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ON OPERATORS FROM ℓ_s TO $\ell_p \widehat{\otimes} \ell_q$ OR TO $\ell_p \widehat{\widehat{\otimes}} \ell_q$

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Abstract. We show that every operator from ℓ_s to $\ell_p \widehat{\otimes} \ell_q$ is compact when $1 \le p, q < s$ and that every operator from ℓ_s to $\ell_p \widehat{\widehat{\otimes}} \ell_q$ is compact when $1/p + 1/q > 1 + 1/s$.

1. Introduction. We recall Pitt's theorem: for $1 \le p < s < \infty$, every operator from ℓ_s to ℓ_p is compact [7], [8]. This result has been extended to different settings. Among the more recent contributions we mention [1] and [3]. The aim of this paper is to show that every operator from ℓ_s to $\ell_p \widehat{\otimes} \ell_q$ is compact when

$$1 \le p, q < s \tag{1.1}$$

and that every operator from ℓ_s to $\ell_p \widehat{\widehat{\otimes}} \ell_q$ is compact when

$$\frac{1}{p} + \frac{1}{q} > 1 + \frac{1}{s}. \tag{1.2}$$

A proof of the injective case, using τ_α-convergence, is given in [1]. Here we use a different method and the same technique in both cases. Let $r = s'$ be the conjugate exponent of s (i.e. $1/s + 1/s' = 1$). We show that under condition (1.1) (resp. (1.2)) the space $[\ell_p \widehat{\otimes} \ell_q] \widehat{\widehat{\otimes}} \ell_r$ (resp. $[\ell_p \widehat{\widehat{\otimes}} \ell_q] \widehat{\widehat{\otimes}} \ell_r$) does not contain a subspace isomorphic to c_0. The conclusions will then follow from [11].

2. Notation. We shall make use of standard Banach space facts and terminology as may be found in [6], [7].

The term *operator* means bounded linear operator. *Subspace* means closed linear subspace.

Let E, F be Banach spaces. We denote by:

- $\mathcal{L}(E,F)$ the space of operators from E to F.
- $\mathcal{N}(E,F)$ the space of nuclear operators from E to F, and by $\|u\|_{\text{nuc}}$ the nuclear norm of a nuclear operator u.

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- $\mathcal{B}(E,F)$ the space of continuous bilinear forms on $E \times F$.
- $E \widehat{\otimes} F$ the completion of $E \otimes F$ endowed with the projective norm [4], [5].
- $E \widehat{\widehat{\otimes}} F$ the completion of $E \otimes F$ endowed with the injective norm [4], [5].
- $\mathcal{J}(E,F)$ the space of bilinear integral forms on $E \times F$. We have $\mathcal{J}(E,F) = [E \widehat{\widehat{\otimes}} F]^*$. The norm of an integral form φ is denoted by $\|\varphi\|_{\mathrm{int}}$.
- ℓ_p^m the m-dimensional space $\ell_p(\{1,\ldots,m\})$.

Let r be a real number ≥ 1; we define

$$\mathrm{sl}_r(E) = \Big\{ x = (x_n)_{n\geq 1}\,;\ \text{for all } n \geq 1,\ x_n \in E,\ \text{and for all } x^* \in E^*,\ \sum_{n=1}^{\infty} |x^*(x_n)|^r < \infty \Big\}.$$

We recall that for $x = (x_n)_n \in \mathrm{sl}_r(E)$ we have

$$\|x\| = \sup_{\|x^*\|\leq 1} \Big[\sum_{n=1}^{\infty} |x^*(x_n)|^r \Big]^{1/r} < \infty.$$

The space $(\mathrm{sl}_r(E), \|\ \|)$ is a Banach space. For every integer m, let R_m be the projection of $\mathrm{sl}_r(E)$ defined, for every $x = (x_k)_k$, by $R_m(x) = (x_1,\ldots,x_m,0,0,\ldots)$. The subspace

$$F_r(E) = \{ x \in \mathrm{sl}_r(E)\,;\ x = \lim_{m\to\infty} R_m(x) \}$$

of $\mathrm{sl}_r(E)$ is isometrically isomorphic to $\ell_r \widehat{\widehat{\otimes}} E$ (see [9]). We shall use this isometric isomorphism without any reference.

