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AN EXTENSION PROPERTY FOR BANACH SPACES

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Abstract. A Banach space X has property (E) if every operator from X into c_0 extends to an operator from X^{**} into c_0; X has property (L) if whenever $K \subseteq X$ is limited in X^{**}, then K is limited in X; X has property (G) if whenever $K \subseteq X$ is Grothendieck in X^{**}, then K is Grothendieck in X. In all of these, we consider X as canonically embedded in X^{**}. We study these properties in connection with other geometric properties, such as the Phillips properties, the Gelfand–Phillips and weak Gelfand–Phillips properties, and the property of being a Grothendieck space.

Introduction. All Banach spaces enjoy the Hahn–Banach property, the most basic extension property for bounded linear operators between Banach spaces. At the other extreme, perhaps, is the property of being an injective space. This paper introduces a new extension/lifting property lying between these two, called property (E), as well as two related properties, property (G) and property (L), all of which are, in some sense, "anti-Phillips" properties ([6], [16]). We explore these new properties and their relationships with other geometric properties, such as the property (V*) of Pełczyński, and the property of being a Grothendieck space. Property (E) is also related to the $C(K)$-EP of [18] (or EP of [7]).

A Banach space X has property (E) if every bounded linear operator from X into c_0 extends to a bounded linear operator from X^{**} into c_0, where we consider X as canonically embedded in X^{**}. The space X has property (L) (resp. property (G)) if whenever a subset $K \subseteq X$ is limited (resp. Grothendieck) in X^{**}, then K is limited (resp. Grothendieck) in X (the converses are always true). For a closed subspace $F \subseteq X$, the pair (F, X) has the c_0-extension property (c_0-EP) if every operator from F into c_0 extends to an operator from X into c_0.

The main results can be summarized as follows.

(a) X has property (E) if and only if there exists a space Z and a surjective map $Q : Z \to X^{**}$ such that the pair $(Q^{-1}(X), Z)$ has the c_0-EP.

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(b) In particular, let $Q : \ell_1(\Gamma) \to X^{**}$ be a surjective map. If $Q^{-1}(X)$ is separable, then X has property (E).

(c) If X^{**}/X is separable, then X has the hereditary property (E).

(d) If X has property (E), then every operator from X into c_0 is unconditionally converging. (The converse is false.)

(e) X has the Phillips property and property (E) if and only if X is finite-dimensional.

(f) X has the weak Phillips property and property (E) if and only if X is a Grothendieck space. In particular, a C^*-algebra has property (E) if and only if it is a Grothendieck space. A subspace X of $K(\mathcal{H})$ has property (E) if and only if X is reflexive.

(g) If $F \subseteq X$ is a subspace such that both F and X/F have property (E), and in addition the pair $(F^{\perp\perp}, X^{**})$ has the c_0-EP, then X has property (E).

(h) If X has property (E), then X has both properties (L) and (G).

(i) If X has property (V*), then X has the hereditary property (G) (the converse is false).

(j) If X is a subspace of a space with an unconditional basis, then X has property (V*) if and only if X has the hereditary property (G).

(k) If $\ell_1 \not\subseteq X^*$, then X has the hereditary properties (G) and (L).

(l) $\mathcal{L}_1$-spaces have the hereditary properties (G) and (L).

1. Notation and background. In general, X, Y and Z denote Banach spaces over $\mathbb{C}$ or $\mathbb{R}$, assumed to be infinite-dimensional unless stated otherwise; we consider X as canonically embedded in X^{**}, but also write $\iota : X \to X^{**}$ for this canonical embedding when convenient; we set $p = \iota^*$. Viewing X^* as canonically embedded in X^{***}, we can consider p as a projection. In keeping with other authors, we denote the quotient space X^{**}/X by $H(X)$. We denote the closed unit ball of X by B_X. By "operator", or "map", or "mapping", we mean a bounded linear operator; by "subspace", a closed infinite-dimensional subspace, unless stated otherwise; by "space", a Banach space. When $F \subseteq X$ is a subspace of X, we identify the bidual F^{**} with the subspace $F^{\perp\perp} \subseteq X^{**}$. As usual, "rwc" means "relatively weakly compact". We denote the space of all operators from X into Y by $\mathcal{B}(X,Y)$, and the space of compact operators on a Hilbert space $\mathcal{H}$ by $K(\mathcal{H})$.

All measure spaces (Ω, Σ, μ) are assumed to be localizable, i.e., such that $L^1(\mu)^* = L^\infty(\mu)$. The term "$L^1$-space" will mean a space $L^1(\mu)$ for some measure μ, and similarly for "L^∞-space". In general, Γ will denote a nonempty set, possibly uncountable.

For a Banach space property $(\mathcal{P})$, we say X has the *hereditary* property $(\mathcal{P})$ if every subspace of X has property $(\mathcal{P})$.

