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RANK α OPERATORS ON THE SPACE $C(T,X)$

BY

DUMITRU POPA (Constanţa)

Abstract. For $0 \le \alpha < 1$, an operator $U \in L(X,Y)$ is called a rank α operator if $x_n \xrightarrow{\tau_\alpha} x$ implies $Ux_n \to Ux$ in norm. We give some results on rank α operators, including an interpolation result and a characterization of rank α operators $U : C(T,X) \to Y$ in terms of their representing measures.

Let X be a Banach space and $0 \le \alpha < 1$; a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called τ_α-*convergent to* 0, written $x_n \xrightarrow{\tau_\alpha} 0$, if there exists a constant $c \ge 0$ such that $\|\sum_{n\in B} x_n\| \le c|B|^\alpha$ for all finite subsets $B \subset \mathbb{N}$, or equivalently, $\|\sum_{n\in B} \lambda_n x_n\| \le c|B|^\alpha$ for all finite subsets $B \subset \mathbb{N}$ and $\lambda_n \in K = \mathbb{R}$ or $\mathbb{C}$ with $|\lambda_n| \le 1$ (the constant c may vary). Here $|B|$ is the cardinality of B. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called τ_α-*convergent to* x, written $x_n \xrightarrow{\tau_\alpha} x$, if $x_n - x \xrightarrow{\tau_\alpha} 0$.

For $0 \le \alpha < 1$, an operator $U \in L(X,Y)$ is called a *rank α operator* if $x_n \xrightarrow{\tau_\alpha} x$ implies $Ux_n \to Ux$ in norm. We denote by $R_\alpha(X,Y)$ the Banach space of all rank α operators from X to Y. A Banach space has *rank α* if each τ_α-convergent sequence is norm convergent. The notions of τ_α-convergence and rank α spaces have been first introduced by A. Pełczyński [8]. Observe that rank 0 operators coincide with unconditionally converging operators. In the following proposition we give some results concerning rank α operators.

PROPOSITION 1. (a) R_α *is an operator ideal in the sense of A. Pietsch* [9], *for each* $0 \le \alpha < 1$.

(b) *If* $0 \le \alpha \le \beta < 1$, *then* $R_\beta(X,Y) \subset R_\alpha(X,Y)$.

(c) $\mathrm{DP}(X,Y) \subset R_\alpha(X,Y)$ *for each* $0 \le \alpha < 1$, *where* DP *denotes the ideal of Dunford–Pettis operators.*

(d) R_α *is a closed ideal of operators for each* $0 \le \alpha < 1$.

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Proof. (a) is clear; (b) follows from the fact that if $\alpha \leq \beta$, then τ_α-convergence implies τ_β-convergence; (c) follows from the fact that τ_α-convergence implies weak convergence.

(d) If $x_n \overset{\tau_\alpha}{\to} 0$ then there is a constant $c > 0$ such that $\sup_{n\in\mathbb{N}} \|x_n\| \leq c$. If $U_k \in R_\alpha(X,Y)$ for each $k \in \mathbb{N}$ and $U_k \to U$ in norm then for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\|U_k - U\| < \varepsilon/(2c)$. Since $U_k \in R_\alpha(X,Y)$, we have $\|U_k(x_n)\| \to 0$ as $n \to \infty$, so there exists $n_\varepsilon \in \mathbb{N}$ such that $\|U_k(x_n)\| < \varepsilon/2$ for each $n \geq n_\varepsilon$; hence $\|U(x_n)\| < \varepsilon$ for each $n \geq n_\varepsilon$, i.e. $U \in R_\alpha(X,Y)$.

Now we indicate in what conditions a diagonal operator has rank α, which shows in particular that the inclusions (b) and (c) from Proposition 1 are strict.