3. Lemmas. Let $1 \leq p,q,r < \infty$. We denote by $(P_m)_m$ the natural projections associated to the unit vector basis of ℓ_p and by $(Q_m)_m$ the natural projections associated to the unit vector basis of ℓ_q. We denote by $\widetilde{P}_m, \widetilde{Q}_m$ the norm 1 projections of $\ell_r \widehat{\widehat{\otimes}} (\ell_p \widehat{\otimes} \ell_q)$ or $\ell_r \widehat{\widehat{\otimes}} (\ell_p \widehat{\widehat{\otimes}} \ell_q)$ which are defined by $\widetilde{P}_m = I_{\ell_r} \otimes (P_m \otimes I_{\ell_q})$ and $\widetilde{Q}_m = I_{\ell_r} \otimes (I_{\ell_p} \otimes Q_m)$. For every $x = (x_k)_k \in F_r(\ell_p \widehat{\otimes} \ell_q)$ or $x = (x_k)_k \in F_r(\ell_p \widehat{\widehat{\otimes}} \ell_q)$ we have

$$\widetilde{P}_m(x) = ((P_m \otimes I_{\ell_q})(x_1), \ldots, (P_m \otimes I_{\ell_q})(x_k), \ldots),$$

$$\widetilde{Q}_m(x) = ((I_{\ell_p} \otimes Q_m)(x_1), \ldots, (I_{\ell_p} \otimes Q_m)(x_k), \ldots).$$

For all m,n we have $\widetilde{P}_m \circ R_n = R_n \circ \widetilde{P}_m$, $\widetilde{Q}_m \circ R_n = R_n \circ \widetilde{Q}_m$ and $\widetilde{P}_m \circ \widetilde{Q}_n = \widetilde{Q}_n \circ \widetilde{P}_m$.

It is well known that, if $(\pi_m)_m$ is a sequence of operators on a Banach space E such that $\lim_{m\to\infty} \pi_m(x) = x$ for every $x \in E$, then for every

Banach space F and for every $u \in E \widehat{\otimes} F$ (resp. $u \in E \widehat{\widehat{\otimes}} F$) we have $\lim_{m\to\infty}(\pi_m \otimes I_F)(u) = u$. This remark leads to the following lemma:

LEMMA 3.1. *For every $x \in F_r(\ell_p \widehat{\otimes} \ell_q)$ and every $x \in F_r(\ell_p \widehat{\widehat{\otimes}} \ell_q)$ we have*

$$x = \lim_{m\to\infty} \widetilde{P}_m(x) = \lim_{m\to\infty} \widetilde{Q}_m(x).$$

LEMMA 3.2. *For every integer m, $\widetilde{P}_m[F_r(\ell_p \widehat{\otimes} \ell_q)]$ and $\widetilde{P}_m[F_r(\ell_p \widehat{\widehat{\otimes}} \ell_q)]$ are isomorphic to $\ell_r \widehat{\widehat{\otimes}} \ell_q$.*

Proof. It is easy to show that

$$\widetilde{P}_m[F_r(\ell_p \widehat{\otimes} \ell_q)] = F_r[(P_m \otimes I_{\ell_q})(\ell_p \widehat{\otimes} \ell_q)].$$

We have $(P_m \otimes I_{\ell_q})(\ell_p \widehat{\otimes} \ell_q)$ isomorphic to $\ell_p^m \widehat{\otimes} \ell_q$. It is well known that $\ell_p^m \widehat{\otimes} \ell_q$ is isomorphic to the m-product $[\ell_q]^m$ of ℓ_q and so to ℓ_q. Hence, $F_r[(P_m \otimes I_{\ell_q})(\ell_p \widehat{\otimes} \ell_q)]$ is isomorphic to $\ell_r \widehat{\widehat{\otimes}} \ell_q$. With the same argument we show that $\widetilde{P}_m[F_r(\ell_p \widehat{\widehat{\otimes}} \ell_q)]$ is isomorphic to $\ell_q \widehat{\widehat{\otimes}} \ell_r$.