Any other unexplained notation is as found in [2], [3], or [11].

We now review the definitions of the main properties used in the paper.

(a) *Limited subsets.* A bounded subset $K \subseteq X$ is a limited subset of X if for every weak*-null sequence (f_n) in X^*, $\langle x, f_n\rangle \to 0$ uniformly for $x \in K$. Equivalently, every operator $T : X \to c_0$ maps K into a relatively compact subset of c_0.

(b) *Grothendieck subsets.* A bounded subset $K \subseteq X$ is a Grothendieck subset of X if for every operator $T : X \to c_0$, the image $T(K)$ is rwc.

(c) *Grothendieck spaces.* A space X is a Grothendieck space if every weak*-convergent sequence in X^* is weakly convergent. Two obviously equivalent properties are that every operator from X into c_0 is weakly compact, and that B_X is a Grothendieck subset of X.

(d) (V)-*sets.* A bounded subset $K \subseteq X^*$ is said to be a (V)-set if for every wuC series $\sum x_n$ in X, we have

$$\lim_{n\to\infty} \langle x_n, f\rangle = 0 \qquad \text{uniformly for } f \in K.$$

Equivalently, for every operator $S : c_0 \to X$, the set $S^*(K)$ is a relatively (weakly) compact subset of ℓ_1.

(e) *Property* (V) *and property* (V1). A space X has property (V) if every (V)-set is rwc. Equivalently, for every Banach space Y, every unconditionally converging operator $T : X \to Y$ is weakly compact. A space X has property (V1) if every unconditionally converging operator $T : X \to c_0$ is weakly compact.

(f) (V*)-*sets.* A bounded subset $B \subseteq X$ is said to be a (V*)-set if for every wuC series $\sum f_n$ in X^*, we have

$$\lim_{n\to\infty} \langle f_n, x\rangle = 0 \qquad \text{uniformly for } x \in B.$$

Equivalently, every operator from X into ℓ_1 maps B into a relatively (weakly) compact set.

(g) *Property* (V*). A space X has property (V*) if every (V*)-set is rwc. Equivalently, a bounded subset B of X is rwc if and only if, for every operator T from X into ℓ_1, the image $T(B)$ is relatively (weakly) compact.

(h) *The separable complementation property.* A space X has the separable complementation property (SCP) if every separable subspace of X is contained in a separable complemented subspace of X. SCP is enjoyed by all L^1-spaces, all weakly compactly generated spaces and their subspaces, and more generally, all countably determined spaces, and all dual spaces with the RNP.

(i) *The Phillips properties.* A space X has the (weak) Phillips property if the canonical projection $p : X^{***} \to X^*$ is sequentially weak*-(weak) norm continuous. These properties are studied extensively in the papers [6] and [16].

2. Property (E) and the c_0-EP. We recall that a space X has the (weak) Phillips property if and only if for every operator $T : X^{**} \to c_0$, the mapping $T\iota : X \to c_0$, that is, the restriction of T to X, is (weakly) compact [6]. This fact is the original motivation for the following definition of property (E), an extension property which is a kind of "anti-Phillips" property.

2.1. Definition. (a) A space X has *property* (E) if every operator from X into c_0 extends to an operator from X^{**} into c_0.

(b) Let $F \subseteq X$ be a subspace of X. The pair (F, X) has the c_0-*extension property* (c_0-EP) if every operator from F into c_0 extends to an operator from X into c_0.

We first make some simple observations and related remarks about these newly defined properties.

(1) Clearly, X has property (E) if and only if the pair (X, X^{**}) has the c_0-EP.

(2) X has the (weak) Phillips property and property (E) if and only if X is finite-dimensional (a Grothendieck space).

(3) Any Grothendieck space, or any space complemented in its second dual, has property (E).

(4) If the pair (X, X^{**}) has the $C(K)$-EP [18], then X has property (E), although the converse is false: There is a $C(K)$ space which is a Grothendieck space but is not complemented in its second dual [15].

(5) The collection of spaces with property (E) is a Banach space ideal. In particular, property (E) passes to complemented subspaces.

(6) [14, Theorem 1.1] implies the following generalization of Sobczyk's Theorem: Let X be a space with subspace F. If X/F is separable, then (F, X) has the c_0-EP. We make extensive use of this in what follows.

(7) If X has the SCP and property (E), then every separable subspace of X has property (E).

(8) In comparison to the $C(K)$-EP [18], if there exists a sequentially weak*-weak* continuous "selection" function $s : X^* \to X^{***}$ such that for every $f \in X^*$ we have $p(sf) = f$, then X has property (E).

