \to Y$, $\pi f \sigma : X \to Y$ is not an isomorphism. If P and Q are projective Λ-modules, $f : P \to Q$ is a radical morphism if and only if $\mathrm{Im}\, f \subset \mathrm{rad}\, Q$.

2. Complexes of fixed size. We denote by $\mathbf{C}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$ the category of bounded complexes over $\mathrm{proj}\,\Lambda$. We recall that there is an exact structure on this category given by the exact sequences of complexes $0 \to X \xrightarrow{f} E \xrightarrow{g} Y \to 0$. Note that for all such sequences with Y a complex of projective Λ-modules, the exact sequences $0 \to X^i \xrightarrow{f^i} E^i \xrightarrow{g^i} Y^i \to 0$, $i \in \mathbb{Z}$, split. Here the corresponding projective and injective objects coincide, moreover a morphism of complexes $u : Z \to W$ is homotopic to zero if and only if u factorizes through an injective object. The injective objects are direct sums of complexes of the form $\cdots \to 0 \to P \xrightarrow{\mathrm{id}_P} P \to 0 \to \cdots$. Thus if X and Y are non-injective indecomposable objects in $\mathbf{C}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$, they are homotopic if and only if they are isomorphic as complexes.

Let Y be a complex in $\mathbf{C}_m(\mathrm{Proj}\,\Lambda)$. We denote by $E_C(Y)$ the endomorphism ring of Y in the category of complexes and by $E_K(Y)$ the endomorphism ring in the homotopy category. The ring $E_K(Y)$ is a quotient ring of $E_C(Y)$.

An object $Y \in \mathbf{C}_m(\operatorname{Proj} \Lambda)$ is called *endofinite* if for all $i \in \mathbb{Z}$, Y^i has finite length as $E_C(Y)$-module; in this case we put

$$\operatorname{endol}(Y) = \sum_{i=1}^{m} \operatorname{length}_{E_C(Y)} Y^i.$$

In case $Y \in \mathbf{C}_m(\operatorname{proj} \Lambda)$ we say that Y is *finite*. Now an object $X \in \mathbf{C}_m(\operatorname{Proj} \Lambda)$ is called *generic* if it is endofinite, indecomposable and not finite. For $X \in \mathbf{C}_m(\operatorname{proj} \Lambda)$, we put

$$\operatorname{length}(X) = \sum_{i=1}^{m} \operatorname{length}_k X^i.$$

We need the following two results.

LEMMA 2.1. *Suppose $Y \in \mathbf{C}^1_m(\operatorname{Proj} \Lambda)$ is such that for some $u \in [2, m]$ we have*

$$\operatorname{length}_{E_K(Y)} H^{u-1}(Y) \le c \quad \textit{and} \quad \operatorname{length}_{E_C(Y)} Y^u \le d_u.$$

Then

$$\operatorname{length}_{E_C(Y)} Y^{u-1} \le (d_u + c)L \quad \textit{with} \quad L = \operatorname{length}_k \Lambda.$$

Proof. We have

$$\operatorname{length}_{E_C(Y)} Y^{u-1}/\operatorname{Ker} d_Y^{u-1} = \operatorname{length}_{E_C(Y)} \operatorname{Im} d_Y^{u-1} \le d_u,$$

and moreover $\operatorname{length}_{E_C(Y)} \operatorname{Ker} d_Y^{u-1}/\operatorname{Im} d_Y^{u-2} \le c$. Therefore

$$\operatorname{length}_{E_C(Y)} Y^{u-1}/\operatorname{Im} d_Y^{u-2} \le c + d_u.$$

Here $\operatorname{Im} d_Y^{u-2} \subset \operatorname{rad} Y^{u-1}$, thus

$$\operatorname{length}_{E_C(Y)} Y^{u-1}/\operatorname{rad} Y^{u-1} \le \operatorname{length}_{E_C(Y)} Y^{u-1}/ \operatorname{Im} d_Y^{u-2}.$$

Consequently, $\operatorname{length}_{E_C(Y)} Y^{u-1} \le (c + d_u)L$. ∎

LEMMA 2.2. *Let $Y \in \mathbf{C}^1_m(\operatorname{Proj} \Lambda)$ be such that $\operatorname{Ker} d_Y^1 \subset \operatorname{rad} Y^1$ and for some fixed c and all $j \in [2, m]$, we have $\operatorname{length}_{E_K(Y)} H^j(Y) \le c$. Then Y is an endofinite object and*

$$\operatorname{endol}(Y) < c(mL + (m-1)L^2 + (m-2)L^3 + \cdots + 2L^{m-1} + L^m).$$

Proof. Here $Y^{m+1} = 0$, so by the previous lemma, $\operatorname{length}_{E_C(Y)} Y^m \le cL$. Then again by Lemma 2.1 we have

$$\begin{aligned} \operatorname{length}_{E_C(Y)} Y^{m-1} &\le c(L + L^2),\\ \operatorname{length}_{E_C(Y)} Y^{m-2} &\le c(L + L^2 + L^3), \quad \ldots,\\ \operatorname{length}_{E_C(Y)} Y^{2} &\le c(L + L^2 + \cdots + L^{m-1}). \end{aligned}$$

Thus

$$\operatorname{length}_{E_C(Y)} Y^1/\operatorname{Ker} d_Y^1 = \operatorname{length}_{E_C(Y)} \operatorname{Im} d_Y^1 \le c(L + \cdots + L^{m-1}).$$

By our assumptions, $\operatorname{Ker} d_Y^1 \subset \operatorname{rad} Y^1$, therefore

$$\operatorname{length}_{E_{C(Y)}} Y^1 \leq \operatorname{length}_{E_C}(Y^2)L < c(L + \cdots + L^m).$$

From this we obtain our result. ∎

One can see, using similar arguments, that the statements in Lemmas 2.1 and 2.2 are true for $Y \in \mathbf{C}_m^1(\operatorname{proj} \Lambda)$ if we take $\operatorname{length}_k Y^i$ instead of $\operatorname{length}_{E_{C(Y)}} Y^i$, $\operatorname{length}_k H^i(Y)$ instead of $\operatorname{length}_{E_{K(Y)}} H^i(Y)$ and $\operatorname{length}(Y)$ instead of $\operatorname{endol}(Y)$.

Definition 2.3. The category $\mathbf{C}_m(\operatorname{proj} \Lambda)$ is called *endo-discrete* (respectively *discrete*) if for all natural numbers d there are only a finite number of isomorphism classes of indecomposable objects X with $\operatorname{endol}(X) \leq d$ (respectively $\operatorname{length}(X) \leq d$).

