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ABELIAN GROUPS OF ZERO ADJOINT ENTROPY

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Abstract. The notion of adjoint entropy for endomorphisms of an Abelian group is somehow dual to that of algebraic entropy. The Abelian groups of zero adjoint entropy, i.e. ones whose endomorphisms all have zero adjoint entropy, are investigated. Torsion groups and cotorsion groups satisfying this condition are characterized. It is shown that many classes of torsionfree groups contain groups of either zero or infinite adjoint entropy. In particular, no characterization of torsionfree groups of zero adjoint entropy is possible. It is also proved that the mixed groups of a wide class all have infinite adjoint entropy.

Introduction. The notion of algebraic entropy, introduced in 1965 by Adler, Konheim and McAndrew [AKM] and recently investigated in a series of papers [DGSZ], [SZ1], [SZ2], [AADGH] and [G], has a sort of dual version if, instead of using finite subgroups in its definition, one uses subgroups of finite index and the natural modifications in the defining dual process.

This dual notion has recently been developed in [DGS] and is called "adjoint entropy". The reason for this name is two-fold. First, the adjoint entropy is not a real dual notion with respect to the algebraic entropy. Second, and more important, the adjoint entropy $\mathrm{ent}^*(\phi)$ of an endomorphism ϕ of the group G coincides with the algebraic entropy $\mathrm{ent}(\phi^*)$ of the adjoint endomorphism ϕ^* of the Pontryagin dual G^* of G (see [DGS]).

In the investigation of the algebraic entropy in [DGSZ], a relevant feature was the study of the Abelian groups G such that the algebraic entropy of every $\phi \in \mathrm{End}(G)$, the ring of endomorphisms of G, is zero. This condition is expressed by saying that G has zero algebraic entropy. Section 5 in [DGSZ] was devoted to those groups. The analogous problem for the adjoint algebraic entropy seems of interest, too. So our goal in this paper is to investigate the Abelian groups of zero adjoint entropy, i.e., those G such that $\mathrm{ent}^*(G) = 0$ (which means $\mathrm{ent}^*(\phi) = 0$ for all $\phi \in \mathrm{End}(G)$).

In Section 1 we fix the notation and recall some preliminary results, mostly taken from [DGS]. Section 2 is devoted to the study of the tor-

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sion case. The main result states that a reduced torsion group G satisfies $\mathrm{ent}^*(G) = 0$ exactly if all its primary components are finite.

Section 3 deals with the torsionfree case. We prove that some conditions on a torsionfree group G are enough to ensure that $\mathrm{ent}^*(G) = 0$, for instance if G is the direct sum of a rigid system of groups. But we will see that, in general, no characterization of the torsionfree groups G of zero adjoint entropy is possible. In particular, the assumption that G is indecomposable with $\mathrm{End}(G)$ countable, or that $\mathrm{End}(G)$ is commutative and not algebraic over $\mathbb{Q}$, or that G itself is countable and superdecomposable, or that G has a preassigned cardinality, are all compatible with either $\mathrm{ent}^*(G) = 0$ or $\mathrm{ent}^*(G) = \infty$. We also exhibit a torsionfree group G with $2^{\aleph_0}$ subgroups of finite index satisfying $\mathrm{ent}^*(G) = 0$. This example is motivated by the fact, proved in [DGS], that a group G with countably many cofinite subgroups has zero adjoint entropy.

In Section 4 we examine the relationship between the property $\mathrm{ent}^*(G) = 0$ and the similar property for the localizations G_p of G, for p a prime number. We prove that a reduced cotorsion group C satisfies $\mathrm{ent}^*(C) = 0$ if and only if it is algebraically compact with finitely generated p-adic components, that is, if it is compact in the natural topology. Furthermore, we show that all groups G in a vast class of nonsplitting mixed Abelian groups investigated in [FG] satisfy $\mathrm{ent}^*(G) = \infty$. This result leads us to conjecture that if a mixed group G has zero adjoint entropy, then all its primary components are finite, analogously to the torsion case.

1. Preliminaries. All groups considered in this paper are Abelian. For standard notions and results in Abelian group theory, we refer to Fuchs's classical volumes [F1], [F2].