Now since operators from a space X into c_0 correspond to weak*-null sequences in X^*, it is easy to see that property (E) can also be considered a lifting property: X has property (E) if and only if every weak*-null sequence in X^* lifts to a weak*-null sequence in X^{***}, that is, for every weak*-null sequence (f_n) in X^*, there exists a weak*-null sequence (G_n) in X^{***} such that $pG_n = f_n$ for all n. We give a number of additional characterizations of property (E) as the first result.

2.2. THEOREM. *The following are equivalent*:

(a) *X has property (E).*

(b) *Weak***-null sequences in X^* lift to weak***-null sequences in X^{***}.*

(c) *For every $T : X \to c_0$, there exists an operator $U : H(X) \to \ell_\infty$ such that $T^{**} - U\pi : X^{**} \to c_0 \subseteq \ell_\infty$, where $\pi : X^{**} \to H(X)$ is the canonical quotient map.*

(d) *For every $T : X \to c_0$, there exists a space Y with property (E) such that T factors through Y.*

(e) *For every space Z and any $T : X \to Z$, there exists a space Y with property (E) such that T factors through Y.*

(f) *There exists a space Y with property (E) such that $X \subseteq Y$ and the pair (X, Y) has the c_0-EP.*

(g) *There exists a space Y with property (E) such that $X \subseteq Y$ and Y/X is separable.*

(h) *There exists a space Z and a surjective map $Q : Z \to X^{**}$ such that the pair $(Q^{-1}(X), Z)$ has the c_0-EP.*

(i) *There exists a space Y with property (E) and a surjective map $Q : Y^{**} \to X^{**}$ such that $Q^{-1}(X) = Y$.*

Proof. The equivalence of (a), (b), and (c) is easy to see, as are the implications (a)⇒(e)⇒(d) and (a)⇒(f)⇒(d). We show that (d)⇒(a). Let $T : X \to c_0$, and choose Y with property (E) such that T factors through Y. Let $R : X \to Y$ and $S : Y \to c_0$ be such that $T = SR$. Let $Q : Y^{**} \to c_0$ be an extension of S. The mapping $QR^{**} : X^{**} \to c_0$ is then easily seen to be an extension of T. The implication (a)⇒(g) is trivial, and the implication (g)⇒(f) holds by [14, Theorem 1.1], so (a)–(g) are all equivalent.

Since the implications (a)⇒(i)⇒(h) are obvious, to complete the proof, it suffices to show that (h)⇒(a). Here, we use an idea from the proof of [7, Prop. 1.1]: Let $T : X \to c_0$, and let $J : Q^{-1}(X) \hookrightarrow Z$ be the inclusion mapping. By hypothesis, the operator $TQJ : Q^{-1}(X) \to c_0$ extends to an operator $S : Z \to c_0$. Now, if $z \in \ker(Q)$, then $Sz = 0$, so we obtain a mapping $\widetilde{S} : X^{**} = Z/\ker(Q) \to c_0$. For every $x \in X$, let $z \in Z$ be such that $Qz = x$. Then $\widetilde{S}(z + \ker(Q)) = Sz = TQJ(z) = Tx$. Hence, $\widetilde{S}$ extends T, proving that (h)⇒(a). ∎

It follows immediately from part (f) that if X has property (E) and F is a subspace such that (F, X) has the c_0-EP, then F has property (E).

Before deriving some corollaries from this theorem, we mention that there is yet another way to think about property (E) and the c_0-EP using the language of exact sequences. For example, a space X has property (E) if and only if the contravariant functor $\mathcal{B}(\cdot, c_0)$ is exact when applied to the canonical short exact sequence

$$0 \to X \to X^{**} \to H(X) \to 0.$$

Now, for spaces X, Y, and Z, the space X is said to be an *extension* of Y by Z (or a *twisted sum* of Y with Z) if there exists a short exact sequence

$$0 \to Y \xrightarrow{i} X \xrightarrow{\pi} Z \to 0.$$

The twisted sum X *splits* if there exists a map $r : Z \to X$ such that πr is the identity on Z. In this case, it follows that $X = i(Y) \oplus r(Z)$, the latter space being clearly isomorphic to $Y \oplus Z$.

2.3. Corollary. *Let $Q : \ell_1(\Gamma) \to X^{**}$ be a surjective map. If either of the following equivalent conditions is satisfied, then both X and $Q^{-1}(X)$ have property (E):*

(1) *The pair $(Q^{-1}(X), \ell_1(\Gamma))$ has the c_0-EP.*
(2) *Every twisted sum of c_0 with $H(X)$ splits.*

Proof. The equivalence of (1) and (2) follows from [8, Proposition 3.1]. On the other hand, if (1) holds, then clearly so does condition (h) of the previous theorem, implying that X has property (E). ∎

Thus, it is not true that $(F, \ell_1(\Gamma))$ has the c_0-EP for every closed subspace $F \subseteq \ell_1(\Gamma)$; a knowledge of which pairs $(F, \ell_1(\Gamma))$ have the c_0-EP may help us to learn which spaces do have property (E). For example, with regard to the following corollary, $(F, \ell_1(\Gamma))$ has the c_0-EP whenever F or $\ell_1(\Gamma)/F$ is separable, by the fact that $\ell_1(\Gamma)$ has the SCP, and by [14, Theorem 1.1], respectively.