For a projective Λ-module P we consider the objects $J_u(P)$ for $u \in [1, m-1]$, $T(P)$ and $S(P)$ in $\mathbf{C}_m(\operatorname{Proj} \Lambda)$, defined as follows: $J_u(P)^i = 0$ for $i \neq u, u+1$, and $J_u(P)^u = J_u(P)^{u+1} = P$, $d^u_{J_u(P)} = \operatorname{id}_P$; $S(P)^i = 0$ for $i \neq 1$ and $S(P)^1 = P$; $T(P)^i = 0$ for $i \neq m$ and $T(P)^m = P$. If $h : P \to Q$ is a morphism between projective Λ-modules, then $S(h) : S(P) \to S(Q)$ is the morphism with $S(h)^1 = h$.

Consider in $\mathbf{C}_m(\operatorname{Proj} \Lambda)$ the class $\mathcal{E}$ of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that for all $i \in [1, m]$ the sequences $0 \to X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i \to 0$ are exact. Then the pair $(\mathbf{C}_m(\operatorname{Proj} \Lambda), \mathcal{E})$ is an exact category.

The indecomposable $\mathcal{E}$-projectives (respectively $\mathcal{E}$-injectives) are the complexes $T(P)$ and $J_u(P), u \in [1, m-1]$, (respectively $J_u(P), S(P)$) with P an indecomposable projective Λ-module (see Corolary 3.3 of [1]).

Observe that a complex X in $\mathbf{C}_m(\operatorname{Proj} \Lambda)$ is in $\mathbf{C}_m^1(\operatorname{Proj} \Lambda)$ if and only if X has no direct summands of the form $J_u(P)$ for $u \in [1, m-1]$ and P a projective Λ-module.

Throughout this paper we denote by $\mathbf{K}^{\leq m, \mathrm{b}}(\operatorname{Proj} \Lambda)$ the homotopy category of those complexes X over $\operatorname{Proj} \Lambda$ such that $H^i(X) = 0$ for almost all i and $X^j = 0$ for $j > m$.

Let $F : \mathbf{K}^{\leq m, \mathrm{b}}(\operatorname{Proj} \Lambda) \to \mathbf{C}_m(\operatorname{Proj} \Lambda)$ be the functor given on objects by $F(X)^i = 0$ for $i < 1$ and $F(X)^j = X^j$ for $j \geq 1$. If $f : X \to Y$ is a morphism in $\mathbf{K}^{\leq m, b}(\operatorname{Proj} \Lambda)$, $F(f) : F(X) \to F(Y)$ is defined by $F(f)^s = f^s$ for $s \in [1, m]$.

We know from Corollary 5.7 of [1] that F induces an equivalence

$$\overline{F} : \mathcal{L}_m \to \overline{\mathbf{C}}_m(\operatorname{Proj} \Lambda)$$

where $\mathcal{L}_m$ is the full subcategory of $\mathbf{K}^{\leq m, \mathrm{b}}(\operatorname{Proj} \Lambda)$ whose objects are those X with $H^i(X) = 0$ for $i \leq 1$. The category $\overline{\mathbf{C}}_m(\operatorname{Proj} \Lambda)$ is the category with the same objects as $\mathbf{C}_m(\operatorname{Proj}\Lambda)$ and morphisms being the morphisms in $\mathbf{C}_m(\operatorname{Proj}\Lambda)$ modulo those which factorize through $\mathcal{E}$-injectives.

LEMMA 2.4. *Let Y be an indecomposable object in $\mathcal{L}_m$ and $E_K(Y)$ the endomorphism ring of Y in the category $\mathcal{L}_m$. Assume that for all $i \in \mathbb{Z}$, $H^i(Y)$ has finite length as $E_K(Y)$-module. Then $F(Y) = \overline{Y} \oplus W$, with $\overline{Y}$ an indecomposable complex in $\mathbf{C}^1_m(\operatorname{Proj}\Lambda)$ and W an $\mathcal{E}$-injective complex in $\mathbf{C}_m(\operatorname{Proj}\Lambda)$. Moreover*:

(1) $\operatorname{length}_{E_K(\overline{Y})} H^i(\overline{Y}) = \operatorname{length}_{E_K(Y)} H^i(Y)$ *for all* $i \in [2,m]$;
(2) $\operatorname{endol}(\overline{Y}) < c(mL + (m-1)L + \cdots + L^m)$ *with* $c = \max\{\operatorname{length}_{E_K(Y)} H^i(Y) \mid i \in [2,m]\}$.

Proof. By (i) and (iii) of Proposition 6.6 of [1], we have $F(Y) = \overline{Y} \oplus W$ where $\overline{Y} \in \mathbf{C}^1_m(\operatorname{Proj}\Lambda)$ has no $\mathcal{E}$-injective direct summands and W is an $\mathcal{E}$-injective complex. We know that $\overline{F}$ is an equivalence and Y is an indecomposable object in $\mathcal{L}_m$, thus $\overline{Y}$ is an indecomposable complex. By the definition of the functor F, for $i > 1$ we have $H^i(Y) = H^i(F(Y))$. Denote by $\sigma : \overline{Y} \to F(Y)$ the canonical inclusion and by $\pi : F(Y) \to \overline{Y}$ the canonical projection. We have $W = W_1 \oplus W_2$, with W_1 a direct sum of objects of the form $S(P)$ and W_2 a direct sum of objects of the form $J_u(P)$ with $u \in [1, m-1]$. Then for $i > 1$, $H^i(W_1) = 0$ and $H^i(W_2) = 0$. Therefore for $i > 1$, $H^i(\sigma) : H^i(\overline{Y}) \to H^i(F(Y))$ is an isomorphism with inverse $H^i(\pi)$. On the other hand, $\overline{F}$ induces an isomorphism $\mu : E_K(Y) \to \overline{\operatorname{End}}_{\mathbf{C}_m(\operatorname{Proj}\Lambda)}(\overline{Y})$. For the homotopy class $\underline{f}$ of a morphism $f : Y \to Y$ of complexes we have $\mu(\underline{f}) = \overline{\pi F(f)\sigma}$, the image of $\pi F(f)\sigma$ in $\overline{\operatorname{End}}_{\mathbf{C}_m(\operatorname{Proj}\Lambda)}(\overline{Y})$. Observe that if $u : \overline{Y} \to \overline{Y}$ is a morphism of complexes which factorizes through some $\mathcal{E}$-injective, then $H^i(u) = 0$ for $i > 1$, thus $H^i(\overline{Y})$ is an $\overline{\operatorname{End}}_{\mathbf{C}_m(\operatorname{Proj}\Lambda)}(\overline{Y})$-module.