EXAMPLE 2. *Let $1 < p < \infty$, $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in l_\infty$ and $D_\lambda : l_p \to l_p$ be the diagonal operator associated to λ, i.e. $D_\lambda(x_n) = (\lambda_n x_n)$. Then*:

(a) *For $1/p \leq \alpha < 1$, $D_\lambda \in R_\alpha(l_p,l_p)$ if and only if $\lambda \in c_0$.*

(b) *For $0 \leq \alpha < 1/p$, $D_\lambda \in R_\alpha(l_p,l_p)$ if and only if $\lambda \in l_\infty$.*

Proof. (a) If $D_\lambda \in R_\alpha(l_p,l_p)$, then since $1/p \leq \alpha$, Proposition 1(b) implies that $D_\lambda \in R_{1/p}(l_p,l_p)$. Since $e_n \overset{\tau_{1/p}}{\to} 0$ (e_n is the canonical basis of l_p), we find that $\|D_\lambda(e_n)\| \to 0$, i.e. $\lambda_n \to 0$ and so $\lambda \in c_0$. Conversely, if $\lambda \in c_0$, then the operator D_λ is compact, so D_λ has rank α.

(b) For $0 \leq \alpha < 1/p$, the space l_p has rank α (see [1], Proposition 2.3(2), or [8]), so by the ideal property of rank α operators, $D_\lambda \in R_\alpha(l_p,l_p)$ for each $\lambda \in l_\infty$.

It is also easy to prove the following:

PROPOSITION 3. *For each compact Hausdorff space T,*

$$W(C(T),X) = \mathrm{DP}(C(T),X) = R_0(C(T),X) = R_\alpha(C(T),X)$$

for each $0 \leq \alpha < 1$.

Proof. The first two equalities are well known ([5], Theorem 15, pp. 159–160), and Proposition 1(c)&(b) assures that $\mathrm{DP}(C(T),X) \subset R_\alpha(C(T),X) \subset R_0(C(T),X)$.

Now we prove that a certain composition operator is a rank α operator.

PROPOSITION 4. *Let $A \in R_\alpha^{\mathrm{dual}}(X,Y)$, $B \in \mathrm{DP}(Z,T)$ and define $h : L(Y,Z) \to L(X,T)$ by $h(U) = BUA$. Then h is a rank α operator.*

Proof. Let $U_n \overset{\tau_\alpha}{\to} 0$. For $n \in \mathbb{N}$, let $x_n \in X$ with $\|x_n\| \leq 1$ be such that

$$\|h(U_n)\| - 1/n < \|h(U_n)(x_n)\| = \|(BU_nA)(x_n)\|. \tag{1}$$

If $z^* \in Z^*$, since $U_n \overset{\tau_\alpha}{\to} 0$, we obtain $z^* \circ U_n \overset{\tau_\alpha}{\to} 0$. Now $A \in R_\alpha^{\mathrm{dual}}(X,Y)$ so $A^*(z^* \circ U_n) \to 0$ in norm, or $z^* \circ U_n \circ A \to 0$ in norm of X^*. Hence

$(z^* \circ U_n \circ A)(x_n) \to 0$, i.e. $\langle (U_n \circ A)(x_n), z^* \rangle \to 0$ and since $z^* \in Z^*$ is arbitrary, $(U_n \circ A)(x_n) \to 0$ weakly. As $B \in \mathrm{DP}(Z,T)$ we have $B((U_n \circ A)(x_n)) \to 0$ in norm of T, i.e. $(B \circ U_n \circ A)(x_n) \to 0$ in norm of T and the relation (1) implies $\|h(U_n)\| \to 0$, i.e. h is a rank α operator.

Corollary 5. *Let $0 \le \alpha < 1$, $U \in R_\alpha^{\mathrm{dual}}(X, X_1)$ and $V \in \mathrm{DP}^{\mathrm{dual}}(Y, Y_1)$. Then the projective tensor product $U \widetilde{\otimes}_\pi V$ is in $R_\alpha^{\mathrm{dual}}(X \widetilde{\otimes}_\pi Y, X_1 \widetilde{\otimes}_\pi Y_1)$.*

Proof. Since $h = (U \widetilde{\otimes}_\pi V)^* : L(X_1, Y_1^*) \to L(X, Y^*)$ acts as $h(\psi) = V^* \psi U$, it suffices to apply Proposition 4.