2.4. Corollary. *Let $Q : \ell_1(\Gamma) \to X^{**}$ be a surjective map.*

(a) *If $Q^{-1}(X)$ is separable, or $\ell_1(\Gamma)/Q^{-1}(X)$ is separable, then both X and $Q^{-1}(X)$ have property (E).*

(b) *If X has the Phillips property, then the pair $(Q^{-1}(X), \ell_1(\Gamma))$ fails to have the c_0-EP. In particular, $Q^{-1}(X)$ is nonseparable.*

We obtain the somewhat surprising fact that although there is a surjection of ℓ_1 onto c_0, if $Q : \ell_1(\Gamma) \to \ell_\infty$ is a surjective map, then $Q^{-1}(c_0) \neq \ell_1$, and in fact, $Q^{-1}(c_0)$ is not even separable, nor is it complemented in $\ell_1(\Gamma)$. (The same is true of course if we take any predual of ℓ_1 in place of c_0.) In fact, if Y is any complemented subspace of $\ell_1(\Gamma)$ such that $Q^{-1}(c_0) \subseteq Y$, then the quotient space $Y/Q^{-1}(c_0)$ is nonseparable. More generally, if A is any C^*-algebra, and if $Q : \ell_1(\Gamma) \to A^{**}$ is a surjective map, then neither $Q^{-1}(A)$ nor $\ell_1(\Gamma)/Q^{-1}(A)$ is separable, for by Corollary 3.10 below, a C^*-algebra A has property (E) if and only if it is a Grothendieck space, and of course $\ell_1(\Gamma)/Q^{-1}(A)$ is isomorphic to A^{**}/A.

Now we consider a class of spaces which have the hereditary property (E), the coseparable spaces. For the remainder of this section, F denotes a subspace of X.

2.5. Definition. A space X is *coseparable* if $H(X) = X^{**}/X$ is separable.

Observe that if X is coseparable, then both F and X/F are coseparable; to see this, we need only apply the exact functor $H(\cdot)$ to the canonical short exact sequence

$$0 \to F \to X \to X/F \to 0.$$

It follows immediately from this fact, together with the preceding results, that if X is coseparable, then every quotient space of X has the hereditary property (E). Here, we use a result of Valdivia [17] to give a simple alternative proof.

2.6. Corollary. *If X is coseparable, then X has the hereditary property (E).*

Proof. If X is coseparable, then $X = R \oplus S$ where R is reflexive and S is separable [17]. Since separability is a three-space property, S^{**} is separable, so by the separable injectivity of c_0, the space S, and hence X, has property (E). ∎

Observe that if X is coseparable, and also coreflexive, i.e., $H(X)$ is both separable and reflexive, then all dual spaces of X are coseparable, and so all quotient spaces of all dual spaces of X have the hereditary property (E).

Of course, quotient spaces of spaces with property (E) need not have property (E); every Banach space is a quotient of some $\ell_1(\Gamma)$. So, it is certainly of interest to find conditions under which a quotient space of a space with property (E) will also have property (E). We examine this next. Then we prove a result which says that property (E) is "almost" a three-space property. These and similar results can be proved using commutative diagrams, though we do not do this explicitly here.

2.7. Theorem. *Assume X has property (E). If the pair $(H(F), H(X))$ has the c_0-EP, then X/F has property (E).*

Proof. To simplify the notation a little, if Y is a space and $E \subseteq Y$ is a subspace, let $j_E : E \hookrightarrow Y$ be the inclusion map, and let $\pi_E : Y \to Y/E$ be the canonical quotient map. Let $T : X/F \to c_0$. Then $T\pi_F : X \to c_0$ extends to $S : X^{**} \to c_0$ such that $Sj_F^{**} : F^{**} \to c_0$ satisfies $Sj_F^{**}\restriction_F = 0$. Hence, we obtain $Q : H(F) \to c_0$ such that for every $\phi \in F^{**}$, $Q(\phi + F) = Sj_F^{**}(\phi)$. By hypothesis, Q extends to a map $U : H(X) \to c_0$. The mapping $S - U\pi_X : X^{**} \to c_0$ satisfies $(S - U\pi_X)(\phi) = 0$ for all $\phi \in F^{\perp\perp}$. Hence, there exists a map $W : X^{**}/F^{\perp\perp} \to c_0$ such that $W\pi_{F^{\perp\perp}} = S - U\pi_X : X^{**} \to c_0$. For any

$x \in X$, we have $W(\iota x + F^{\perp\perp}) = (S - U\pi_X)(\iota x) = S\iota x = T\pi_F(x) = T(x+F)$. Thus, W defines an extension of T, proving that X/F has property (E). ∎

It follows immediately that if X has property (E) and $H(F)$ is complemented in $H(X)$, then X/F has property (E). This occurs in particular if F is reflexive, since then $H(F)$ is trivial. We can also see again that every quotient space of a coseparable space has property (E), as remarked above.

