COLLOQUIUM MATHEMATICUM

VOL. 91 2002 NO. 2

DENSE RANGE PERTURBATIONS OF HYPERCYCLIC OPERATORS

BY

LUIS BERNAL-GONZÁLEZ (Sevilla)

Abstract. We show that if (T_n) is a hypercyclic sequence of linear operators on a locally convex space and (S_n) is a sequence of linear operators such that the image of each orbit under every linear functional is non-dense then the sequence $(T_n + S_n)$ has dense range. Furthermore, it is proved that if T, S are commuting linear operators in such a way that T is hypercyclic and all orbits under S satisfy the above non-denseness property then $T - S$ has dense range. Corresponding statements for operators and sequences which are hypercyclic in a weaker sense are shown. Our results extend and improve a result on denseness due to C. Kitai.

1. Introduction and notation. It is evident that every hypercyclic operator has dense range. In this paper we study the preservation of this property under suitable perturbations. Therefore, this note can be placed in the general setting of the research of properties which are stable for operators whenever a "small" change is performed on them. Perturbation theory for linear operators was created by Rayleigh and Schrödinger and it still attracts the attention of many mathematicians. The topics that have been dealt with under the optics of that theory are diverse: spectrum, normal operators, compact operators, semi-Fredholm operators, surjectivity, denseness of range, convergence problems, semigroups of operators, ... A detailed treatment of the topic can be found in the classical book [Ka]. For a recent paper that deals with dense perturbations of operators with dense range, see [CK].

Let us fix some notation and terminology. Throughout this paper X will be a (Hausdorff) topological linear space over $\mathbb{K}$, $\mathbb{K}$ being either the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$. $\mathbb{N}$ denotes the set of positive integers. By an *operator* on X we will mean a continuous linear selfmapping on X. A se-

2000 *Mathematics Subject Classification*: Primary 47A16; Secondary 46A03, 47A05, 47A55.

Key words and phrases: hypercyclic and almost hypercyclic operators and sequences, totally weakly non-dense, totally weakly nowhere dense, perturbation, generalized kernel, one-to-one sequence, dense range sequence.

This work is supported in part by D.G.E.S. Grant #BFM2001–2717 and the Junta de Andalucía.

quence (T_n) of operators is said to be *hypercyclic* (resp. *densely hypercyclic*) if the set $\mathrm{HC}((T_n)) = \{$*hypercyclic vectors for* $(T_n)\} := \{x \in X :$ the orbit $\{T_n x : n \in \mathbb{N}\}$ is dense$\}$ is not empty (is dense in X, respectively). An operator T on X is called *hypercyclic* whenever the sequence of powers (T^n) is hypercyclic, where $T^1 = T$, $T^2 = T \circ T$, and so on; a vector $x \in X$ is said to be *hypercyclic for* T if and only if it is hypercyclic for (T^n). We denote by $\mathrm{HC}(T)$ the set of hypercyclic vectors for T. It is easy to see that (T^n) is densely hypercyclic whenever T is hypercyclic. For a good up-to-date survey of concepts and results on hypercyclicity, the reader is referred to [Gr].

It cannot be expected that the sum of a hypercyclic operator and a non-hypercyclic one is hypercyclic. Indeed, on any separable infinite-dimensional Banach space X there is a hypercyclic operator of the form $T = I + K$, where I is the identity operator and K is a compact operator (see [An] or [B1]). Then $I = T - K$ has dense range but it is not hypercyclic, and K is not hypercyclic because it is compact (see [Ki]). In fact, in the complex case, K is quasi-nilpotent, i.e., its spectrum $\sigma(K)$ reduces to $\{0\}$ or, equivalently, by Gelfand's formula, its sequence of norms satisfies $\|K^n\|^{1/n} \to 0$ $(n \to \infty)$. In particular, each orbit $\{K^n x : n \in \mathbb{N}\}$ $(x \in X)$ is bounded. In this order of ideas, we recall that in 1982 C. Kitai [Ki] essentially showed that if T is hypercyclic on a complex locally convex space X and $\lambda \in \mathbb{C}$ then $T + \lambda I$ has dense range. In fact, $P(T)$ has dense range for every polynomial P on $\mathbb{C}$ (see [Bo]). (We point out that the latter result holds for real locally convex spaces [Be].) Moreover, if X is a real or complex Banach space then $f(T)$ also has dense range for every non-constant entire function f; under certain conditions, f can even be assumed only analytic on certain sets [B2].