Through the isomorphism $\mu : E_K(Y) \to \overline{\operatorname{End}}_{\mathbf{C}_m(\operatorname{Proj}\Lambda)}(\overline{Y})$, $H^i(\overline{Y})$ becomes an $E_K(Y)$-module and $H^i(\sigma) : H^i(\overline{Y}) \to H^i(F(Y))$ is an isomorphism of $E_K(Y)$-modules for $i > 1$, thus

$$\operatorname{length}_{\overline{\operatorname{End}}_{\mathbf{C}_m(\operatorname{Proj}\Lambda)}(\overline{Y})} H^i(\overline{Y}) = \operatorname{length}_{E_K(Y)} H^i(Y).$$

Now we recall that a morphism $f : \overline{Y} \to \overline{Y}$ is homotopic to zero if and only if f is a sum of morphisms which factorize through objects of the form $J_u(P)$. But there are no non-zero morphisms from $J_u(P)$ to $\overline{Y}$ if $u < 1$ and there are no non-zero morphisms from $\overline{Y}$ to $J_u(P)$ if $u \geq m$. Thus f is homotopic to zero if and only if f is a sum of morphisms which factorize through objects $J_u(P)$ with $1 \leq u \leq m-1$; these objects are $\mathcal{E}$-injective objects in $\mathbf{C}_m(\operatorname{Proj}\Lambda)$. Therefore we have an epimorphism

$$E_K(\overline{Y}) \to \overline{\operatorname{End}}_{\mathbf{C}_m(\operatorname{Proj}\Lambda)}(\overline{Y}).$$

The above morphism induces a structure of $E_K(\overline{Y})$-module on $H^i(\overline{Y})$ for $i > 1$ which coincides with its natural structure of $E_K(\overline{Y})$-module. Therefore

$$\operatorname{length}_{E_K(\overline{Y})} H^i(\overline{Y}) = \operatorname{length}_{\overline{\operatorname{End}}_{\mathbf{C}_m(\operatorname{Proj} \Lambda)}(\overline{Y})} H^i(\overline{Y}) = \operatorname{length}_{E_K(Y)} H^i(Y)$$

for all $i > 1$, so we obtain (1).

Finally, if $\operatorname{Ker} d^1_{\overline{Y}}$ is not contained in $\operatorname{rad} \overline{Y}^1$, then $\overline{Y}$ has a direct summand of the form $S(P)$, which is not the case because $\overline{Y}$ is indecomposable. Thus $\operatorname{Ker} d^1_{\overline{Y}} \subset \operatorname{rad} \overline{Y}^1$, and from Lemma 2.2 we obtain (2). ∎

Proposition 2.5. *If Λ is an artin algebra, then*:

(i) $\mathcal{D}^{\mathrm{b}}(\Lambda)$ *is endo-discrete* (*resp. discrete*) *if and only if* $\mathbf{C}_m(\operatorname{proj} \Lambda)$ *is endo-discrete* (*resp. discrete*) *for all* m.

(ii) $\mathcal{D}^{\mathrm{b}}(\operatorname{Mod} \Lambda)$ *has a generic complex if and only if* $\mathbf{C}_m(\operatorname{Proj} \Lambda)$ *has a generic object for some* m.

Proof. Assume $\mathcal{D}^{\mathrm{b}}(\Lambda)$ is endo-discrete. Take a family $\{Y_s\}_{s\in I}$ of pairwise non-isomorphic indecomposable objects in some $\mathbf{C}_m(\operatorname{proj} \Lambda)$ with $\operatorname{endol}(Y_s) \leq d$ for a fixed d and all $s \in I$. We are going to prove that I is a finite set. Since there are only finitely many isomorphism classes of indecomposable $\mathcal{E}$-injective complexes in $\mathbf{C}_m(\operatorname{proj} \Lambda)$, we may assume that the complex Y_s is not $\mathcal{E}$-injective for all $s \in I$. Here $\operatorname{endol}(Y_s) = \sum_{i=1}^m \operatorname{length}_{E_C(Y_s)}(Y_s^i) \leq d$. As $\operatorname{Ker} d^i_{Y_s}$ and $\operatorname{Im} d^{i-1}_{Y_s}$ are $E_C(Y_s)$-submodules of Y_s^i, this implies that $\operatorname{length}_{E_C(Y_s)} \operatorname{Ker} d^i_{Y_s} \leq d$ and

$$\operatorname{length}_{E_C(Y_s)}(\operatorname{Ker} d^i_{Y_s}/\operatorname{Im} d^{i-1}_{Y_s}) = \operatorname{length}_{E_K(Y_s)} H^i(Y_s) \leq d.$$

The objects Y_s are pairwise non-isomorphic in $\mathbf{C}^{\mathrm{b}}(\operatorname{proj} \Lambda)$, thus they are pairwise non-homotopic. Now the Y_s are complexes of projective Λ-modules, so they are pairwise non-isomorphic in $\mathcal{D}^{\mathrm{b}}(\Lambda)$. Since $\mathcal{D}^{\mathrm{b}}(\Lambda)$ is endo-discrete, the set I is finite; this proves that $\mathbf{C}_m(\operatorname{proj} \Lambda)$ is endo-discrete for all m.

Conversely, suppose that for all m, $\mathbf{C}_m(\operatorname{proj} \Lambda)$ is endo-discrete. Take a family $\{Z_s\}_{s\in I}$ of pairwise non-isomorphic indecomposable complexes in $\mathcal{D}^{\mathrm{b}}(\Lambda)$ with fixed homology endolength $\mathbf{h} = (h_i)_{i\in\mathbb{Z}}$. Set $c = \max\{h_i\}_{i\in\mathbb{Z}}$. After a shift we may assume that for some m, $h_i = 0$ for $i \notin [2, m]$. For each $s \in I$ take a quasi-isomorphism $P_s \to Z_s$ with P_s an indecomposable object in $\mathbf{K}^{\leq m,\mathrm{b}}(\operatorname{proj} \Lambda)$. Now $\overline{F} : \mathcal{L}_m \to \overline{\mathbf{C}}_m(\operatorname{Proj} \Lambda)$ is an equivalence, which implies that the objects $Y_s = \overline{F}(Z_s)$ are pairwise non-isomorphic indecomposables in $\overline{\mathbf{C}}_m(\operatorname{proj} \Lambda)$. We have an isomorphism $E_K(P_s) \cong \operatorname{End}_{\mathcal{D}^{\mathrm{b}}(\Lambda)}(Z_s)$, thus $H^i(Z_s)$ becomes an $E_K(P_s)$-module and $H^i(Z_s) \cong H^i(P_s)$ as $E_K(P_s)$-modules, so

$$\begin{aligned}\operatorname{length}_{E_K(P_s)} H^i(P_s) &= \operatorname{length}_{E_K(P_s)} H^i(Z_s) \\ &= \operatorname{length}_{\operatorname{End}_{\mathcal{D}^{\mathrm{b}}(\Lambda)}(Z_s)} H^i(Z_s) = h_i.\end{aligned}$$