2.8. Corollary. *If R is a reflexive space, and Y fails to have property (E), then every twisted sum of R with Y also fails to have property (E).*

For example, any twisted sum of ℓ_2 with c_0, which need not split by [8], fails to have property (E). On the other hand, if $F \subseteq \ell_1$ such that $\ell_1/F = c_0$, then F has property (E), and ℓ_1 is a twisted sum of F with c_0.

Now, we show that property (E) is at least almost a three-space property using a technique similar to the proof of [7, Lemma 3.1].

2.9. Theorem. *Assume both F and X/F have property (E). Suppose in addition that the pair $(F^{\perp\perp}, X^{**})$ has the c_0-EP. Then X has property (E).*

Proof. Let $T : X \to c_0$ be given; let $j : F \hookrightarrow X$ be the inclusion map; and let $S : F^{**} \to c_0$ be an extension of the map $Tj : F \to c_0$. By hypothesis, there exists a map $U : X^{**} \to c_0$ extending S. Set $W = U\iota - T : X \to c_0$. It is easy to check that $Wj = 0$, so we may define a map $\widetilde{W} : X/F \to c_0$ by $\widetilde{W}(x+F) = W(x)$. By hypothesis, there exists an extension V of $\widetilde{W}$, with $V : (X/F)^{**} \to c_0$. Let $\pi : X^{**} \to X^{**}/F^{\perp\perp}$ be the quotient map. The map $U - V\pi : X^{**} \to c_0$ is then an extension of T, since for every $x \in X$ we have

$$\begin{aligned} U\iota x - V\pi\iota x &= U\iota x - V(\iota x + F^{\perp\perp}) = U\iota x - \widetilde{W}(x+F) \\ &= U\iota x - W(x) = U\iota x - (U\iota x - Tx) = Tx. \end{aligned}$$

Hence, X has property (E), as desired. ∎

2.10. Corollary. *If F has property (E), and $(X/F)^{**}$ is separable, then X has property (E).*

Proof. The hypotheses clearly imply that X/F is coseparable, and $(F^{\perp\perp}, X^{**})$ has the c_0-EP by [14, Theorem 1.1]. ∎

In particular, if F has property (E), and if $X^{**} = F^{**} \oplus S$, where S is separable, then X has property (E). Of course, the space S is then a separable bidual. Also, if Y has property (E), and R is any separable reflexive space, then any twisted sum of Y with R has property (E). This is reminiscent of the fact that if Y is complemented in Y^{**}, and R is reflexive, then any twisted sum of Y with R is complemented in its bidual [2, Prop. 3.7.d].

Remark. The proof of [7, Lemma 3.1] implies that if both (F, X^{**}) and $(X/F, X^{**}/F)$ have the c_0-EP, then X has property (E).

3. Property (G), property (L), and the Phillips properties. We recall from [6] that a space X has the (weak) Phillips property if and only if B_X is a (Grothendieck) limited subset of X^{**}. This motivates the following definitions of what are, like property (E), in some sense, "anti-Phillips" properties.

3.1. Definition. X has *property* (G) (resp. *property* (L)) if for every bounded subset $K \subseteq X$, if K is a Grothendieck (resp. limited) subset of X^{**}, then K is a Grothendieck (resp. limited) subset of X.

First, some basic observations.

(1) Like property (E), properties (G) and (L) pass to complemented subspaces.

(2) X has the Phillips property and property (L) if and only if X is finite-dimensional; X has the weak Phillips property and property (G) if and only if X is a Grothendieck space.

We now show the simple fact that property (E) implies both properties (L) and (G).

3.2. Lemma. *If X has property (E), then X has both property (L) and property (G).*

Proof. Let $K \subseteq X$ be a bounded subset which is limited (resp. Grothendieck) in X^{**}. Let $T : X \to c_0$, and let $S : X^{**} \to c_0$ be an extension of T. Then $S(\iota K) = T(K)$ is relatively (resp. weakly) compact in c_0, and hence K is limited (resp. Grothendieck) in X. ∎

Next, we give some characterizations of properties (L) and (G) similar to Theorem 2.2.