Consequently, it is natural to ask whether the property of dense range is preserved under the addition of certain "soft" operators. This problem will be treated in this paper both for operators and for sequences of operators. The hypothesis of hypercyclicity can be weakened. For this, we will introduce the notion of "almost hypercyclicity".

2. Generalized kernel, dense range and almost hypercyclicity. We start this section with a purely set-theoretic concept, which will be needed later. If $T_n : A \to B$ $(n \in \mathbb{N})$ are mappings between two arbitrary sets A, B then we say that *the sequence* (T_n) *is one-to-one* if given $x, y \in A$ with $x \neq y$ there exists $n \in \mathbb{N}$ such that $T_j x \neq T_j y$ for every $j \geq n$.

We recall that if T is an operator on X, its *generalized kernel* is

$$\mathcal{K}(T) = \bigcup_{n=1}^{\infty} \mathrm{Ker}(T^n) = \bigcup_{n=1}^{\infty} \{x \in X : T^n x = 0\}.$$

This notion appears in connection to hypercyclicity in, for instance, [BM], [BP] and [HS]. If (T_n) is a sequence of operators then we define its *generalized* *kernel* as

$$\mathcal{K}((T_n)) = \liminf_{n\to\infty}(\mathrm{Ker}(T_n)) = \bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty}\mathrm{Ker}(T_j).$$

It is clear that $\mathcal{K}((T_n))$ is always a linear submanifold of X, and that $\mathcal{K}(T) = \mathcal{K}((T^n))$. Observe also that (T_n) is one-to-one if and only if $\mathcal{K}((T_n)) = \{0\}$. The range of an operator S on X is $R(S) = \{Sx : x \in X\}$, and S is said to have dense range if $\overline{R(S)} = X$, where $\overline{A}$ denotes the closure of any subset $A \subset X$. Observe that T has dense range if and only if $\overline{\mathrm{span}\,(\bigcup_{j=n}^{\infty} R(T^j))} = X$ $(n \in \mathbb{N})$, because if $\overline{R(T)} = X$ then also $\overline{R(T^m)} = X$ for each $m \in \mathbb{N}$ and, trivially, $\overline{\mathrm{span}\,(\bigcup_{j=m}^{\infty} R(T^j))} = R(T^m)$ for every m. Accordingly, we say that a sequence (T_n) of operators on X *has dense range* whenever

$$\overline{\mathrm{span}\Big(\bigcup_{j=n}^{\infty} R(T_j)\Big)} = X \qquad (\forall n \in \mathbb{N}).$$

Note that if (T_n) has dense range, $m \in \mathbb{N}$ and S is an operator with dense range, then the sequences $\{T_n : n \geq m\}$ and (ST_n) also have dense range.

PROPOSITION 1. *If (T_n) is a hypercyclic sequence of operators then it has dense range.*

Proof. Choose a vector $x \in \mathrm{HC}((T_n))$. Then its orbit $\{T_n x : n \in \mathbb{N}\}$ is dense and, due to the fact that each finite set $\{T_1 x, \ldots, T_n x\}$ is closed and with empty interior, one sees that $\{T_j x : j \geq n\}$ $(n \in \mathbb{N})$ is also dense. But, trivially, $\{T_j x : j \geq n\} \subset \mathrm{span}(\bigcup_{j=n}^{\infty} R(T_j))$, so the closure of the latter set is X for all $n \in \mathbb{N}$, as required. ∎

Stronger versions of Proposition 1 will be given in Theorem 3(4) and Corollary 5.