A natural question is: is the ideal of all dual rank α operators projective tensor stable? The answer is no. For $2 \le p < \infty$, take the identity operator $i : l_q \to l_q$ $(1/p + 1/q = 1)$, the dual of which has rank α for each $0 \le \alpha < 1/p$ (see [1], Proposition 2.3, or [8]). But the dual of $i \widetilde{\otimes}_\pi i : l_q \widetilde{\otimes}_\pi l_q \to l_q \widetilde{\otimes}_\pi l_q$ is the identity operator on $L(l_q, l_p)$, which, because $2 \le p < \infty$, $q \le p$, contains a copy of c_0 and hence has no rank.

For $0 \le \alpha < 1$, X a Banach space, let $E_\alpha(X) = \{(x_n)_{n\in\mathbb{N}} \subset X \mid x_n \xrightarrow{\tau_\alpha} 0\}$, which is evidently a Banach space for the norm

$$\|\xi\| = \sup\left\{ \frac{\|\sum_{n\in B} x_n\|}{|B|^\alpha} \,\middle|\, B \text{ finite} \subset \mathbb{N},\ B \ne \emptyset \right\},$$

where $\xi = (x_n)_{n\in\mathbb{N}} \in E_\alpha(X)$. Observe that $U \in L(X,Y)$ is a rank α operator if and only if for each sequence $(x_n)_{n\in\mathbb{N}} \in E_\alpha(X)$, the sequence $(Ux_n)_{n\in\mathbb{N}}$ is in $c_0(Y)$. In addition the operator $h : E_\alpha(X) \to c_0(Y)$ given by $h((x_n)_{n\in\mathbb{N}}) = (Ux_n)_{n\in\mathbb{N}}$ is linear and continuous.

Now we prove an interpolation result for rank α operators. We recall that given a Banach interpolation couple $\mathbf{Y} = (Y_0, Y_1)$ and $0 < \theta < 1$, $[Y_0, Y_1]_\theta$ is the interpolation space obtained by the complex method of Calderón (see [12], 1.9.3, for details).

Proposition 6. *Let $0 < \theta < 1$, X a Banach space, $\mathbf{Y} = (Y_0, Y_1)$ a Banach interpolation couple and $0 \le \alpha < 1$. Then*

$$[R_\alpha(X, Y_0), L(X, Y_1)]_\theta \subset R_\alpha(X, [Y_0, Y_1]_\theta).$$

Proof. For $\xi = (x_n)_{n\in\mathbb{N}} \in E_\alpha(X)$ we define the operator

$$h_\xi : R_\alpha(X, Y_0) + L(X, Y_1) \to l_\infty(Y_0 + Y_1), \qquad h_\xi(U) = (Ux_n)_{n\in\mathbb{N}}.$$

Then using the definition of rank α operators (see the above discussion), we obtain two continuous linear operators: $h_\xi : R_\alpha(X, Y_0) \to c_0(Y_0)$ and $h_\xi : L(X, Y_1) \to l_\infty(Y_1)$, with

$$\|h_\xi : R_\alpha(X, Y_0) \to c_0(Y_0)\| \le \|\xi\|, \qquad \|h_\xi : L(X, Y_1) \to l_\infty(Y_1)\| \le \|\xi\|,$$

hence by interpolation,

$$h_\xi : [R_\alpha(X, Y_0), L(X, Y_1)]_\theta \to [c_0(Y_0), l_\infty(Y_1)]_\theta$$

is also a continuous linear operator and

$$\|h_\xi : [R_\alpha(X,Y_0), L(X,Y_1)]_\theta \to [c_0(Y_0), l_\infty(Y_1)]_\theta\| \\ \le \|h_\xi : R_\alpha(X,Y_0) \to c_0(Y_0)\|^{1-\theta} \|h_\xi : L(X,Y_1) \to l_\infty(Y_1)\|^\theta \le \|\xi\|.$$