3.3. Theorem. *The following are equivalent*:

(a) *X has property (L) (resp. property (G)).*

(b) *For every $T : X \to c_0$, there exists a space Y with property (L) (resp. property (G)) such that T factors through Y.*

(c) *For every space Z and any $T : X \to Z$, there exists a space Y with property (L) (resp. property (G)) such that T factors through Y.*

(d) *There exists a space Y with property (L) (resp. property (G)) such that $X \subseteq Y$ and Y/X is separable.*

(e) *There exists a space Y with property (L) (resp. property (G)) such that $X \subseteq Y$ and (X, Y) has the c_0-EP.*

Proof. We only prove the case of property (L), the proof for property (G) being the same, *mutatis mutandis*.

The implications (a)⇒(c)⇒(b) are obvious. We show that (b)⇒(a). Let $K \subseteq X$ be a bounded subset which is limited in X^{**}, and let $T : X \to c_0$.

Let Y be a space with property (L) such that T factors through Y, and let $R : X \to Y$ and $S : Y \to c_0$ be such that $T = SR$. Since K is limited in X^{**}, the subset $R^{**}(\iota K)$ is limited in Y^{**}, and hence $R(K)$ is limited in Y. It follows that $T(K) = S(R(K))$ is limited in c_0, hence $T(K)$ is relatively compact. Hence, K is limited in X, proving the equivalence of (a)–(c).

Now, the implication (a)⇒(d) is obvious, and (d)⇒(e) holds by [14, Theorem 1.1], so we need only show that (e)⇒(a). Suppose K is a bounded subset of X which is limited in X^{**}. Let $X \subseteq Y$ where Y has property (L), and (X, Y) has the c_0-EP. Since $X^{**} = X^{\perp\perp} \subseteq Y^{**}$, it follows that K is limited in Y^{**}, and hence, K is limited in Y. Since (X, Y) has the c_0-EP, K is limited in X, proving that X has property (L). ■

Thus, if X has property (G) or (L), and (F, X) has the c_0-EP, then F has property (G) or (L), respectively, as is the case for property (E).

We now consider some equivalent necessary conditions for X to have property (G) or (L), and hence for X to have property (E).

3.4. Theorem. *Any space X with property (G) or (L) satisfies the following equivalent conditions*:

(a) *X contains no complemented copy of c_0.*

(b) *Every operator from X into any separable space is unconditionally converging.*

(c) *Every operator from X into c_0 is unconditionally converging.*

(d) *Every relatively weak*-sequentially compact subset of X^* is a* (V)-*set.*

Proof. If X has property (G) or (L), then condition (a) is satisfied, so we need only show the equivalence of (a)–(d).

(a)⇒(b). Arguing by contradiction, suppose that there exists a separable space Y and an operator $T : X \to Y$ which is not unconditionally converging. By [3, p. 54], T fixes a copy of c_0, that is, there exists a subspace F of X, which is isomorphic to c_0, such that if $i : F \hookrightarrow X$ is the inclusion map, then $Ti : F \to Y$ is an isomorphism of F into Y. Since $Ti(F)$ is isomorphic to c_0 and Y is separable, there exists a projection $Q : Y \to Ti(F)$. Set $R = i(Ti)^{-1}QT$. It is easy to see that R is a projection from X into X, with $R(X) = F$, implying that X contains a complemented copy of c_0, a contradiction.

(b)⇒(c). Trivial.

(c)⇒(d). It is clearly enough to show that if (f_j) is any weak*-null sequence in X^*, then the set $\{f_j : j \geq 1\}$ is a (V)-set. So, let $\sum x_n$ be a wuC series in X, and let (f_j) be a weak*-null sequence in X^*. Let $T : X \to c_0$ be the mapping which sends $x \in X$ to the sequence $(\langle x, f_n\rangle)$. Since T is unconditionally converging by hypothesis, the series

$$\sum_{n=1}^{\infty} Tx_n = \sum_{n=1}^{\infty} (\langle x_n, f_j\rangle)_{j=1}^{\infty}$$

is unconditionally converging. In particular,

$$\|(\langle x_n, f_j\rangle)_{j=1}^{\infty}\| \to 0,$$

i.e., $\lim_n \sup_j |\langle x_n, f_j\rangle| = 0$. Thus, the set $\{f_j : j \geq 1\}$ is a (V)-set.

(d)⇒(a). Suppose $X = c_0 \oplus F$ for some subspace $F \subseteq X$, and let (e_n) denote the canonical basis of c_0. It is easy to see that $\sum e_n$ is wuC in X. Letting (f_n) denote the canonical basis of $c_0^* = \ell_1$, it is also easy to see that (f_n) is weak*-null in X^*, but is not a (V)-set, since $\langle e_n, f_n\rangle = 1$ for all n. This contradiction shows that (a) must hold, and completes the proof. ∎

3.5. Corollary. *Assume B_{X^*} is weak*-sequentially compact. If X has property (G) or (L), then $c_0 \not\subseteq X$. Hence, for all $1 < p < \infty$, $K(\ell_p)$ is not complemented in any dual space.*

Proof. If X has property (G) or (L), we may apply part (d) of the previous theorem to deduce that B_{X^*} is a (V)-set. Hence, if $\sum x_n$ is any wuC series in X, then

$$\lim_n \|x_n\| = \lim_n \sup_{f \in B_{X^*}} |\langle x_n, f\rangle| = 0.$$

It follows that $\sum x_n$ is unconditionally converging, and hence X contains no isomorphic copy of c_0 [3, p. 45].