It is well known (see, for instance, [Ho]) that if X is locally convex then T has dense range if and only if its adjoint operator T^* on the topological dual space X^* is one-to-one. The same is true for sequences of operators, as the following lemma asserts. It will be employed in Section 3.

LEMMA 2. *If (T_n) is a sequence of operators on a locally convex space X, then (T_n) has dense range if and only if the sequence (T_n^*) is one-to-one.*

Proof. Firstly, assume that (T_n) has dense range and that, to obtain a contradiction, (T_n^*) is not one-to-one. Then there exists $\varphi \in X^* \setminus \{0\}$ such that $\varphi \in \mathcal{K}((T_n^*))$. Therefore a positive integer m can be found in such a way that $T_j^*\varphi = 0$ for all $j \geq m$, i.e., $\varphi(T_j x) = 0$ for all $j \geq m$ and all $x \in X$. Hence φ is zero on $\bigcup_{j=m}^{\infty} R(T_j)$, so it is also zero on the closure A of the linear span of the latter set by linearity and continuity. Then $A \neq X$ because φ is not identically zero. This is absurd since (T_n) has dense range.

Conversely, suppose that (T_n^*) is one-to-one. Again to obtain a contradiction, suppose that (T_n) does not have dense range. Then there exists $m \in \mathbb{N}$ such that $A \neq X$, where A is as before. Pick a vector $x_0 \in X \setminus A$. Since A is closed and convex, the Hahn–Banach theorem guarantees the existence of $\varphi \in X^*$ with $\varphi(x_0) = 1$ and $\varphi(y) = 0$ for all $y \in \bigcup_{j=m}^{\infty} R(T_j)$, which tells us that $\varphi \neq 0$ and $\varphi(T_j x) = 0$ for every $x \in X$ and every $j \geq m$, i.e., $T_j^* \varphi = 0$ $(j \geq m)$. Thus,

$$\varphi \in \bigcap_{j=m}^{\infty} \operatorname{Ker}(T^j) \subset \mathcal{K}((T^n)),$$

which is absurd. ∎

In another order of ideas, A. Peris [Pe] has proposed the following interesting problem, related to multi-hypercyclicity: If $x \in X$ and T is an operator on X such that $\overline{\{T^n x : n \in \mathbb{N}\}}$ has nonempty interior, does this set equal the whole space X? As far as I know, this problem remains unsolved to date. Consequently, we introduce the following definition. Assume that (T_n) is a sequence of operators on X. Then we say that (T_n) is *almost hypercyclic* whenever there is some vector $x \in X$ whose orbit under (T_n) is somewhere dense. Similarly, an operator T on X is *almost hypercyclic* whenever its sequence of powers (T^n) is almost hypercyclic. We could define accordingly the corresponding almost hypercyclic vectors, denoting by $\mathrm{AHC}((T_n))$ and $\mathrm{AHC}(T)$ their respective sets. Trivially, hypercyclicity implies almost hypercyclicity. The example $T_n : x \in \mathbb{R} \mapsto q_n x \in \mathbb{R}$ $(n \in \mathbb{N})$, where (q_n) is an enumeration of the rational numbers in $[0, 1]$, shows that the converse is false.

In order to state our results, two new concepts are needed. If $A \subset X$ then we say that A is *totally weakly non-dense* (resp. *totally weakly nowhere dense*) whenever $\varphi(A)$ is non-dense (nowhere dense, resp.) in $\mathbb{K}$ for every $\varphi \in X^*$. Clearly, if A is totally weakly nowhere dense then it is totally weakly non-dense. For instance, every bounded set is totally weakly non-dense (but not necessarily totally weakly nowhere dense), and every sequence of the form $\{\lambda^n x_0 : n \in \mathbb{N}\}$ ($x_0 \in X$ and $\lambda \in \mathbb{K}$ are fixed) is totally weakly nowhere dense: take into account the continuity of each φ together with the fact that no sequence (λ^n) is somewhere dense in $\mathbb{K}$. But even a nowhere dense set does not need to be totally weakly non-dense; for instance, the set $A := \{(x, 0) : x \in \mathbb{R}\}$ is nowhere dense in $X := \mathbb{R}^2$ but $\varphi(A)$ is, trivially, dense in $\mathbb{R}$ for all φ in X^* with the sole exception of φ being a scalar multiple of the x-projection $(x, y) \mapsto x$.