But $[c_0(Y_0),\, l_\infty(Y_1)]_\theta = c_0([Y_0,Y_1]_\theta)$ (see [12], 1.18, Observation 3), thus

$$h_\xi : [R_\alpha(X,Y_0), L(X,Y_1)]_\theta \to c_0([Y_0,Y_1]_\theta)$$

is also a continuous linear operator, i.e. for each $U \in [R_\alpha(X,Y_0), L(X,Y_1)]_\theta$ and each $\xi = (x_n)_{n\in\mathbb{N}} \in E_\alpha(X)$, $h_\xi(U) = (Ux_n)_{n\in\mathbb{N}} \in c_0([Y_0,\, Y_1]_\theta)$ and

$$\|h_\xi(U)\| = \|(Ux_n)\|_{c_0([Y_0,Y_1]_\theta)} \le \|\xi\| \cdot \|U\|_{[R_\alpha(X,Y_0),L(X,Y_1)]_\theta}.$$

Thus $U \in R_\alpha(X,[Y_0,Y_1]_\theta)$ and $\|U\|_{R_\alpha(X,[Y_0,Y_1]_\theta)} \le \|U\|_{[R_\alpha(X,Y_0),L(X,Y_1)]_\theta}$.

For Banach spaces X and Y we denote by $X \mathbin{\widetilde{\otimes}_\varepsilon} Y$ the injective tensor product of X and Y, i.e. the completion of the algebraic tensor product $X \otimes Y$ with respect to the injective cross-norm $\varepsilon(u) = \sup\{|\langle x^* \otimes y^*, u\rangle| \mid \|x^*\| \le 1, \|y^*\| \le 1\}$ for $u \in X \otimes Y$ (see [5], Chapter VIII). If $U \in L(Z \mathbin{\widetilde{\otimes}_\varepsilon} X, Y)$, for each $z \in Z$ we consider the operator $U^\# z : X \to Y$ given by $(U^\# z)(x) = U(z \otimes x)$ for $x \in X$; evidently, $U^\# : Z \to L(X,Y)$ is linear and continuous.

Proposition 7. *If $U \in R_\alpha(Z \mathbin{\widetilde{\otimes}_\varepsilon} X, Y)$, then $U^\# \in R_\alpha(Z, R_\alpha(X,Y))$.*

Proof. For $z \in Z$, define $V_z : X \to Z \mathbin{\widetilde{\otimes}_\varepsilon} X$ by $V_z(x) = z \otimes x$. Then by the hypothesis and the ideal property of the rank α operators it follows that $U^\# z = UV_z$ is a rank α operator. Let $z_n \xrightarrow{\tau_\alpha} 0$. For $n \in \mathbb{N}$, let $\|x_n\| \le 1$ be such that

$$\|U^\# z_n\| - 1/n < \|(U^\# z_n)(x_n)\| = \|U(z_n \otimes x_n)\|.$$

For every finite subset $B \subset \mathbb{N}$ we have

$$\varepsilon\Big(\sum_{n\in B} z_n \otimes x_n\Big) = \sup_{\|x^*\|\le 1} \Big\|\sum_{n\in B} z_n x^*(x_n)\Big\| \le c|B|^\alpha,$$

since $|x^*(x_n)| \le 1$, hence $z_n \otimes x_n \xrightarrow{\tau_\alpha} 0$. As U is a rank α operator, we have $\|U(z_n \otimes x_n)\| \to 0$, so $\|U^\# z_n\| \to 0$ and hence $U^\# \in R_\alpha(Z, R_\alpha(X,Y))$.