We have $P_s \in \mathcal{L}_m$, so by Lemma 2.4, $Y_s = Y_{s,0} \oplus Y_{s,1}$ with $Y_{s,1}$ an $\mathcal{E}$-injective complex and $Y_{s,0}$ an indecomposable complex which is not $\mathcal{E}$-injective with $\operatorname{endol}(Y_{s,0}) < c(mL + (m-1)L^2 + \cdots + L^m)$. Here $\overline{F}$ is an equivalence, so the complexes $Y_{s,0}$ with $s \in I$ are pairwise non-isomorphic indecomposable complexes in $\mathbf{C}_m(\operatorname{proj} \Lambda)$ which is endo-discrete, hence I is finite. This proves (i); the corresponding statement for discrete categories is proved in a similar way.

Let Y be a generic complex in $\mathcal{D}^{\mathrm{b}}(\operatorname{Mod} \Lambda)$. As before we may assume that if $\mathbf{h}\operatorname{endol}(Y) = \mathbf{h} = (h_i)_{i\in\mathbb{Z}}$, then there is an m such that $h_i = 0$ for $i \notin [2, m]$. Consider a quasi-isomorphism $P_Y \to Y$ with P_Y an indecomposable object in $\mathbf{K}^{\leq m,\mathrm{b}}(\operatorname{Proj} \Lambda)$. As in the proof of (i) we have

$$\operatorname{length}_{E_K(P_Y)} H^i(P_Y) = \operatorname{length}_{\operatorname{End}_{\mathcal{D}^{\mathrm{b}}(Y)}} H^i(Y) = h_i \quad \text{for all } i \in \mathbb{Z},$$

thus $P_Y \in \mathcal{L}_m$. Take $F(P_Y) \in \mathbf{C}_m(\operatorname{Proj} \Lambda)$. By Lemma 2.4 we have $F(P_Y) = \overline{P}_Y \oplus W$ with $\overline{P}_Y$ an indecomposable complex in $\mathbf{C}^1_m(\operatorname{Proj} \Lambda)$ which is not $\mathcal{E}$-injective, and W an $\mathcal{E}$-injective complex. Moreover by Lemma 2.4(2), $\overline{P}_Y$ is an endofinite complex, therefore it is generic in $\mathbf{C}_m(\operatorname{Proj} \Lambda)$.

Conversely, assume Y is a generic object in some $\mathbf{C}_m(\operatorname{Proj} \Lambda)$. We may assume $H^j(Y) = 0$ for all $j \leq 1$. If for some j, the image of d_Y^{j-1} is not contained in $\operatorname{rad} Y^j$, then Y has a direct summand of the form $J_{j-1}(P)$ for some indecomposable projective Λ-module P. But Y is indecomposable, so $Y \cong J_{j-1}(P)$, which is not the case because Y is not a finite object. Therefore $Y \in \mathbf{C}^1_m(\operatorname{Proj} \Lambda)$, and consequently Y is a non-zero object in the category $\mathbf{K}^{\leq m,\mathrm{b}}(\operatorname{Proj} \Lambda)$. We know that Y^i has finite length over $E_C(Y)$ for all $i \in \mathbb{Z}$, and so $\operatorname{length}_{E_C(Y)} H^i(Y) = \operatorname{length}_{E_K} H^i(Y)$ is finite for all $i \in \mathbb{Z}$. Consider Y as an object of $\mathcal{D}^{\mathrm{b}}(\operatorname{Mod} \Lambda)$; here Y is a finite complex of projective Λ-modules. Then $E_K(Y) = \operatorname{End}_{\mathcal{D}^{\mathrm{b}}(\operatorname{Mod} \Lambda)}(Y)$. Therefore Y is indecomposable in $\mathcal{D}^{\mathrm{b}}(\operatorname{Mod} \Lambda)$ and $H^i(Y)$ has finite length as $\operatorname{End}_{\mathcal{D}^{\mathrm{b}}(\operatorname{Mod} \Lambda)}(Y)$-module for all $i \in \mathbb{Z}$. Suppose that Y is isomorphic to some object $X \in \mathcal{D}^{\mathrm{b}}(\Lambda)$. Here $H^i(Y) \cong H^i(X)$ for all $i \in \mathbb{Z}$. Thus $H^j(X) = 0$ for all $j > m$, therefore there is a complex P_X quasi-isomorphic to X with $P_X \in \mathbf{K}^{\leq m,\mathrm{b}}(\operatorname{Proj} \Lambda)$ and such that P^i_X is a finitely generated projective Λ-module for all $i \in \mathbb{Z}$. Then Y and P_X are quasi-isomorphic complexes both in $\mathbf{K}^{\leq m,\mathrm{b}}(\operatorname{Proj} \Lambda)$, therefore they are isomorphic objects in the category $\mathcal{L}_m$; thus we have $Y = F(Y) \cong F(P_X)$ in the category $\overline{\mathbf{C}}_m(\operatorname{Proj} \Lambda)$. By Lemma 2.4, $F(P_X) = \overline{P}_X \oplus W$, with W an $\mathcal{E}$-injective object and $\overline{P}_X \in \mathbf{C}^1_m(\operatorname{Proj} \Lambda)$. Here Y is not a finite object and it is indecomposable, therefore Y is not $\mathcal{E}$-injective. On the other hand, $\overline{P}_X$ is a finite object which is not $\mathcal{E}$-injective. Thus $\overline{P}_X = \overline{P}_1 \oplus W_1$ with $\overline{P}_1$ without $\mathcal{E}$-injective direct summands and W_1 an $\mathcal{E}$-injective object. Therefore $Y \cong \overline{P}_1$ as complexes, which is a contradiction because Y is not a finite object, and hence Y is a generic complex in $\mathcal{D}^{\mathrm{b}}(\operatorname{Mod} \Lambda)$. ∎

3. A category of morphisms. For $m \geq 1$, we consider the following category $\mathcal{M}_m$ of morphisms in $\mathbf{C}^1_m(\operatorname{Proj} \Lambda)$. The objects of $\mathcal{M}_m$ are radical morphisms $f : S(P) \to X$ in $\mathbf{C}^1_m(\operatorname{Proj} \Lambda)$ with P a projective Λ-module and X an object in $\mathbf{C}^1_m(\operatorname{Proj} \Lambda)$. The morphisms from $f : S(P) \to X$ to $f' : S(P') \to X'$ are given by pairs of morphisms $u = (u_1, u_2)$, $u_1 : P \to P'$, $u_2 : X \to X'$, such that $u_2 f = f' S(u_1)$. If $u = (u_1, u_2)$ is a morphism from $f : S(P) \to X$ to $f' : S(P') \to X'$ and $v = (v_1, v_2)$ is a morphism from $f' : S(P') \to X'$ to $f'' : S(P'') \to X''$, then $vu = (v_1 u_1, v_2 u_2)$. The identity morphism of the object $f : S(P) \to X$ is given by the pair $(\mathrm{id}_P, \mathrm{id}_X)$.