Since $K(\ell_p)$ is separable and contains a copy of c_0, $K(\ell_p)$ has neither property (G), nor property (L), and so in particular is not complemented in any dual space. ∎

Since properties (E), (G), and (L) are designed to be "anti-Phillips" properties, it is not surprising that it is unusual or even impossible for a (infinite-dimensional) space to have both one of the Phillips properties and one of the properties (E), (G), or (L), as partially noted above.

3.6. Theorem. *For a space X, the following are equivalent*:

(a) *X has property (E) and the weak Phillips property.*
(b) *X has property (G) and the weak Phillips property.*
(c) *X has property (E) and property* (V1).
(d) *X has property (G) and property* (V1).
(e) *X has property (L) and property* (V1).
(f) *X is a Grothendieck space.*

Proof. The equivalence of (a), (b), and (f) is clear. Clearly, (f)⇒(c), and (c) implies both (d) and (e) by Lemma 3.2. On the other hand, both (d) and (e) separately imply (f), for if (d) or (e) holds, then by Theorem 3.4

above, every operator from X into c_0 is unconditionally converging. Since X has property (V1), every operator from X into c_0 is weakly compact, completing the proof. ■

3.7. THEOREM. *X has both the Phillips property and property (L) or property (G), if and only if X is finite-dimensional.*

Proof. Assume that X has the Phillips property. Then X^* has the Schur property, B_X is limited in X^{**}, and X has the weak Phillips property [6]. But if X has property (G), then X is a Grothendieck space by the previous theorem, while if X has property (L), then B_X is limited in X. In either case, X is finite-dimensional by the Josefson–Nissenzweig theorem [3, Chapter 12]. ■

REMARKS. (a) As easy corollaries, we obtain:

(1) [6, Theorem 2.13] A Banach space with the Phillips property is not complemented in any dual space.
(2) The well known result that no separable C^*-algebra is complemented in its second dual.
(3) A dual space has the weak Phillips property if and only if it is a Grothendieck space.

(b) It is unknown if every space with property (L) and the weak Phillips property is a Grothendieck space.

Since L^1-spaces have the SCP, every separable $\mathcal{L}_1$-space has property (E). On the other hand, if X is an $\mathcal{L}_\infty$-space, then X^{**} is complemented in an L^∞-space, and hence, X^{**} is a Grothendieck space. Hence, X has property (E) if and only if X is a Grothendieck space. It follows that every separable $\mathcal{L}_\infty$-space fails to have property (E).

3.8. COROLLARY. *Every separable $\mathcal{L}_1$-space has property (E). Every separable $\mathcal{L}_\infty$-space fails to have property (E).*

The converse of Theorem 3.4 holds for C^*-algebras and for subspaces of $K(\mathcal{H})$ as seen below. However, the following example shows that the converse of the theorem (and of Corollary 3.5) is false in general.

3.9. EXAMPLE (A separable space which is somewhat reflexive, and in particular, contains no copy of c_0, yet has neither property (L) nor property (G)). If Y is the space constructed in [1], then $Y^* = \ell_1$, and so Y has the Phillips property, although Y contains no copy of c_0. By Theorem 3.7 above, Y has neither property (L) nor property (G).

3.10. COROLLARY. *Let X be a C^*-algebra or a subspace of $K(\mathcal{H})$. Then the following are equivalent*:

(a) *X has property* (E).
(b) *X has property* (G).
(c) *X has property* (L).
(d) *X is a Grothendieck space.*
(e) *X contains no complemented copy of* c_0.

If X is a subspace of $K(\mathcal{H})$, then (a)–(e) *are equivalent to*

(f) *X is reflexive.*

Hence, no C^*-subalgebra of $K(\mathcal{H})$ has property (E).

Proof. In both cases, X has property (V) by [13] and [6]. Hence, X has the weak Phillips property, so by Theorem 3.6, (a), (b), and (d) are equivalent. Now, (a)⇒(c)⇒(e) by Lemma 3.2 and Theorem 3.4, and since X has property (V), condition (e) implies that X is a Grothendieck space. This proves the equivalence of (a)–(e). If X is a subspace of $K(\mathcal{H})$, then since $\ell_1 \not\subseteq X$, if X is a Grothendieck space, then X must be reflexive [4]. ∎

By [13], the predual of any von Neumann algebra has property (V*) since its dual space has property (V). The following result then implies that the predual of any von Neumann algebra has the hereditary property (G). In particular, any abstract L-space has the hereditary property (G). The result will also allow us to obtain a partial converse to Theorem 3.4 for subspaces of spaces with unconditional bases. We remind the reader that property (V*) is hereditary, and remark that spaces with the hereditary property (V) have the hereditary weak Phillips property [6].