If T is a compact Hausdorff space and X is a Banach space we denote by $C(T,X)$ the Banach space of all continuous X-valued functions defined on T, equipped with the supremum norm. Also if T is a compact space, we denote by Σ the σ-field of Borel subsets of T, and if X is Banach space, $B(\Sigma,X)$ is the Banach space of totally measurable X-valued functions equipped with the supremum norm. It is well known that every continuous linear operator $U : C(T,X) \to Y$ has a representing measure $G : \Sigma \to L(X,Y^{**})$ such that $U(f) = \int_T f\,dG$ for $f \in C(T,X)$ and there is a canonical extension $\widehat{U} : B(\Sigma,X) \to Y^{**}$ of U to the space $B(\Sigma,X)$ given by $\widehat{U}(f) = \int_T f dG$ for $f \in B(\Sigma,X)$ (see [3], Representation Theorem 2.2, or [6], Theorem 9, p. 398).

Also we denote by $\|G\|(E) = \sup\{|G_{y^*}(E)| \mid \|y^*\| \le 1\}$ the semivariation of the representing measure G, for $E \in \Sigma$, where $G_{y^*}(E) = \langle y^*, G(E)x\rangle$; we say that the semivariation $\|G\|$ is *continuous at* $\emptyset$ if $\|G\|(E_k) \to 0$ for $E_k \searrow \emptyset$, $E_k \in \Sigma$. As is well known, $\|G\|$ is continuous at $\emptyset$ if and only if there exists a Borel measure $\alpha \ge 0$ on Σ such that $\lim_{\alpha(E)\to 0} \|G(E)\| = 0$.

Since $C(T, X) = C(T) \widetilde{\otimes}_\varepsilon X$, from Proposition 7 we have:

COROLLARY 8. *If $U \in R_\alpha(C(T, X), Y)$, then $G(E) \in R_\alpha(X, Y)$ for each $E \in \Sigma$ and the semivariation $\|G\|$ is continuous at $\emptyset$.*

Proof. Using Propositions 7 and 3 we infer that

$$U^\# \in R_\alpha(C(T), R_\alpha(X, Y)) = W(C(T), R_\alpha(X, Y)),$$

hence the representing measure F of $U^\#$ is countably additive ([5], Theorem 5 (Bartle–Dunford–Schwartz), p. 153), so F has the semivariation continuous at $\emptyset$. But the representing measure F of $U^\#$ under our hypothesis coincides with that of U ([3], Theorem 4.4), hence G takes its values in $R_\alpha(X, Y)$ and the semivariation $\|G\|$ is continuous at $\emptyset$.

With the help of Corollary 8 the proof of the following proposition is analogous to that of Theorem 3 from [2], so we omit it.

PROPOSITION 9. *Let X, Y be Banach spaces, T a compact Hausdorff space, $U : C(T, X) \to Y$ a continuous linear operator, and $\widehat{U} : B(\Sigma, X) \to Y^{**}$ the canonical extension of U. Then $U \in R_\alpha(C(T, X), Y)$ if and only if $\widehat{U}$ takes its values in Y and $\widehat{U} \in R_\alpha(B(\Sigma, X), Y)$.*

Now for a given closed operator ideal $\mathcal{A}$ we indicate a way to construct a continuous linear operator on $C(T, X)$ with representing measure having natural properties. Compare this result with that of [10], Proposition 1.

PROPOSITION 10. *Let $\mathcal{A}$ be a closed operator ideal, and $(U_n)_{n\in\mathbb{N}} \subset \mathcal{A}(X, Y)$ a sequence such that $\sum_{n=1}^\infty \|y^* U_n\| < \infty$ for each $y^* \in Y^*$. If T is a compact space on which there exists a purely non-atomic regular probability Borel measure λ, and $(r_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in $L_2(\lambda)$ with $\sup_{n\in\mathbb{N}} \sup_{t\in T} |r_n(t)| < \infty$, then the operator $U : C(T, X) \to Y$ given by $U(f) = \sum_{n=1}^\infty U_n(\int_T f r_n \, d\lambda)$ is linear and continuous and its representing measure G has the properties: $G(E) \in \mathcal{A}(X, Y)$ for each Borel subset E and $\|G\|$ is continuous at $\emptyset$.*