We denote by $\mathbb{Z}_p$ the localization of the ring of integers $\mathbb{Z}$ at the prime p, by J_p the ring of p-adic integers, and by $\mathbb{F}_p$ the field with p elements. Of course, any Abelian p-group is canonically a $\mathbb{Z}_p$-module and a J_p-module. For G an Abelian group, $t(G)$ denotes its torsion subgroup, and $t_p(G)$ the p-primary component of $t(G)$. Moreover, $d(G)$ denotes the maximal divisible subgroup of G, and $\mathrm{End}(G)$ its endomorphism ring. The cotorsion hull of G is denoted by $G^\bullet$.

A description of the notion of *algebraic entropy* for Abelian groups and of its basic properties may be found in [DGSZ]. Generalizations of this concept to modules over commutative domains were investigated in [SZ2].

For G an Abelian group, we denote by $\mathcal{C}(G)$ the set of subgroups of G of finite index. We recall the definition of *adjoint entropy*, first introduced in [DGS].

Let G be an Abelian group, $\phi \in \mathrm{End}(G)$, and $N \in \mathcal{C}(G)$. We define the nth *cotrajectory* of N with respect to ϕ as

$$C_n(\phi, N) = \frac{G}{N \cap \phi^{-1}N \cap \cdots \cap \phi^{-n+1}N}$$

(note that $N \cap \phi^{-1}N \cap \cdots \cap \phi^{-n+1}N \in \mathcal{C}(G)$).

Then the *adjoint entropy* of ϕ is defined as

$$\mathrm{ent}^*(\phi) = \sup\{\lim_{n\to\infty} |C_n(\phi, N)|/n : N \in \mathcal{C}(G)\}.$$

If all the endomorphisms $\phi : G \to G$ satisfy the condition $\mathrm{ent}^*(\phi) = 0$, we say that G *has zero adjoint entropy*, and we write $\mathrm{ent}^*(G) = 0$. Trivially, a divisible group D has zero adjoint entropy, since $\mathcal{C}(D) = \{D\}$. More generally, we define $\mathrm{ent}^*(G)$ to be the supremum of $\mathrm{ent}^*(\phi)$ over $\phi \in \mathrm{End}(G)$. Recall that in [DGS] it is proved that for any Abelian group G either $\mathrm{ent}^*(G) = 0$ or $\mathrm{ent}^*(G) = \infty$ (see Theorem 1.1 below).

The various properties and results on adjoint entropy we will need are taken from the paper [DGS]. However, we emphasize the following two properties, that are used throughout the present paper.

(a) Let $G = H \oplus K$ be an Abelian group, and $\phi \in \mathrm{End}(G)$ an endomorphism such that H, K are ϕ-invariant. Then $\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\phi|_H) + \mathrm{ent}^*(\phi|_K)$.

(b) Let G be an abelian group that has a direct summand H such that $\mathrm{ent}^*(H) = \infty$. Then $\mathrm{ent}^*(G) = \infty$.

The proof of property (a) may be found in [DGS]; property (b) is a straightforward consequence of (a).

For the convenience of the reader, and for further reference, we reformulate here the fundamental Theorems 7.5 and 7.6 of [DGS] (see also Corollary 7.7 of that paper).

THEOREM 1.1 ([DGS]). *If G is an Abelian group and $\phi \in \mathrm{End}(G)$, then either $\mathrm{ent}^*(\phi) = 0$ or $\mathrm{ent}^*(\phi) = \infty$. Moreover $\mathrm{ent}^*(\phi) = 0$ if and only if, for every prime number p, the induced endomorphism $\bar\phi \in \mathrm{End}(G/pG)$ is algebraic over $\mathbb{F}_p$.*

From the above theorem it obviously follows that $\mathrm{ent}^*(G) = \infty$ if and only if $\mathrm{ent}^*(\phi) = \infty$ for some $\phi \in \mathrm{End}(G)$.

The next consequence of Theorem 1.1 shows that, for torsionfree groups, the property of having adjoint entropy zero can be detected by just looking at the endomorphism ring. As shown in the forthcoming paper [GS], the analogous result is not true for the so-called rank-entropy, that was investigated in [SZ1].