To finish this section, we recall that a subset A of a topological space Y is said to be of *first category* whenever it is a countable union of nowhere dense subsets. A is said to be *of second category* if it is not of first category. Y is called a *Baire space* if each non-empty open subset is of second category. In a Baire space Y, a subset A is said to be *residual* whenever its complement

$Y \setminus A$ is of first category. For instance, every complete metric space is a Baire space (see [Ox, Chap. 9]).

3. Dense range perturbations. We are now ready to establish our main results.

THEOREM 3. *Assume that* (T_n), (S_n) *are two sequences of operators on a locally convex space* X. *We have*:

(1) *If* (T_n) *is hypercyclic and every orbit* $\{S_n x : n \in \mathbb{N}\}$ $(x \in X)$ *is totally weakly non-dense then the sequence* $(T_n + S_n)$ *has dense range.*

(2) *If* (T_n) *is densely hypercyclic*, X *is Baire metrizable and there is a subset* $M \subset X$ *of second category such that every orbit* $\{S_n x : n \in \mathbb{N}\}$ $(x \in M)$ *is totally weakly non-dense then the sequence* $(T_n + S_n)$ *has dense range.*

(3) *If* (T_n) *is almost hypercyclic and every orbit* $\{S_n x : n \in \mathbb{N}\}$ $(x \in X)$ *is totally weakly nowhere dense then the sequence* $(T_n + S_n)$ *has dense range.*

(4) *If* (T_n) *is almost hypercyclic then it has dense range.*

Proof. Part (4) is a consequence of (3), just by taking $S_n = 0$ $(n \in \mathbb{N})$. As for (1)–(3), it is clear that we may replace $(T_n + S_n)$ by $(T_n - S_n)$. Suppose that $(T_n - S_n)$ does not have dense range. Then the sequence $(T_n^* - S_n^*)$ is not one-to-one by Lemma 2, so there exists a non-zero linear functional $\varphi \in \mathcal{K}((T_n^* - S_n^*))$. This leads to the existence of a positive integer m with $\varphi(T_j x) - \varphi(S_j x) = 0$ $(x \in X,\ j \geq m)$.

Under the hypothesis of (1), we choose a vector $z \in \mathrm{HC}((T_n))$. Then $\varphi(T_j z) = \varphi(S_j z)$ $(j \geq m)$. But the set $\{\varphi(S_j z) : j \geq m\}$ is not dense in $\mathbb{K}$, by hypothesis. However, φ is onto since $\varphi \neq 0$, and $\{T_j z : j \geq m\}$ is dense (see beginning of the proof of Proposition 1). Hence the set $\{\varphi(T_j z) : j \geq m\}$ is dense in $\mathbb{K}$. This contradiction proves the result.

Under the hypothesis of (2), we know that X is metrizable and separable (this is guaranteed by the existence of a hypercyclic vector), so that $\mathrm{HC}((T_n))$ is a G_δ-subset (see [Gr, p. 349]). Since it is, in addition, dense, it turns out that it is residual in the Baire space X. Therefore $M \cap \mathrm{HC}((T_n))$ is non-empty. If we choose a vector z in that intersection then we can conclude the desired result in the same way as above.