Proof. First observe that the hypothesis and the closed graph theorem imply that

$$\sup_{\|y^*\|\le 1} \sum_{n=1}^\infty \|y^* U_n\| = M < \infty.$$

For $f \otimes x \in C(T) \otimes X$, we have

$$U(f \otimes x) = \sum_{n=1}^{\infty} U_n(x) \int_T f r_n \, d\lambda.$$

Using the orthonormality of the sequence $(r_n)_{n\in\mathbb{N}}$ it follows that for each $E \in \Sigma$, $\int_E r_n \, d\lambda \to 0$. Since

$$\sum_{n=1}^{\infty} |y^*(U_n x)| \leq M\|y^*\| \cdot \|x\| < \infty$$

for each $y^* \in Y^*$, $x \in X$, the series $\sum_{n=1}^{\infty} U_n(x) \int_T f r_n \, d\lambda$ is norm convergent (see [4], Theorem 6, p. 44). Also for $f \in C(T, X)$ and $n \in \mathbb{N}$ we have

$$\Big\| \sum_{k=1}^{n} U_k \Big(\int_T f r_k \, d\lambda \Big) \Big\| \leq L\|f\| \sup_{\|y^*\|\leq 1} \sum_{k=1}^{n} \|y^* U_k\| \leq LM\|f\|,$$

where $L = \sup_{n\in\mathbb{N}} \sup_{t\in T} |r_n(t)|$. Now the Banach–Steinhaus theorem assures that the series $\sum_{n=1}^{\infty} U_n(\int_T f r_n \, d\lambda)$ is norm convergent for each $f \in C(T, X)$ and the operator U is linear and continuous.

If G is the representing measure of U then

$$G(E)(x) = \sum_{n=1}^{\infty} \Big(\int_E r_n \, d\lambda \Big) U_n(x),$$

i.e. $G(E) = \sum_{n=1}^{\infty} \alpha_n(E) U_n$, where $\alpha_n(E) = \int_E r_n \, d\lambda$. Also for $E \in \Sigma$ and $x \in X$ with $\|x\| \leq 1$ we have

$$\begin{aligned} \Big\| G(E)x - \sum_{k=1}^{n} \alpha_k(E) U_n(x) \Big\| &\leq (\sup_{k\geq n} |\alpha_k(E)|) \sup_{\|y^*\|\leq 1} \sum_{n=1}^{\infty} \|y^* U_n\| \\ &= M \sup_{k\geq n} |\alpha_k(E)|, \end{aligned}$$

i.e. $\|G(E) - \sum_{k=1}^{n} \alpha_k(E) U_k\| \leq M \sup_{k\geq n} |\alpha_k(E)| \to 0$. Since the ideal $\mathcal{A}$ is closed it follows that $G(E) \in \mathcal{A}(X, Y)$. Also, the well known Nikodym convergence theorem implies that $G : \Sigma \to L(X, Y)$ is countably additive and so $\|G\|$ is continuous at $\emptyset$.

Remark 11. Let $(x_n)_n \subset X^*$ be a bounded sequence, and let $(y_n)_{n\in\mathbb{N}} \subset Y$ with $\sum_{n=1}^{\infty} |y^*(y_n)| < \infty$ for each $y^* \in Y^*$. Then taking $U_n = x_n^* \otimes y_n$, we have

$$\sum_{n=1}^{\infty} \|y^* U_n\| \leq (\sup_{n\in\mathbb{N}} \|x_n^*\|) \sum_{n=1}^{\infty} |y^*(y_n)| < \infty$$

for each $y^* \in Y^*$, so we can apply Proposition 10.