An object $f : S(P) \to X$ is called *endofinite* if P and all X^i have finite length as $E(f) = \operatorname{End}_{\mathcal{M}_m}(f)$-modules. In this case $\operatorname{endol}(f) = \operatorname{length}_{E(f)} P + \sum_i \operatorname{length}_{E(f)} X^i$. The object $f : S(P) \to X$ is called *finite* if P is a finitely generated Λ-module and X is an object in $\mathbf{C}^1_m(\operatorname{proj} \Lambda)$. We put

$$\operatorname{length}(f) = \operatorname{length}_k P + \sum_i \operatorname{length}_k X^i.$$

PROPOSITION 3.1. *There is a functor $\Theta : \mathcal{M}_m \to \mathbf{C}^1_{m+1}(\operatorname{Proj} \Lambda)$ with the following properties*:

(i) *Θ is an equivalence of categories.*

(ii) *$f \in \mathcal{M}_m$ is endofinite* (*respectively finite*) *if and only if $\Theta(f)$ is endofinite* (*respectively finite*) *and*

$$\operatorname{endol}(f) = \operatorname{endol}(\Theta(f)) \quad (\textit{respectively} \quad \operatorname{length}(f) = \operatorname{length} \Theta(f)).$$

Proof. Take an object $f : S(P) \to X$ in $\mathcal{M}_m$. We have the morphism $f^1 : P \to X^1$, f is a radical morphism, thus $\operatorname{Im} f^1 \subset \operatorname{rad} X^1$, moreover f is a morphism of complexes, so that $d^1_X f^1 = f^2 d^1_{S(P)} = 0$. Therefore we have the complex $\Theta(f)$ in $\mathbf{C}^1_{m+1}(\operatorname{Proj} \Lambda)$ given by $\Theta(f)^i = 0$ for $i \notin [1, m+1]$, $\Theta(f)^1 = P$, $\Theta(f)^{i+1} = X^i$ for $i \in [1, m]$, $d^1_{\Theta(f)} = f^1$, $d^{i+1}_{\Theta(f)} = d^i_X$ for $i \in [1, m]$.

Now if $u = (u_1, u_2)$ is a morphism from $f : S(P) \to X$ to $f' : S(P') \to X'$, we define $\Theta(u)$ in the following way: $\Theta(u)^i = 0$ for $i \notin [1, m+1]$, $\Theta(u)^1 = u_1 : \Theta(f)^1 = P \to \Theta(f')^1 = P'$, $\Theta(u)^{i+1} = u_2^i : \Theta(f)^{i+1} = X^i \to \Theta(f')^{i+1} = (X')^i$ for $i \in [1, m]$.

We have $d^1_{\Theta(f')} \Theta(u)^1 = (f')^1 u_1 = (u_2)^1 f = \Theta(u)^2 d^1_{\Theta(f)}$. For $i \in [1, m]$ we have $d^{i+1}_{\Theta(f')} \Theta(u)^{i+1} = d^i_{X'} u_2^i = u_2^{i+1} d^i_X = \Theta(u)^{i+2} d^{i+1}_{\Theta(f)}$. From this we conclude that $\Theta(u) : \Theta(f) \to \Theta(f')$ is a morphism of complexes. We have $\Theta(\mathrm{id}_f) = \mathrm{id}_{\Theta(f)}$. Now if v is a morphism from $f' : S(P') \to X'$ to $f'' : S(P'') \to X''$, then $\Theta(v)\Theta(u) = \Theta(vu)$. Clearly Θ is a full, faithful dense functor and (ii) holds. ∎

Let P be an indecomposable in $\operatorname{proj} \Lambda$. We recall (see Proposition 8.7 of [1]) that for $u \in [1, m-1]$ there is a left almost split morphism $s : J_u(P) \to$

$L_u(P)$ in $\mathbf{C}_m(\operatorname{proj}\Lambda)$ with $L_u(P)$ an indecomposable object in $\mathbf{C}^1_m(\operatorname{proj}\Lambda)$. Therefore for any complex X in the category $\mathbf{C}^1_m(\operatorname{proj}\Lambda)$ we have an epimorphism $\operatorname{Hom}(s,1) : \operatorname{Hom}_{\mathbf{C}_m(\operatorname{proj}\Lambda)}(L_u(P),X) \to \operatorname{Hom}_{\mathbf{C}_m(\operatorname{proj}\Lambda)}(J_u(P),X)$. If $X = \bigoplus_{i\in I} X_i$ is a complex in the category $\mathbf{C}^1_m(\operatorname{Proj}\Lambda)$, with $X_i \in \mathbf{C}^1_m(\operatorname{proj}\Lambda)$, then any morphism $f : J_u(P) \to X$ is of the form σf_0, where $f_0 : J_u(P) \to \bigoplus_{i\in I_0} X_i$, $\sigma : \bigoplus_{i\in I_0} X_i \to X$, I_0 is a finite subset of I and σ is the canonical inclusion. Thus $f_0 = gs$ for some $g : L_u(P) \to \bigoplus_{i\in I_0} X_i$, therefore $f = \sigma gs$. This implies that if X is a complex in $\mathbf{C}^1_m(\operatorname{Proj}\Lambda)$ which is an arbitrary sum of objects in $\mathbf{C}^1_m(\operatorname{proj}\Lambda)$ then

$$\operatorname{Hom}(s,1) : \operatorname{Hom}_{\mathbf{C}_m(\operatorname{Proj}\Lambda)}(L_u(P),X) \to \operatorname{Hom}_{\mathbf{C}_m(\operatorname{Proj}\Lambda)}(J_u(P),X)$$

is an epimorphism.