If we start from (3) then we can choose a vector $z \in \mathrm{AHC}((T_n))$. Then $\varphi(T_j z) = \varphi(S_j z)$ $(j \geq m)$. This time the set $\{\varphi(S_j z) : j \geq m\}$ is nowhere dense. But the closure of $\{T_j z : j \in \mathbb{N}\}$ contains some non-empty open set U, so the closure of $\{T_j z : j \geq m\}$ also does. Since $\varphi \neq 0$, it is an open mapping, so $\varphi(U)$ is a non-empty subset of $\mathbb{K}$. The continuity of φ yields

$$\overline{\varphi(\{T_j z : j \geq m\})} \supset \varphi(\overline{\{T_j z : j \geq m\}}) \supset \varphi(U),$$

which contradicts the hypothesis. The proof is finished. ∎

Note that Theorem 3 can be applied if, for instance, $S_n = \lambda_n I$ $(n \in \mathbb{N})$ with (λ_n) a non-dense/nowhere dense sequence of scalars (for (1), (3), resp.), or if every orbit $\{S_n x : n \in \mathbb{N}\}$ $(x \in X)$ is bounded (for (1)), which in turn holds whenever X is a normed space and the sequence $(\|S_n\|)$ of norms is bounded. In order to illustrate Theorem 3, let us give the following rather natural example. Let X be a separable normed space and fix a vector $u \in X$ with norm $\|u\| = 1$. Then each vector x in X can be uniquely written as $x = au + y$ with $a \in \mathbb{K}$, $y \in Y$, where Y is a complementary linear subspace to $\langle u \rangle$, the set of scalar multiples of u. The projection $Px := au$ is an operator on X. Fix a dense sequence (x_n) in X. Define the sequence (T_n) of operators by $T_n x = a x_n$, that is, T_n is the rank-one operator projecting u to x_n. It is clear that (T_n) is hypercyclic; it is in fact densely hypercyclic since $\mathrm{HC}((T_n)) \supset \{au + y : a \in \mathbb{K} \setminus \{0\},\ y \in Y\}$. Now, consider the sequence $n_1 < n_2 < n_3 < \ldots$ of positive integers satisfying

$$\{x_{n_j} : j \in \mathbb{N}\} = \{x_n : n \in \mathbb{N}\} \cap \{x \in X : \|x\| \le 1\}.$$

Define the sequence (S_n) of operators by

$$S_n = \begin{cases} -T_{n_j} & \text{if } n = n_j, \\ 0 & \text{if } n \neq n_j \text{ for all } j \in \mathbb{N}. \end{cases}$$

Evidently, $\|S_n\| \le \|P\|$ for all $n \in \mathbb{N}$. Therefore the sequence $(T_n + S_n)$ has dense range by Theorem 3(1). Nevertheless, it is not hypercyclic, because if $x = 0 \cdot u + y$ then x is trivially non-hypercyclic and if $x = au + y$ with $a \neq 0$ then the orbit $(T_n x + S_n x)$ cannot approximate any non-zero vector in the ball $\{\|x\| < |a|\}$.

In our next result, we deal with preservation of the dense range property of a single operator T if the "perturbing" operator S commutes with T (see parts (1)–(4)). Part (5) improves some of the results contained in Theorems 3–4 of [B2]. We denote, as usual, by $\sigma(T)$ the spectrum of T and by $\sigma_{\mathrm{P}}(T)$ the point spectrum of T, that is, $\sigma_{\mathrm{P}}(T)$ is the set of eigenvalues of T.

THEOREM 4. *Assume that T, S are commuting operators on a locally convex space X. We have*:

(1) *If T is hypercyclic and for every $x \in X$ the orbit $\{S^n x : n \in \mathbb{N}\}$ is totally weakly non-dense then $T - S$ has dense range.*

(2) *If X is Baire metrizable, T is hypercyclic and there exists a subset $M \subset X$ of second category such that for every $x \in M$ the orbit $\{S^n x : n \in \mathbb{N}\}$ is totally weakly non-dense then $T - S$ has dense range.*

(3) *If T is almost hypercyclic and for every $x \in X$ the orbit $\{S^n x : n \in \mathbb{N}\}$ is totally weakly nowhere dense then $T - S$ has dense range.*