PROPOSITION 12. *The following assertions about a Banach space X are equivalent*:

(i) *X has rank α.*

(ii) *For any compact Hausdorff space T and any Banach space Y, a continuous linear operator $U : C(T,X) \to Y$ has rank α if and only if its representing measure G has the properties*: *$G(E) \in R_\alpha(X,Y)$ for each $E \in \Sigma$ and $\|G\|$ is continuous at $\emptyset$.*

Proof. (i)$\Rightarrow$(ii). Using Corollary 8 we have to prove that if X has rank α and $U : C(T,X) \to Y$ is linear and continuous with $G(E) \in R_\alpha(X,Y)$ for each $E \in \Sigma$ and with $\|G\|$ continuous at $\emptyset$, then U is a rank α operator. Let $(f_n)_{n\in\mathbb{N}} \subset C(T,X)$ with $f_n \stackrel{\tau_\alpha}{\to} 0$. Then for each $t \in T$ and a finite subset $B \subset \mathbb{N}$ we have

$$\Big\| \sum_{n\in B} f_n(t) \Big\| \le \Big\| \sum_{n\in B} f_n \Big\| \le c|B|^\alpha,$$

$f_n(t) \stackrel{\tau_\alpha}{\to} 0$ and since X has rank α, $f_n(t) \to 0$ in norm for each $t \in T$.

Now the proof is similar to that of Theorem 2.1 of [11], and uses the fact that if the semivariation $\|G\|$ is continuous at $\emptyset$, then G has a positive control measure; we omit the details.

(ii)$\Rightarrow$(i). Let $x_n \stackrel{\tau_\alpha}{\to} 0$. Then there exist $x_n^* \in X^*$ with $\|x_n^*\| \le 1$ and $x_n^*(x_n) = \|x_n\|$. Let T be a non-dispersed compact Hausdorff space. Then there is a purely non-atomic regular probability Borel measure λ on T (see [7], Theorem 2.8.10). Now we can construct a Haar system $\{A_i^n \mid 1 \le i \le 2^n, n \ge 0\}$ in Σ (that is, $A_1^0 = T$; for each n, $\{A_i^n \mid 1 \le i \le 2^n\}$ is a partition of T; $A_i^n = A_{2i}^{n+1} \cup A_{2i+1}^{n+1}$ and $\lambda(A_i^n) = 1/2^n$ for $1 \le i \le 2^n$ and $n \ge 0$). Let $r_n = \sum_{i=1}^{2^n} (-1)^i \chi_{A_i^n}$. Clearly $(r_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in $L_2(\lambda)$. Now by Remark 11 we can construct a $U : C(T,X) \to c_0$ associated to $(x_n^*)_{n\in\mathbb{N}} \subset X^*$ and $(e_n)_{n\in\mathbb{N}} \subset c_0$, i.e.

$$U(f) = \Big(\int_T x_n^* f r_n \, d\lambda \Big)_{n\in\mathbb{N}}, \qquad f \in C(T,X).$$

By (ii), U is a rank α operator, hence by Proposition 9, the canonical extension $\widehat{U} : B(\Sigma,X) \to c_0$ of U is also a rank α operator. But $\widehat{U}(f) = (\int_T x_n^* f r_n \, d\lambda)_{n\in\mathbb{N}}$ for $f \in B(\Sigma,X)$ and obviously $r_n \otimes x_n \stackrel{\tau_\alpha}{\to} 0$, hence $\|\widehat{U}(r_n \otimes x_n)\| \to 0$. Now by the orthonormality of the sequence $(r_n)_{n\in\mathbb{N}}$ we have $\widehat{U}(r_n \otimes x_n) = \|x_n\| e_n$, hence $\|x_n\| \to 0$, i.e. X has rank α.

OBSERVATION 13. In Proposition 12, we can replace the non-dispersed compact Hausdorff space T by the Cantor group $\Delta = \{-1,1\}^N$ and let λ be the Haar measure on Δ and $r_n \in C(\Delta)$ the nth Rademacher function on Δ, i.e. $r_n(\delta) = \delta_n$ for each $\delta \in \Delta$. In this case, it is not necessary to use the space $B(\Sigma,X)$ to prove the result.

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Department of Mathematics
University of Constanţa
8700 Constanţa, Romania
E-mail: dpopa@univ-ovidius.ro

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