In the following we shall consider sequences of block operators. A sequence $(T_n)_n$ of operators from ℓ_p to $\ell_{q'}$ is called a *sequence of block operators* if there exist two strictly increasing sequences $(i_n)_n$ and $(j_n)_n$ of integers such that $i_0 = j_0 = 0$ and, for every integer $n \geq 1$, we have

$$T_n = (Q^*_{j_n} - Q^*_{j_{n-1}}) \circ T_n \circ (P_{i_n} - P_{i_{n-1}}).$$

We write as lemmas the results of Tong [10] that we will use below.

LEMMA 3.3. *Let $(T_n)_n$ be a sequence of block operators from ℓ_p to $\ell_{q'}$. Suppose that $\|T_n\| = 1$ for every n. Then, for every integer N and for every finite sequence $(\alpha_n)_{1\leq n\leq N}$ of scalars, we have*

$$\Big\|\sum_{n=1}^{N} \alpha_n T_n\Big\| = \begin{cases} \Big[\sum_{n=1}^{N} |\alpha_n|^{\frac{pq'}{p-q'}}\Big]^{\frac{p-q'}{pq'}} & \text{if } 1 \leq q' < p < \infty, \\ \max_{1\leq n\leq N} |\alpha_n| & \text{if } 1 \leq p \leq q' \leq \infty, \\ \Big[\sum_{n=1}^{N} |\alpha_n|^{q'}\Big]^{1/q'} & \text{if } 1 \leq q' < p = \infty. \end{cases}$$

LEMMA 3.4. *Let $(T_n)_n$ be a sequence of block operators from ℓ_p to $\ell_{q'}$. Suppose that $\|T_n\|_{\text{nuc}} = 1$ for every n. Then, for every integer N and for every finite sequence $(\alpha_n)_{1\leq n\leq N}$ of scalars, we have*

$$\Big\|\sum_{n=1}^{N}\alpha_n T_n\Big\|_{\text{nuc}} = \begin{cases} \Big[\sum_{n=1}^{N}|\alpha_n|^{\frac{pq'}{pq'+p-q'}}\Big]^{\frac{pq'+p-q'}{pq'}} & \textit{if } 1\le p<q'<\infty,\\ \max_{1\le n\le N}|\alpha_n| & \textit{if } p=1 \textit{ and } q'=\infty,\\ \Big[\sum_{n=1}^{N}|\alpha_n|^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}} & \textit{if } 1<p<q'=\infty,\\ \sum_{n=1}^{N}|\alpha_n| & \textit{if } 1\le q'\le p\le\infty. \end{cases}$$

The following lemma is a direct consequence of the proof of the theorem of [9].

Lemma 3.5. *Let X be an infinite-dimensional subspace of $\ell_p \,\widehat{\widehat{\otimes}}\, \ell_q$. If $q' > p$, then X contains a subspace isomorphic to ℓ_σ where $\sigma = p$ or $\sigma = q$ or $\sigma = \frac{pq}{p+q-pq} = \frac{pq'}{q'-p}$, and if $q' \le p$, then X contains a subspace isomorphic to c_0.*

4. Operators from ℓ_s into $\ell_p \widehat{\otimes} \ell_q$. For every $b \in \mathcal{B}(E,F)$ we denote by $T_b \in \mathcal{L}(E,F^*)$ the operator defined by $(T_b(x))(y) = b(x,y)$ for every $x \in E$ and $y \in F$. We recall that the operator $b \mapsto T_b$ is an isometric isomorphism from $\mathcal{B}(E,F)$ onto $\mathcal{L}(E,F^*)$.