3.11. THEOREM. *If X has property* (V*), *then X has the hereditary property* (G).

Proof. Since property (V*) is hereditary, it is enough to show that if X has property (V*), then X has property (G). Let $K \subseteq X$ be a bounded subset of X which is a Grothendieck subset of X^{**}. Let $\tau : X \to \ell_1$, and let $q : \ell_\infty^* \to \ell_1$ be the canonical projection. Then $q\tau^{**} : X^{**} \to \ell_1$, and since K is a Grothendieck subset of X^{**}, the image $q\tau^{**}(\iota K) = \tau(K)$ is a Grothendieck subset of ℓ_1, and hence is relatively compact. Hence, K is a (V*)-set, so K is rwc, and is therefore a Grothendieck subset of X. Hence, X has property (G). ∎

3.12. COROLLARY. *Assume X has property* (V*) *and the Schur property. Then X has the hereditary properties* (G) *and* (L).

Proof. Let $K \subseteq X$ be limited in X^{**}. Then K is certainly a Grothendieck subset of X^{**}, and so by the proof of the theorem, K is a (V*)-set, and hence is rwc. But since X has the Schur property, K is relatively compact, and thus is limited in X. Since both the Schur property and property (V*) are hereditary, we are done. ∎

REMARKS. (a) The converse of Theorem 3.11 is false: The predual B of the James Tree space has the hereditary properties (G) and (L) by Theorem 3.14 below. Nevertheless, B is not weakly sequentially complete, and so fails to have property (V*).

(b) It is unknown whether property (V*) implies property (E) or property (L).

(c) There is a separable $\mathcal{L}_\infty$-space with the Schur property [1], which by Corollary 3.8 above has neither property (G) nor property (L).

(d) It can be shown that for spaces with the SCP, properties (G) and (L) are determined separably.

Now, if X is a subspace of a space with an unconditional basis, and if $c_0 \not\subseteq X$, then X has property (V*) by [11, Theorem 1.c.13] and [12]. If X itself has an unconditional basis, then X is isomorphic to a dual space [3, p. 53]. Combining these facts with Lemma 3.2 and Theorems 3.11 and 3.4, we obtain the following corollary.

3.13. COROLLARY. *Assume X is a subspace of a space with an unconditional basis. The following are equivalent*:

(a) *X has property* (V*).
(b) *X has property* (G).
(c) $c_0 \not\subseteq X$.

If X itself has an unconditional basis, *then* (a)–(c) *are equivalent to*

(d) *X has property* (E).

Recall that a space X has the (*weak*) *Gelfand–Phillips property* if every (Grothendieck) limited subset of X is relatively (weakly) compact. We abbreviate these as GPP and WGP below. See [9] for details on the WGP. Since the GPP and the WGP are both hereditary [5], [9], it follows that if X^{**} has both the GPP and the WGP, then X has the hereditary properties (G) and (L). We use this fact to prove the final result.

3.14. THEOREM. *Suppose that*

(1) *X is an $\mathcal{L}_1$-space*, *or*
(2) $\ell_1 \not\subseteq X^*$, *or*
(3) *B_{X^*} is weak***-sequentially compact and there exists no surjection of X onto c_0.*

Then X has the hereditary properties (G) and (L). In particular, *coreflexive spaces have the hereditary properties (G) and (L).*

Proof. If X is an $\mathcal{L}_1$-space, then X^{**} is complemented in an L^1-space, so X^{**} has both the GPP and the WGP, since L^1-spaces have both these properties.

If condition (2) or (3) holds, then $B_{X^{***}}$ has a weak*-dense, weak*-CSC subset, namely B_{X^*}, from which it again follows that X^{**} has both the GPP and the WGP by [5] and [9].

If X is coreflexive, then X^* is coreflexive, and hence every subspace of X^* is coreflexive. Hence condition (b) is satisfied. ■

Thus, in particular, the predual of the James Tree space has the hereditary properties (G) and (L).

We end the paper with what seem to be the most interesting open problems.

Open problems.

(1) Are properties (E), (G), and (L) three-space properties?

(2) If X has both property (L) and property (G), then X has property (E).

(3) Coreflexive spaces have property (E).

(4) If every separable subspace of X has property (E), then X has property (E).

(5) If X is any subspace of a space with an unconditional basis, then (a)–(d) of Corollary 3.14 are all equivalent.

(6) If X has property (V*), then X has property (E).

(7) If X^* is weakly sequentially complete, then X^* has the hereditary property (G).

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