(4) *If X is an F-space, T is onto and almost hypercyclic and there exists a residual subset $M \subset X$ such that for every $x \in M$ the orbit $\{S^n x : n \in \mathbb{N}\}$ is totally weakly nowhere dense, then $T - S$ has dense range.*

(5) *Assume that X is a Banach space and that T is almost hypercyclic. If X is complex and f is an analytic function on an open set $\Omega \subset \mathbb{C}$ containing $\sigma(T)$ and f is non-constant on every connected component of Ω then $f(T)$ has dense range. If X is real and f is a real-analytic function that extends holomorphically to some open set $\Omega \subset \mathbb{C}$ containing $\sigma(\widetilde{T})$, where $\widetilde{T}$ is the complexification of T, and f is non-constant on every component of Ω, then $f(T)$ has dense range.*

Proof. Let us prove (1). To obtain a contradiction, assume that $T - S$ does not have dense range, so its adjoint operator $T^* - S^*$ is not one-to-one. Consequently, there is $\varphi \in X^* \setminus \{0\}$ with $(T^* - S^*)\varphi = 0$, i.e., $\varphi(Tx) = \varphi(Sx)$ for all $x \in X$. Since T and S commute, we obtain

$$\varphi(T^2x) = \varphi(TTx) = \varphi(STx) = \varphi(TSx) = \varphi(SSx) = \varphi(S^2x),$$

so $\varphi(T^2x) = \varphi(S^2x)$ for all $x \in X$. An induction procedure leads to $\varphi(T^nx) = \varphi(S^nx)$ for all $n \in \mathbb{N}$. Pick a hypercyclic vector z for T. Then the set $\{\varphi(S^nz) : n \in \mathbb{N}\}$ is not dense in $\mathbb{K}$, the orbit $\{T^nz : n \in \mathbb{N}\}$ is dense in X and $\varphi(T^nz) = \varphi(S^nz)$ $(n \in \mathbb{N})$. This gives us the desired contradiction because $\varphi \neq 0$ (hence onto) and, consequently, the set $\{\varphi(S^nx) : n \in \mathbb{N}\}$ should be dense in $\mathbb{K}$.

The proof of (2) is analogous, just by taking into account that X must be separable and, as in the proof of Theorem 3, the set $\mathrm{HC}(T)$ of T-hypercyclic vectors is dense and G_δ, so residual in the Baire space X. Then the vector z can be chosen in M and the remainder of the argument is similar to case (1).

As for (3), by following the way of contradiction of (1), we would obtain $\varphi(T^nz) = \varphi(S^nz)$ for all $n \in \mathbb{N}$, where $z \in \mathrm{AHC}(T)$. Since $\varphi \neq 0$, it is open. This property together with the continuity of φ shows that the closure of $\{\varphi(T^nz) : n \in \mathbb{N}\}$ contains a non-empty open set, which contradicts the fact that the closure of $\{\varphi(S^nz) : n \in \mathbb{N}\}$ has empty interior.

Assume now that the hypotheses of (4) are fulfilled, and let $z \in \mathrm{AHC}(T)$. Then there is a non-empty open subset U with $\overline{\{T^nz : n \in \mathbb{N}\}} \supset U$. If $m \in \mathbb{N}$, we have

$$\overline{\{T^n(T^mz) : n \in \mathbb{N}\}} = \overline{\{T^m(T^nz) : n \in \mathbb{N}\}} \supset T^m(\overline{\{T^nz : n \in \mathbb{N}\}}) \supset T^m(U).$$

Since T is onto, T^m is also onto, so it is open by the Open Mapping Theorem, because X is an F-space. Then $T^m(U)$ is a non-empty open set, whence $T^mz \in \mathrm{AHC}(T)$ for every m. Therefore $\mathrm{AHC}(T)$ contains the orbit of z, so $\mathrm{AHC}(T)$ is somewhere dense; in particular, it is of second category. This implies that $M \cap \mathrm{AHC}(T) \neq \emptyset$. Thus, we can choose a vector z in that intersection, and the remainder of the argument is as in case (3).