Theorem 4.1. *Let $1 \le p,q,r$ be real numbers such that $1 \le r$, $1 \le p < r'$ and $1 \le q < r'$. Then the space $(\ell_p \,\widehat{\otimes}\, \ell_q) \,\widehat{\widehat{\otimes}}\, \ell_r$ does not contain a subspace isomorphic to c_0.*

Proof. By Grothendieck's result [5] the space $\ell_p \widehat{\otimes} \ell_q$ is the dual of $\ell_{p'} \widehat{\widehat{\otimes}} \ell_{q'}$ (with c_0 in place of ℓ_∞ when p or $q = 1$). Therefore the space $\ell_p \,\widehat{\otimes}\, \ell_q$ is a separable dual, hence, by [2], it does not contain a subspace isomorphic to c_0.

We assume that $F_r(\ell_p \widehat{\otimes} \ell_q)$ contains a subspace isomorphic to c_0; we shall show that this leads to a contradiction. We shall construct a normalized basic sequence $(x_n)_n$ of $F_r(\ell_p \,\widehat{\otimes}\, \ell_q)$ equivalent to the unit vector basis of c_0 and three strictly increasing sequences of integers $(i_n)_n, (j_n)_n, (k_n)_n$ such that $i_0 = j_0 = k_0 = 0$ and, for every integer $n \ge 1$,

$$x_n = (R_{k_n} - R_{k_{n-1}})(x_n) = (\widetilde{P}_{i_n} - \widetilde{P}_{i_{n-1}})(x_n) = (\widetilde{Q}_{j_n} - \widetilde{Q}_{j_{n-1}})(x_n). \tag{4.1}$$

This will be done in three stages. We begin with a normalized basic sequence $(u_n)_n$ of $F_r(\ell_p \,\widehat{\otimes}\, \ell_q)$ equivalent to the unit vector basis of c_0.

In the first stage we show that there exists a normalized basic sequence $(v_n)_n$ of $F_r(\ell_p \,\widehat{\otimes}\, \ell_q)$ equivalent to the unit vector basis of c_0 and a strictly increasing sequence $(m_n)_n$ of integers such that $m_0 = 0$ and, for every integer $n \geq 1$,

$$v_n = (R_{m_n} - R_{m_{n-1}})(v_n). \tag{4.2}$$

Let $\varepsilon > 0$. For every integer $m \geq 1$, the subspace $\operatorname{Im} R_m$ of $F_r(\ell_p \widehat{\otimes} \ell_q)$ is isomorphic to $[\ell_p \widehat{\otimes} \ell_q]^m$ so it does not contain a subspace isomorphic to c_0. Due to this remark it is easy to construct by induction a normalized block basic sequence $(u'_n)_n$ of $(u_n)_n$ and a strictly increasing sequence $(m_n)_n$ of integers such that $\|R_{m_1}(u'_1) - u'_1\| \leq \varepsilon/2$ and, for every integer $n \geq 2$, $\|R_{m_{n-1}}(u'_n)\| \leq \varepsilon/2^{n+1}$ and $\|R_{m_n}(u'_n) - u'_n\| \leq \varepsilon/2^{n+1}$. For every integer n we have

$$\|u'_n - (R_{m_n} - R_{m_{n-1}})(u'_n)\| \leq \frac{\varepsilon}{2^n}$$

so, for $\varepsilon > 0$ small enough, the sequence $((R_{m_n} - R_{m_{n-1}})(u'_n))_n$ is seminormalized and equivalent to the unit vector basis of c_0. For every integer n we take

$$v_n = \frac{(R_{m_n} - R_{m_{n-1}})(u'_n)}{\|(R_{m_n} - R_{m_{n-1}})(u'_n)\|}.$$

The sequence $(v_n)_n$ is a normalized basic sequence of $F_r(\ell_p \widehat{\otimes} \ell_q)$ equivalent to the unit vector basis of c_0 which satisfies condition (4.2).