Corollary 1.2. *Let G be a torsionfree group and $A = \mathrm{End}(G)$ its endomorphism ring. Then* $\mathrm{ent}^*(G) = 0$ *if and only if A is residually algebraic, that is, A/pA is algebraic over $\mathbb{F}_p$ for each prime p.*

Proof. Using the notation of Theorem 1.1, the map $A \to \mathrm{End}(G/pG)$, $\phi \mapsto \bar{\phi}$, has kernel $\mathrm{Hom}(G, pG)$. Since G is torsionfree, we have $\mathrm{Hom}(G, pG) = pA$, hence $\bar{\phi} = \phi + pA$ (note that, without the assumption that G is torsionfree, only the inclusion $pA \subseteq \mathrm{Hom}(G, pG)$ is valid). Thus, for any given $\phi \in A$ and for each prime p, the induced endomorphism $\bar{\phi} \in \mathrm{End}(G/pG)$ is algebraic over $\mathbb{F}_p$ exactly if $\bar{\phi} = \phi + pA$ is an algebraic element of A/pA. Thus the claim follows. ∎

It is worthwhile giving an example of a torsion group G such that $p\,\mathrm{End}(G) \subsetneq \mathrm{Hom}(G, pG)$. Let $G = \bigoplus_{i>0} \langle z_i \rangle$, where $\langle z_i \rangle \cong \mathbb{Z}/p^i\mathbb{Z}$ for all $i > 0$, and let $\phi \in A = \mathrm{End}(G)$ be defined by extending the assignments $x_i \mapsto px_{i+1}$. Then $\phi \in \mathrm{Hom}(G, pG)$ and is injective, hence $\phi \notin pA$, since the elements of pA annihilate the socle of G.

We recall that in [DGS] a group G was called *narrow* if $\mathcal{C}(G)$ is countable. In the following result, which is part of Theorem 3.3 in [DGS], we recall the characterization of narrow groups that we will use later.

Proposition 1.3 ([DGS]). *Let G be a reduced Abelian group. The following are equivalent:*

(1) *G is narrow;*
(2) *G/pG is finite for every prime number p.*

Narrow groups have zero adjoint entropy, as proved in [DGS, Proposition 3.7].

Proposition 1.4 ([DGS]). *Every narrow group G satisfies* $\mathrm{ent}^*(G) = 0$.

The next result follows from the discussion in Section 6 of [DGS] (in particular, see Proposition 6.2).

Proposition 1.5 ([DGS]). *If a p-group G is an infinite direct sum of cyclic groups, then* $\mathrm{ent}^*(G) = \infty$. *Moreover, if a group G is an infinite direct sum of copies of the same group, then also* $\mathrm{ent}^*(G) = \infty$.

We remark that the preceding proposition also follows from Theorem 1.1. In fact, in both the cases considered in Proposition 1.5, one readily finds a $\phi \in \mathrm{End}(G)$ such that $\bar{\phi} \in \mathrm{End}(G/pG)$ is not algebraic over $\mathbb{F}_p$.

We recall the notion of basic submodule for $\mathbb{Z}_p$-modules, the case we are interested in (see [F1, Ch. VI]). Let G be a $\mathbb{Z}_p$-module; then there exists a submodule B of G such that:

(i) B is a direct sum of cyclic $\mathbb{Z}_p$-modules;
(ii) B is pure in G;
(iii) G/B is divisible.

Such a B is said to be a *basic submodule* of G; all basic submodules of G are isomorphic.

2. Adjoint entropy for torsion groups. We start this section with some general results on adjoint entropy, valid for any Abelian group.

PROPOSITION 2.1. *Let G be an Abelian group, and B a pure subgroup of G such that G/B is divisible. Then there is a one-to-one correspondence $\Psi : \mathcal{C}(G) \to \mathcal{C}(B)$ given by the assignment $\Psi : N \mapsto N \cap B$ for $N \in \mathcal{C}(G)$. For $M \in \mathcal{C}(B)$, we have $\Psi^{-1} : M \mapsto M + r_M G$, where $r_M = |B/M|$.*

Proof. If $N \in \mathcal{C}(G)$, then $N \cap B \in \mathcal{C}(B)$, without special assumptions on the subgroup B. Now, we can say more. In fact, since G/B is divisible, for every $N \in \mathcal{C}(G)$ we have $B + N \supseteq B + mG = G$, where $m = |G/N|$. It follows that $B/(N \cap B) \cong G/N$; in particular, $m = r_{N\cap B}$. Then we get $N \cap B + r_{N\cap B}G = N \cap (B + r_{N\cap B}G) = N$, since $N \supseteq r_{N\cap B}G$. For $M \in \mathcal{C}(B)$, we have $(M + r_M G) \cap B = M + r_M G \cap B = M + r_M B = M$, since B is pure in G and $r_M B \subseteq M$. As $G = B + r_M G$, we get

$$\frac{G}{M + r_M G} = \frac{B + r_M G}{M + r_M G} \cong \frac{B}{B \cap (M + r_M G)} = B/M,$$

so $M + r_M G \in \mathcal{C}(G)$. We have also seen that Ψ is a bijection, hence the desired conclusion follows. ∎

It is worth noting that the preceding proposition is not true if we just assume that B is pure, but G/B is not divisible. Easy counterexamples of $M \in \mathcal{C}(B)$ with $M + r_M G \notin \mathcal{C}(G)$ (even in case of p-groups) are provided when $G = B \oplus G'$, and $G'/r_M G'$ is infinite.