Suppose now that $\mathbf{C}_m(\operatorname{proj}\Lambda)$ is endo-discrete. Let $Z_1,\dots,Z_n$ be a complete representative set of the isomorphism classes of indecomposables $X \in \mathbf{C}^1_m(\operatorname{proj}\Lambda)$ with either $\operatorname{endol}(X) \le d$, or for some indecomposable projective Λ-modules P, $X \cong L_u(P)$ for some $u \in [1, m-1]$, or $X \cong T(P)$. Take $Z = Z_1 \oplus \cdots \oplus Z_n$ and let Y be the sum of those Z_i which are isomorphic to some $S(P)$.

Consider now $R = \operatorname{End}_{\mathbf{C}^1_m(\operatorname{proj}\Lambda)}(Z)^{\mathrm{op}}$ and f the projection of Z on Y followed by the corresponding inclusion of Y in Z. Then f is an idempotent and $fRf \cong \operatorname{End}_{\mathbf{C}_m(\operatorname{proj}\Lambda)}(Y)^{\mathrm{op}}$.

Take $A = \begin{pmatrix} fRf & 0 \\ 0 & R\end{pmatrix}$ and consider the following exact sequence of A-A-bimodules:

$$\xi : \quad 0 \to \begin{pmatrix} 0 & f\operatorname{rad}R \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} fRf & f\operatorname{rad}R \\ 0 & R \end{pmatrix} \xrightarrow{\pi} \begin{pmatrix} fRf & 0 \\ 0 & R \end{pmatrix} \to 0.$$

In the following we set F equal to the middle term of the above sequence. We recall that the lift category $\xi(A)$ is defined as follows:

1. The objects in $\xi(A)$ are pairs (P,e) consisting of a projective A-module P and a morphism $e : P \to F \otimes_A P$ of A-modules such that $(\pi \otimes \mathrm{id}_P)e = \mathrm{id}_P$ for $\pi \otimes \mathrm{id}_P : F \otimes_A P \to P$.
2. The morphisms from (P,e) to (P',e') are those morphisms $\theta : P \to P'$ of A-modules such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\theta} & P' \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ F \otimes_R P & \xrightarrow{\mathrm{id}_F \otimes \theta} & F \otimes_R P' \end{array}$$

An object $X = (P,e) \in \xi(A)$ is called *endofinite* if P has finite length as $E(X) = \operatorname{End}_{\xi(A)}(X)$-module; we then put $\operatorname{endol}(X) = \operatorname{length}_{E(X)} P$. An object $Y = (Q,f) \in \xi(A)$ is called *finite* if Q has finite length as k-module, in which case we put $\operatorname{length} Y = \operatorname{length}_k Q$. An object $X \in \xi(A)$ is called *generic* if X is not finite, but indecomposable and endofinite.

DEFINITION 3.2. The category $\xi(A)$ is called *endo-discrete* (respectively *discrete*) if for all natural numbers d, there are only a finite number of isomorphism classes of finite indecomposable objects in $\xi(A)$ having endolength (respectively length) less than or equal to d.

From Theorem 9.5 of [2] we obtain the following.

THEOREM 3.3. *The category $\xi(A)$ is not endo-discrete if and only if it contains a generic object. In case k has infinitely many elements, $\xi(A)$ is not discrete if and only if it contains a generic object.*

In the proof of the next proposition we need the following lemma.

LEMMA 3.4. *Let X be a finitely generated left Λ-module and Y a Λ-B-bimodule with B a k-algebra. Then if Y has finite length as right B-module, the right B-module $\operatorname{Hom}_\Lambda(X,Y)$ has finite length. Moreover*

$$\operatorname{length}_B \operatorname{Hom}_\Lambda(X,Y) \le t(X) \operatorname{length}_B Y,$$

where $t(X)$ is the minimal number of generators of the Λ-module X.

Proof. We have an epimorphism $\Lambda^t \to X$ for $t = t(X)$. Hence we obtain a monomorphism of right B-modules

$$\operatorname{Hom}_\Lambda(X,Y) \to \operatorname{Hom}_\Lambda(\Lambda^t,Y) \cong \bigoplus_{i=1}^{t} Y$$

which yields our result. ■

Assume $\mathbf{C}_m(\operatorname{proj}\Lambda)$ is endo-discrete. We denote by $\mathcal{M}_{m,\mathrm{d}}$ the full subcategory of $\mathcal{M}_m$ whose objects are of the form $f : S(P) \to X$ with X a direct sum of direct summands of Z and $S(P)$ a direct sum of direct summands of Y.

PROPOSITION 3.5. *Suppose Λ is a basic artin algebra. Then there is a functor $U : \mathcal{M}_{m,\mathrm{d}} \to \xi(A)$ with the following properties*:

(i) *U is an equivalence.*
(ii) *For $Y \in \mathcal{M}_{m,\mathrm{d}}$, and $t(Z)$ the maximum of $t(Z^i)$, $i \in [1,m]$, for $Z = (Z^i, d^i_Z)$ and $t(Z^i)$ the minimal number of generators of Z^i as Λ-module we have*

$$\operatorname{endol}(Y) \le \operatorname{endol}(U(Y)) \le t(Z)\operatorname{endol}(Y),$$
$$\operatorname{length}(Y) \le \operatorname{length}(U(Y)) \le t(Z)\operatorname{length}(Y);$$

(iii) *$Y \in \mathcal{M}_{m,\mathrm{d}}$ is a generic object if and only if $U(Y)$ is a generic object in $\xi(A)$.*

Proof. A projective A-module is given by a pair (P_1, P_2) with P_1 a projective fRf-module and P_2 a projective R-module. If (P, e) is an object of $\xi(A)$, then e is given by a morphism $\phi : P_1 \to f \operatorname{rad} P_2$ of fRf-modules.

Now if P_1 and P_2 are as before we have the following natural isomorphisms:

$$\begin{aligned}\operatorname{Hom}_{fRf}(P_1, f \operatorname{rad} P_2) &\cong \operatorname{Hom}_{fRf}(P_1, \operatorname{Hom}_R(Rf, \operatorname{rad} P_2)) \\ &\cong \operatorname{Hom}_R(Rf \otimes_{fRf} P_1, \operatorname{rad} P_2).\end{aligned}$$