In the second stage we show that there exists a normalized basic sequence $(w_n)_n$ of $F_r(\ell_p \widehat{\otimes} \ell_q)$ equivalent to the unit vector basis of c_0 and two strictly increasing sequences of integers $(p_n)_n$ and $(r_n)_n$ such that $p_0 = r_0 = 0$ and, for every integer $n \geq 1$,

$$w_n = (R_{r_n} - R_{r_{n-1}})(w_n) = (\widetilde{P}_{p_n} - \widetilde{P}_{p_{n-1}})(w_n)\,. \tag{4.3}$$

To do this, let $\varepsilon_1 > 0$. By Lemma 3.2, for every integer $p \geq 1$, the space $\widetilde{P}_p[F_r(\ell_p \widehat{\otimes} \ell_q)]$ is isomorphic to $\ell_q \hat{\hat{\otimes}} \ell_r$ with $q < r'$. So, by Lemma 3.5, it does not contain a subspace isomorphic to c_0. It is then easy to construct by induction a normalized block basic sequence $(v'_n)_n$ of $(v_n)_n$ and a strictly increasing sequence $(p_n)_n$ of integers such that $\|v'_1 - \widetilde{P}_{p_1}(v'_1)\| \leq \varepsilon_1/2$ and, for every integer $n \geq 2$,

$$\|\widetilde{P}_{p_{n-1}}(v'_n)\| \leq \frac{\varepsilon_1}{2^{n+1}} \quad \text{and} \quad \|\widetilde{P}_{p_n}(v'_n) - v'_n\| \leq \frac{\varepsilon_1}{2^{n+1}}.$$

For $\varepsilon_1 > 0$ small enough, the sequence $((\widetilde{P}_{p_n} - \widetilde{P}_{p_{n-1}})(v'_n))_n$ is a seminormalized sequence equivalent to the unit vector basis of c_0. For every integer $n \geq 1$ we take

$$w_n = \frac{(\widetilde{P}_{p_n} - \widetilde{P}_{p_{n-1}})(v'_n)}{\|(\widetilde{P}_{p_n} - \widetilde{P}_{p_{n-1}})(v'_n)\|}.$$

It follows from condition (4.2) that there exists a strictly increasing sequence $(r_n)_n$ of integers with $r_0 = 0$ such that $w_n = (R_{r_n} - R_{r_{n-1}})(w_n)$ for every integer n. The sequence $(w_n)_n$ is a normalized basic sequence equivalent to the unit vector basis of c_0 which satisfies condition (4.3).

In the third stage we show that there exists a normalized basic sequence $(x_n)_n$ equivalent to the unit vector basis of c_0 and three strictly increasing sequences of integers $(i_n)_n$, $(j_n)_n$ and $(k_n)_n$ such that $i_0 = j_0 = k_0$ which satisfy condition (4.1).

To do this, we begin with the sequence $(w_n)_n$ satisfying condition (4.3) and we use the same method as in the second stage.

Now we show that the existence of a normalized basic sequence $(x_n)_n$ of $F_r(\ell_p \widehat{\otimes} \ell_q)$ equivalent to the unit vector basis of c_0 satisfying condition (4.1) leads to a contradiction.

For every integer n we have $x_n = (R_{k_n} - R_{k_{n-1}})(x_n)$ so there exists a sequence $(u_l)_l$ in $\ell_p \widehat{\otimes} \ell_q$ such that $x_1 = (u_1, \ldots, u_{k_1}, 0, 0, \ldots)$ and, for every integer $n \geq 2$, $x_n = (0, \ldots, 0, u_{k_{n-1}+1}, \ldots, u_{k_n}, 0, 0, \ldots)$.