Finally, assume that T is almost hypercyclic on a complex Banach space X, and that f is as in the hypothesis of (5). We follow the steps of [B2]. By a special version of the spectral mapping theorem [Ru, Theorem 10.33],

$\sigma_P(f(T^*)) = f(\sigma_P(T^*))$. From part (3) (see Corollary 5) we see that $T - \lambda I$ has dense range for any scalar λ. Then $\sigma_P(T^*)$ cannot contain any scalar λ, i.e., $\sigma_P(T^*) = \emptyset$. Hence $\sigma_P(f(T^*)) = \emptyset$. But $f(T^*) = f(T)^*$, so $\sigma_P(f(T)^*) = \emptyset$. In particular, $0 \notin \sigma_P(f(T)^*)$, so $f(T)^*$ is one-to-one. Consequently, $f(T)$ has dense range.

Assume now that X is real and that f is a real-analytic function as in the hypothesis. Recall that if $\widetilde{X} = X + iX$ is the complexification of X then the complexification $\widetilde{T}$ of T is defined on $\widetilde{X}$ as $\widetilde{T}(x+iy) = Tx + iTy$. Since T is almost hypercyclic, we have $\sigma_P((\widetilde{T})^*) = \emptyset$. Indeed, if $\lambda \in \mathbb{C}$, $\varphi \in (\widetilde{X})^*$ are an eigenvalue and a corresponding eigenvector of $(\widetilde{T})^*$ and $x \in \mathrm{AHC}(T)$, then $\{|\varphi(T^n x)| : n \in \mathbb{N}\} = \{|\varphi(\widetilde{T}^n x)| : n \in \mathbb{N}\} = \{|(\widetilde{T})^{*n}\varphi(x)| : n \in \mathbb{N}\} = \{|\lambda|^n |\varphi(x)| : n \in \mathbb{N}\}$ and, while the closure of the first set contains a non-empty open set of $(0,\infty)$ (because φ is open), the closure of the latter is obviously empty (at this step we have mimicked the proof of a recent result of Bonet and Peris [Bn, p. 592]). Thus, again by the spectral mapping theorem,

$$\sigma_P(f((\widetilde{T})^*)) = f(\sigma_P((\widetilde{T})^*)) = \emptyset.$$

In particular, $f(\widetilde{T})^* = f((\widetilde{T})^*)$ is one-to-one, and so $\widetilde{f(T)} = f(\widetilde{T})$ has dense range. Thus, $f(T)$ also has dense range, which concludes the proof.

Observe that we recover and even improve Kitai's result on denseness (see Section 1). It suffices to take $S = -\lambda I$ in part (3). The following corollary is obtained (compare [Mi, Theorem 4]).

COROLLARY 5. *If T is an almost hypercyclic operator on a locally convex space, then $T + \lambda I$ has dense range for every scalar λ. In particular, T itself has dense range.*

As before, Theorem 4 can be applied if X is a normed space and $\{\|S^n\| : n \in \mathbb{N}\}$ is bounded (for (1)–(2)), which in turn is satisfied if X is a complex Banach space and $\sigma(S)$ is contained in the open unit disk (in particular, this holds whenever S is quasi-nilpotent). In the latter case, (3)–(4) might be applied: indeed, $(S^n x)$ converges to zero for all $x \in X$, so $(\varphi(S^n x))$ tends to zero for all $\varphi \in X^*$, hence each orbit would be totally nowhere dense.

We point out that the statement of Theorem 4 does not hold if X is not locally convex. For instance, if $0 < p < 1$ and $X = L^p[0,1]$ then there are continuous selfmappings $\varphi : [0,1] \to [0,1]$ such that the composition operator $Tf = f \circ \varphi$ is hypercyclic (see [Gr, Remark 4(b)]). Here X is a metrizable non-locally convex topological linear space with $X^* = \{0\}$ (see, for instance, [PU]). Hence, trivially, every subset of X is totally weakly non-dense and by taking $S = T$ in Theorem 4 we would find that the null operator has dense range, which is absurd.