Therefore the category $\xi(A)$ is equivalent to the category $\mathcal{U}$ whose objects are radical morphisms $u : Q_1 \to Q_2$ of R-modules, where Q_1 is a direct sum of direct summands of Rf, and Q_2 is a projective R-module. Now $\mathcal{C}_Z$, the full subcategory of $\mathbf{C}_m(\operatorname{Proj} \Lambda)$ whose objects are arbitrary sums of direct summands of Z, is equivalent to the category $\operatorname{Proj} R$. In a similar way, $\mathcal{C}_Y$, the full subcategory of $\mathcal{C}_Z$ whose objects are arbitrary sums of direct summands of Y, is equivalent to the full subcategory of $\operatorname{Proj} R$, whose objects are arbitrary sums of direct summands of Rf. Consequently, the category $\mathcal{U}$ is equivalent to the category $\mathcal{M}_{m,\mathrm{d}}$ whose objects are radical morphisms $h : S(P) \to X$ in $\mathbf{C}^1_m(\operatorname{Proj} \Lambda)$ with $S(P)$ a sum of direct summands of Y, and X a sum of direct summands of Z. Then we have an equivalence $U : \mathcal{M}_{m,\mathrm{d}} \to \xi(A)$ such that

$$U(h : S(P) \to X) = (\operatorname{Hom}_{\mathbf{C}_m(\operatorname{Proj} \Lambda)}(Y, S(P)) \oplus \operatorname{Hom}_{\mathbf{C}_m(\operatorname{Proj} \Lambda)}(Z, X), e_h).$$

Then if the object $h : S(P) \to X$ is endofinite and $E(h) = \operatorname{End}_{\mathcal{M}_{m,\mathrm{d}}}(h)$, both $S(P)$ and X are Λ-$E(h)^{\mathrm{op}}$-bimodules. Thus by Lemma 3.4 both $\operatorname{Hom}_{\mathbf{C}_m(\operatorname{Proj} \Lambda)}(Y, S(P))$ and $\operatorname{Hom}_{\mathbf{C}_m(\operatorname{Proj} \Lambda)}(Z, X)$ have finite length as $E(h)$-modules, and therefore as $E(U(h))$-modules. Consequently, $U(h : S(P) \to X)$ is endofinite, so if h is generic its image under U is also generic.

Moreover using Lemma 3.4 we have $\operatorname{endol}(U(h)) \le t(Z) \operatorname{endol}(h)$ and if h is a finite object, then $\operatorname{length}(U(h)) \le t(Z) \operatorname{length}(h)$.

Take a decomposition $1 = \sum_j e_j$ into pairwise orthogonal primitive idempotents in Λ. Then for $u \in [1, m-1]$, $L_u(\Lambda e_i) \cong W$ for some $W \in \{Z_1, \dots, Z_n\}$. Denote by f_W the projection of Z on W followed by the inclusion of W in Z. We have

$$f_W \operatorname{Hom}_{\mathbf{C}_m(\operatorname{Proj} \Lambda)}(Z, X) \cong \operatorname{Hom}_{\mathbf{C}_m(\operatorname{Proj} \Lambda)}(L_u(\Lambda e_i), X).$$

On the other hand, we have an epimorphism

$$\begin{aligned}\operatorname{Hom}_{\mathbf{C}_m(\operatorname{Proj} \Lambda)}(L_u(\Lambda e_i), X) &\to \operatorname{Hom}_{\mathbf{C}_m(\operatorname{Proj} \Lambda)}(J_u(\Lambda e_i), X) \\ &\cong \operatorname{Hom}_\Lambda(\Lambda e_i, X^u) \cong e_i X^u.\end{aligned}$$

Thus $\operatorname{length}_{E(h)} e_i X^u \le \operatorname{length}_{E(U(h))} f_W \operatorname{Hom}_{\mathbf{C}_m(\operatorname{Proj} \Lambda)}(Z, X)$.

We recall that $\mathrm{Hom}_{\mathbf{C}_m(\mathrm{Proj}\,\Lambda)}(T(\Lambda e_i), X) \cong \mathrm{Hom}_\Lambda(\Lambda e_i, X^m) \cong e_i X^m$. Taking now $T(\Lambda e_i)$ instead of $L_u(\Lambda e_i)$, we obtain

$$\mathrm{length}_{E(h)}\, e_i X^m = \mathrm{length}_{E(U(h))}\, f_W\, \mathrm{Hom}_{\mathbf{C}_m(\mathrm{Proj}\,\Lambda)}(Z, X)$$

for $W \in \{Z_1, \ldots, Z_n\}$ isomorphic to $T(\Lambda e_i)$.

Since Λ is basic we have

$$\sum_{u=1}^{m} \mathrm{length}_{E(h)}\, X^u \le \mathrm{length}_{E(U(h))}\, \mathrm{Hom}_{\mathbf{C}_m(\mathrm{Proj}\,\Lambda)}(Z, X).$$

In a similar way we obtain

$$\mathrm{length}_{E(h)}\, S(P) \le \mathrm{length}_{E(U(h))}\, \mathrm{Hom}_{\mathbf{C}_m(\mathrm{Proj}\,\Lambda)}(Y, S(P)).$$

Hence $\mathrm{endol}(h) \le \mathrm{endol}(U(h))$. Similarly, $\mathrm{length}(h) \le \mathrm{length}(U(h))$. Therefore if $U(h)$ is endofinite, then so is h, and this implies that if $U(h)$ is generic, then so is h. ∎

THEOREM 3.6. *For $m \ge 1$, the category $\mathbf{C}_m(\mathrm{proj}\,\Lambda)$ is not endo-discrete if and only if $\mathbf{C}_m(\mathrm{Proj}\,\Lambda)$ has a generic object. In case k has infinitely many elements, $\mathbf{C}_m(\mathrm{proj}\,\Lambda)$ is not discrete if and only if $\mathbf{C}_m(\mathrm{Proj}\,\Lambda)$ has a generic object.*

Proof. First we assume that Λ is a basic algebra. For $m = 1$, $\mathbf{C}_1(\mathrm{proj}\,\Lambda) \cong \mathrm{proj}\,\Lambda$, therefore this category is endo-discrete and $\mathbf{C}_1(\mathrm{Proj}\,\Lambda) \cong \mathrm{Proj}\,\Lambda$ does not contain indecomposable objects of infinite length over k, so our result holds in this case.