Let us recall that $[\ell_p \widehat{\otimes} \ell_q]^*$ is isometrically isomorphic to the space $\mathcal{B}(\ell_p, \ell_q)$ ([4], [5]). So, for every integer n, there exists $b_n \in \mathcal{B}(\ell_p, \ell_q)$ such that $\|b_n\| = 1$ and

$$1 = \Big[\sum_{l=k_{n-1}+1}^{k_n} |b_n(u_l)|^r\Big]^{1/r}.$$

It follows from condition (4.1) that for each integer $l \in \{k_{n-1}+1, \ldots, k_n\}$ we have

$$b_n(u_l) = b_n([(P_{i_n} - P_{i_{n-1}}) \otimes (Q_{j_n} - Q_{j_{n-1}})](u_l)). \tag{4.4}$$

Condition (4.4) implies that we may suppose that

$$T_{b_n} = (Q^*_{j_n} - Q^*_{j_{n-1}}) \circ T_{b_n} \circ (P_{i_n} - P_{i_{n-1}}),$$

so $(T_{b_n})_n$ is a sequence of block operators. This last assumption implies that for $n \neq m$ and $l \in \{\, k_{m-1}+1, \ldots, k_m \,\}$, we have $b_n(u_l) = 0$.

Let N be an integer, $\alpha_1, \ldots, \alpha_N$ be scalars and let $b = \alpha_1 b_1 + \cdots + \alpha_N b_N$. We have

$$\Big[\sum_{n=1}^{k_N} |b(u_n)|^r\Big]^{1/r} = [|\alpha_1|^r + \cdots + |\alpha_N|^r]^{1/r},$$

so

$$\begin{aligned}\|x_1 + \cdots + x_N\| &\geq \Lambda(N)\\ &= \sup\{[|\alpha_1|^r + \cdots + |\alpha_N|^r]^{1/r}\,;\, \|\alpha_1 b_1 + \cdots + \alpha_N b_N\| \leq 1\}.\end{aligned}$$

Now we compute $\Lambda(N)$.

In the case $p \leq q'$ we have, by Lemma 3.3,

$$\|\alpha_1 b_1 + \cdots + \alpha_N b_N\| = \|\alpha_1 T_{b_1} + \cdots + \alpha_N T_{b_N}\| = \max_{1 \leq n \leq N} |\alpha_n|,$$

so $\Lambda(N) = N^{1/r}$.

In the case $p > q'$, let $\sigma = pq'/(p - q')$. We also have, by Lemma 3.3,

$$\|\alpha_1 b_1 + \cdots + \alpha_N b_N\| = \|\alpha_1 T_{b_1} + \cdots + \alpha_N T_{b_N}\| = \Big[\sum_{n=1}^{N} |\alpha_n|^{\sigma}\Big]^{1/\sigma}.$$

We have

$$\frac{1}{r} - \frac{1}{\sigma} = \frac{1}{r} - \frac{1}{q'} + \frac{1}{p}$$

and $r < q'$, so $\sigma > r$. Therefore, $\Lambda(N) = N^{\sigma r/(\sigma - r)}$.

In both cases, $\lim_{N\to\infty} \|x_1 + \cdots + x_N\| = \infty$ so $(x_n)_n$ is not equivalent to the unit vector basis of c_0, in contradiction with our construction.

Theorem 4.2. *Let $1 \le p, q, s$ be real numbers such that $1 \le p < s$ and $1 \le q < s$. Then every operator from ℓ_s into $\ell_p \widehat{\otimes} \ell_q$ is compact. The same is true for every operator from c_0 into $\ell_p \widehat{\otimes} \ell_q$.*

Proof. The conclusions follow directly from Corollary 14 of [11].

5. Operators from ℓ_s into $\ell_p \widehat{\widehat{\otimes}} \ell_q$. We recall that if E^* or F^* has the Radon–Nikodym property and one of E^* or F^* has the approximation property then, for every $b \in \mathcal{J}(E, F)$, we have $T_b \in \mathcal{N}(E, F^*)$ and the operator $b \mapsto T_b$ is an isometric isomorphism from $\mathcal{J}(E, F)$ onto $\mathcal{N}(E, F^*)$ [5].