To finish, let us remark that an analogous theorem in the setting of supercyclicity is not possible. We recall that an operator T on a topological linear space is called *supercyclic* whenever the set $\{\lambda T^n x : n \in \mathbb{N},\ \lambda \in \mathbb{K}\}$ is dense in X. It is trivial that supercyclic operators also have dense range. G. Herzog [He] showed that every infinite-dimensional separable Banach space supports a compact supercyclic operator T. But for this operator, $I+T$ is hypercyclic (see [An] or [B1]), so T is quasi-nilpotent. Thus, the null operator, which trivially does not have dense range, equals T (supercyclic) minus T (quasi-nilpotent).

Acknowledgements. The author is grateful to the referee for helpful comments and suggestions.

REFERENCES

[An] S. I. Ansari, *Existence of hypercyclic operators on topological vector spaces*, J. Funct. Anal. 148 (1997), 384–390.
[BM] T. Bermúdez and V. G. Miller, *On operators T such that $f(T)$ is hypercyclic*, Integral Equations Oper. Theory 37 (2000), 332–340.
[B1] L. Bernal-González, *On hypercyclic operators on Banach spaces*, Proc. Amer. Math. Soc. 127 (1999), 1003–1010.
[B2] —, *Universal images of universal elements*, Studia Math. 138 (2000), 241–250.
[Be] J. P. Bès, *Invariant manifolds of hypercyclic vectors for the real scalar case*, Proc. Amer. Math. Soc. 127 (1999), 1801–1804.
[BP] J. P. Bès and A. Peris, *Hereditarily hypercyclic operators*, J. Funct. Anal. 167 (1999), 94–113.
[Bn] J. Bonet and A. Peris, *Hypercyclic operators on non-normable Fréchet spaces*, ibid. 159 (1998), 587–596.
[Bo] P. Bourdon, *Invariant manifolds of hypercyclic vectors*, Proc. Amer. Math. Soc. 118 (1993), 845–847.
[CK] P. G. Casazza and N. J. Kalton, *Generalizing the Paley–Wiener perturbation theory for Banach spaces*, ibid. 127 (1999), 519–527.
[CS] K. C. Chan and J. H. Shapiro, *The cyclic behavior of translation operators on Hilbert spaces of entire functions*, Indiana Univ. Math. J. 40 (1991), 1421–1449.
[Gr] K. G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. 36 (1999), 345–381.
[He] G. Herzog, *On linear operators having supercyclic vectors*, Studia Math. 103 (1992), 295–298.
[HS] G. Herzog and C. Schmoeger, *On operators T such that $f(T)$ is hypercyclic*, ibid. 108 (1994), 209–216.
[Ho] J. Horváth, *Topological Vector Spaces and Distributions*, Vol. I, Addison-Wesley, Reading, 1966.
[Ka] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1976.
[Ki] C. Kitai, *Invariant closed sets for linear operators*, Dissertation, Univ. of Toronto, 1982.
[Mi] V. G. Miller, *Remarks on finitely hypercyclic and finitely supercyclic operators*, Integral Equations Oper. Theory 29 (1997), 110–115.

[Ox] J. C. Oxtoby, *Measure and Category*, 2nd ed., Springer, New York, 1980.
[Pe] A. Peris, *Multi-hypercyclic operators are hypercyclic*, Math. Z. 236 (2001), 779–786.
[PU] E. Pflaumann and H. Unger, *Funktionalanalysis–I*, Bibliographisches Institut AG, Mannheim, 1968.
[Ru] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1991.

Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de Sevilla
Apdo. 1160, Avenida Reina Mercedes
41080 Sevilla, Spain
E-mail: lbernal@us.es

Received 10 April 2000;
revised 13 April 2001 (3912)