Assume our result is proved for m; we shall prove it for $m+1$. Suppose that the category $\mathbf{C}_{m+1}(\mathrm{proj}\,\Lambda)$ is not endo-discrete. Then for some d there are infinitely many isomorphism classes of indecomposables in $\mathbf{C}^1_{m+1}(\mathrm{proj}\,\Lambda)$ with endolength d. If $\mathbf{C}_m(\mathrm{proj}\,\Lambda)$ is not endo-discrete then by the induction hypothesis there is a generic object in $\mathbf{C}_m(\mathrm{Proj}\,\Lambda)$, hence there is one in $\mathbf{C}_{m+1}(\mathrm{Proj}\,\Lambda)$. Therefore we may assume $\mathbf{C}_m(\mathrm{proj}\,\Lambda)$ is endo-discrete. Consider the equivalence $\Theta : \mathcal{M}_n \to \mathbf{C}^1_{m+1}(\mathrm{Proj}\,\Lambda)$. We have an infinite family $\{Y_s\}_{s\in I}$ of pairwise non-isomorphic indecomposable objects in $\mathbf{C}^1_{m+1}(\mathrm{proj}\,\Lambda)$ with $\mathrm{endol}(Y_s) \le d$ for all $s \in I$. For each $s \in I$ there is an object $h_s : S(P_s) \to X_s$ in $\mathcal{M}_m$ with $\Theta(h_s) \cong Y_s$. We have $\mathrm{endol}(X_s) \le \mathrm{length}_{E(h_s)}\, X_s \le d$. Thus each X_s is a finite direct sum of indecomposable finite objects with endolength $\le d$. Therefore each h_s is in $\mathcal{M}_{m,\mathrm{d}}$ and by Proposition 3.5(ii), $\mathrm{endol}(U(h_s)) \le t(Z)d$. Consequently, $\xi(A)$ is not endo-discrete, and by Theorem 3.3 the category $\xi(A)$ contains a generic object G. Then $G \cong U(g)$, and by Proposition 3.5(iii), g is a generic object in $\mathcal{M}_{m,\mathrm{d}}$. But then $\Theta(g)$ is a generic object in $\mathbf{C}_{m+1}(\mathrm{Proj}\,\Lambda)$.

Conversely, suppose Y is a generic object in $\mathbf{C}_{m+1}(\mathrm{Proj}\,\Lambda)$. It has no direct summands of the form $J_u(P)$, so $Y \in \mathbf{C}^1_{m+1}(\mathrm{Proj}\,\Lambda)$ with $\mathrm{endol}(Y) \le d$.

Then $Y \cong \Theta(h)$ with $h : S(P) \to X$ in $\mathcal{M}_m$. Here

$$\operatorname{endol}(X) \leq \operatorname{length}_{E(h)}(X) \leq d.$$

Thus X is endofinite and therefore it is a direct sum of indecomposable objects of endolength $\leq d$. By the induction hypothesis $\mathbf{C}_m(\operatorname{Proj} \Lambda)$ does not contain generic objects, therefore X is a direct sum of direct summands of Z, so $h \in \mathcal{M}_{m,\mathrm{d}}$. By Proposition 3.5(iii), $U(h)$ is a generic object in $\xi(A)$. Again by Theorem 3.3, $\xi(A)$ is not endo-discrete. Thus there is an infinite family $\{Y_s\}_{s\in I}$ of pairwise non-isomorphic finite indecomposable objects in $\xi(A)$ with $\operatorname{endol}(Y_s) \leq u$ for all $s \in I$ and some u. For each Y_s there is a finite object $h_s \in \mathcal{M}_{m,\mathrm{d}}$ such that $U(h_s) \cong Y_s$. By Proposition 3.5(ii), $\operatorname{endol}(h_s) \leq \operatorname{endol}(Y_s) \leq u$. Now $\Theta(h_s)$ is a finite object in $\mathbf{C}_{m+1}(\operatorname{Proj} \Lambda)$ and $\operatorname{endol}(h_s) = \operatorname{endol}(\Theta(h_s))$. Thus $\mathbf{C}_{m+1}(\operatorname{proj} \Lambda)$ is not endo-discrete. The second part of our theorem is proved in a similar way.

To prove our result for a general artin algebra it is enough to show that if Λ_1 and Λ_2 are Morita-equivalent then $\mathbf{C}_m(\operatorname{Proj} \Lambda_1)$ is endo-discrete (respectively discrete) if and only if $\mathbf{C}_m(\operatorname{Proj} \Lambda_2)$ is, and the first category contains a generic object if and only if the second one does. So consider a Morita-equivalence $F = P \otimes_{\Lambda_1} - : \operatorname{Mod} \Lambda_1 \to \operatorname{Mod} \Lambda_2$ with P a Λ_2-Λ_1-bimodule, finitely generated projective on both sides. The functor F induces an equivalence $\widehat{F} : \mathbf{C}_m(\operatorname{Proj} \Lambda_1) \to \mathbf{C}_m(\operatorname{Proj} \Lambda_2)$ such that for $X \in \mathbf{C}_m(\operatorname{Proj} \Lambda_1)$, $\widehat{F}(X)^i = P \otimes_{\Lambda_1} X^i$ and $H^i(\widehat{F}(X)) \cong P \otimes_{\Lambda_1} H^i(X)$. We find that X^i is a Λ-$E_C(X)^{\mathrm{op}}$-bimodule and $\operatorname{length}_{E_C(X)} X^i = \operatorname{length}_{E_C(X)^{\mathrm{op}}} X^i$ for all $i \in [1, m]$. The functor $\widehat{F}$ induces an isomorphism $\nu : E_C(X)^{\mathrm{op}} \to E_C(\widehat{F}(X))^{\mathrm{op}}$ which induces a structure of Λ_2-$E_C(X)^{\mathrm{op}}$-bimodule on $\widehat{F}(X)^i = P \otimes_{\Lambda_1} X^i$, coinciding with the structure of Λ_2-$E_C(X)^{\mathrm{op}}$-bimodule induced by the structure of right $E_C(X)^{\mathrm{op}}$-module of X^i. Therefore, since P_{Λ_1} is a direct summand of some Λ_1^t, we have

$$\operatorname{length}_{E_C(\widehat{F}(X))}(\widehat{F}(X))^i = \operatorname{length}_{E_C(X)} P \otimes X^i \leq t \operatorname{length}_{E_C(X)} X^i.$$

Consequently, $\operatorname{endol}(\widehat{F}(X)) \leq t \operatorname{endol}(X)$. In a similar way we obtain

$$\operatorname{length}_{E_K(\widehat{F}(X))} H^i(\widehat{F}(X)) \leq t \operatorname{length}_{E_K(X)} H^i(X).$$

The above inequalities imply that if the category $\mathbf{C}_m(\operatorname{proj} \Lambda_2)$ is endo-discrete, then so is $\mathbf{C}_m(\operatorname{proj} \Lambda_1)$, and if X is an endofinite complex in $\mathbf{C}_m(\operatorname{Proj} \Lambda_1)$, then $\widehat{F}(X)$ is an endofinite complex in $\mathbf{C}_m(\operatorname{Proj} \Lambda_2)$.

Thus if X is generic, then so is $\widehat{F}(X)$. In a similar way one can prove that if $\mathbf{C}_m(\operatorname{proj} \Lambda_2)$ is discrete, then so is $\mathbf{C}_m(\operatorname{proj} \Lambda_1)$. This proves our result. ∎

Proof of Theorem 1.1. Follows from Proposition 2.5 and Theorem 3.6.

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