Theorem 5.1. *Let $1 \le p, q, r < \infty$. The space $\ell_p \widehat{\widehat{\otimes}} \ell_q \widehat{\widehat{\otimes}} \ell_r$ contains a subspace isomorphic to c_0 if, and only if, $1/p + 1/q + 1/r \le 2$.*

Proof. Suppose there is no subspace isomorphic to c_0 in $\ell_p \widehat{\widehat{\otimes}} \ell_q \widehat{\widehat{\otimes}} \ell_r$. Therefore there is no subspace isomorphic to c_0 in $\ell_p \widehat{\widehat{\otimes}} \ell_q$, hence we have $1/p + 1/q > 1$. The space $\ell_p \widehat{\widehat{\otimes}} \ell_q$ contains a subspace isomorphic to ℓ_σ with $1/\sigma = 1/p + 1/q - 1$. The space $\ell_\sigma \widehat{\widehat{\otimes}} \ell_r$ does not contain a subspace isomorphic to c_0 so we have

$$\frac{1}{\sigma} + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 > 1.$$

Conversely we suppose that $1/p + 1/q + 1/r > 2$ and that $\ell_p \widehat{\widehat{\otimes}} \ell_q \widehat{\widehat{\otimes}} \ell_r$ contains a subspace isomorphic to c_0. We consider $\ell_p \widehat{\widehat{\otimes}} \ell_q \widehat{\widehat{\otimes}} \ell_r$ as the space $F_r(\ell_p \widehat{\widehat{\otimes}} \ell_q)$. We observe that none of the spaces $\ell_p \widehat{\widehat{\otimes}} \ell_q$, $\ell_p \widehat{\widehat{\otimes}} \ell_r$ or $\ell_q \widehat{\widehat{\otimes}} \ell_r$ contain a subspace isomorphic to c_0. Proceeding as in the proof of Theorem 4.1, we can find a normalized basic sequence $(x_n)_n$ of $\ell_p \widehat{\widehat{\otimes}} \ell_q \widehat{\widehat{\otimes}} \ell_r$ equivalent to the unit basis of c_0 and three strictly increasing sequences of integers $(i_n)_n$, $(j_n)_n$ and $(k_n)_n$ such that $i_0 = j_0 = k_0$ and satisfying, for $n = 1, 2, \ldots$, condition (4.1).

Now we show that the existence of these sequences leads to a contradiction. We proceed as in the $\ell_p \widehat{\otimes} \ell_q$ case. For every integer n we have

$x_n = (R_{k_n} - R_{k_{n-1}})(x_n)$ so there exists a sequence $(u_l)_l$ in $\ell_p \mathbin{\widehat{\widehat{\otimes}}} \ell_q$ such that

$$x_n = (0, \ldots, 0, u_{k_{n-1}+1}, \ldots, u_{k_n}, 0, 0, \ldots).$$

For every integer n, there exists $b_n \in [\ell_p \widehat{\widehat{\otimes}} \ell_q]^* = \mathcal{J}(\ell_p, \ell_q)$ such that $\|b_n\|_{\text{int}} = 1$ and

$$\|x_n\| = \Big[\sum_{l=k_{n-1}+1}^{k_n} |b_n(u_l)|^r\Big]^{1/r} = 1.$$

For $n = 1, 2, \ldots$ and $k_{n-1} + 1 \le l \le k_n$ we have

$$b_n(u_l) = b_n([(P_{i_n} - P_{i_{n-1}}) \otimes (Q_{j_n} - Q_{j_{n-1}})](u_l)). \tag{5.1}$$

It follows from condition (5.1) that we may suppose $T_{b_n} = (Q^*_{j_n} - Q^*_{j_{n-1}}) \circ T_{b_n} \circ (P_{i_n} - P_{i_{n-1}})$. This last assumption implies that for $n \neq m$ and $l \in \{k_{m-1} + 1, \ldots, k_m\}$, we have $b_n(u_l) = 0$.

Let N be an integer, $\alpha_1, \ldots, \alpha_N$ be scalars and let $b = \alpha_1 b_1 + \cdots + \alpha_N b_N$. We have $[\sum_{l=1}^{k_N} |b(u_l)|^r]^{1/r} = [\sum_{n=1}^{N} |\alpha_n|^r]^{1/r}$, so

$$\|x_1 + \cdots + x_N\| \ge \Theta(N) = \sup\Big\{\Big[\sum_{n=1}^{N} |\alpha_n|^r\Big]^{1/r} ; \Big\|\sum_{n=1}^{N} \alpha_n b_n\Big\|_{\text{int}} \le 1\Big\}.$$

The integral forms $b_1, \ldots, b_N$ may be considered as integral forms on $\ell_p^{i_N} \times \ell_q^{j_N}$. In this case, $\mathcal{J}(\ell_p^{i_N}, \ell_q^{j_N}) = \mathcal{N}(\ell_p^{i_N}, \ell_{q'}^{j_N})$, so $(T_{b_n})_{1\le n\le N}$ is a finite sequence of nuclear block operators from $\ell_p^{i_N}$ to $\ell_{q'}^{j_N}$.

The assumption $1/p + 1/q + 1/r > 2$ implies $1/p + 1/q > 1$, hence $q' > p$. In the cases $q' < \infty$ or $q' = \infty$ and $1 < p$ we let

$$\sigma = \begin{cases} \dfrac{pq'}{pq' + p - q'} & \text{if } q' < \infty, \\[2ex] \dfrac{p}{p-1} & \text{if } 1 < p < q' = \infty. \end{cases}$$

We observe that always $\sigma > r$. By Lemma 3.4 we have

$$\Big\|\sum_{n=1}^{N} \alpha_n b_n\Big\|_{\text{int}} = \Big\|\sum_{n=1}^{N} \alpha_n T_{b_n}\Big\|_{\text{nuc}} = \Big[\sum_{n=1}^{N} |\alpha_n|^\sigma\Big]^{1/\sigma}$$

and by Lemma 3.3 we have

$$\Theta(N) = \sup\Big\{\Big[\sum_{n=1}^{N} |\alpha_n|^r\Big]^{1/r} ; \Big[\sum_{n=1}^{N} |\alpha_n|^\sigma\Big]^{1/\sigma} \le 1\Big\} = N^{(\sigma-r)/\sigma r}.$$

We deduce that $\|x_1 + \cdots + x_N\| \ge N^{(\sigma-r)/\sigma r}$, in contradiction with $(x_n)_n$ being equivalent to the unit vector basis of c_0.

In the case $p = 1$ and $q' = \infty$, we have

$$\Theta(N) = \Big\|\sum_{n=1}^{N} \alpha_n b_n\Big\|_{\text{int}} = \Big\|\sum_{n=1}^{N} \alpha_n T_{b_n}\Big\|_{\text{nuc}} = \max_{1\le n\le N} |\alpha_n|.$$

In this case, by Lemma 3.3, $\|x_1 + \cdots + x_N\| \geq N^{1/r}$ so the sequence $(x_n)_n$ is not equivalent to the unit vector basis of c_0.

The assumptions that $1/p+1/q+1/r > 2$ and that $\ell_p \hat{\hat{\otimes}} \ell_q \hat{\hat{\otimes}} \ell_r$ contains a subspace isomorphic to c_0 lead to a contradiction. The theorem is proved.

Corollary 14 of [11] implies:

THEOREM 5.2. *Let $1 \leq p, q, s$ be real numbers such that $1/p + 1/q > 1 + 1/s$. Then every operator from ℓ_s into $\ell_p \hat{\hat{\otimes}} \ell_q$ is compact. The same is true for every operator from c_0 into $\ell_p \hat{\hat{\otimes}} \ell_q$.*

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