PROPOSITION 2.2. *Let G be an Abelian group, $\phi \in \mathrm{End}(G)$, and B a ϕ-invariant pure subgroup of G. Then $\mathrm{ent}^*(\phi) \geq \mathrm{ent}^*(\phi|_B)$.*

Proof. In view of Theorem 1.1, it clearly suffices to show that $\mathrm{ent}^*(\phi) = 0$ implies $\mathrm{ent}^*(\phi|_B) = 0$. Now, the induced endomorphism $\bar{\phi} \in \mathrm{End}(G/pG)$ is algebraic over $\mathbb{F}_p$, again by Theorem 1.1. Equivalently, there exists a monic polynomial $f(X) \in \mathbb{Z}[X]$ such that $f(\phi)(G) \subseteq pG$. Hence, for every $b \in B$ we have $f(\phi)(b) \in pG \cap B = pB$ (since B is pure in G and ϕ-invariant). It follows that $\phi|_B$ induces an endomorphism of B/pB algebraic over $\mathbb{F}_p$, and hence $\mathrm{ent}^*(\phi|_B) = 0$. ∎

EXAMPLE 2.3. It is easy to show that the inequality in Proposition 2.2 is no longer true without the assumption that B is a pure subgroup. For instance, let C be a cyclic group, and D its divisible envelope. Let $B = \bigoplus_{i>0} C_i$ and $G = \bigoplus_{i>0} D_i$, where $C_i \cong C$ and $D_i \cong D$ for all i. Let us take any $\psi \in \mathrm{End}(B)$ such that $\mathrm{ent}^*(\psi) = \infty$ (ψ exists by Proposition 1.5), and consider its extension $\phi \in \mathrm{End}(G)$. Then $\mathrm{ent}^*(\phi) = 0$, since G is divisible, hence we have $\mathrm{ent}^*(\phi) < \mathrm{ent}^*(\phi|_B) = \mathrm{ent}^*(\psi)$.

In the case when B is pure in G and G/B is divisible, we have a result stronger than Proposition 2.2.

PROPOSITION 2.4. *Let G be an Abelian group, $\phi \in \mathrm{End}(G)$, and B a ϕ-invariant pure subgroup of G such that G/B is divisible. Then* $\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\phi|_B)$.

Proof. To simplify the notation, we will write $\psi = \phi|_B$. Proposition 2.1 shows that there is a one-to-one correspondence $\Psi : \mathcal{C}(G) \to \mathcal{C}(B)$ given by $M = N \cap B$ and $N = M + r_M G$, for $N \in \mathcal{C}(G)$ and $M \in \mathcal{C}(B)$. Since G/B is divisible, we have $G = B + mG$ for every nonzero integer m. Hence we also get $G = B + L$ for all $L \in \mathcal{C}(G)$.

Pick any $M \in \mathcal{C}(B)$. We consider the nth cotrajectory of $N = M + r_M G$ with respect to ϕ,

$$C_n(\phi, N) = \frac{G}{N \cap \phi^{-1}N \cap \cdots \cap \phi^{-n+1}N}$$

and the nth cotrajectory of $M = N \cap B$ with respect to ψ,

$$D_n(\psi, M) = \frac{B}{M \cap \psi^{-1}M \cap \cdots \cap \psi^{-n+1}M}\ .$$

Our purpose is to show that $|D_n(\psi, M)| \geq |C_n(\phi, N)|$.

It is readily checked that $\phi^{-k}N \cap \phi^{-k}B = \phi^{-k}M$ for every $k \geq 0$, hence

$$\bigcap_{0\le i<n} \phi^{-i}M = \bigcap_{0\le i<n} \phi^{-i}N \cap \bigcap_{0\le i<n} \phi^{-i}B = \Big(\bigcap_{0\le i<n} \phi^{-i}N\Big) \cap B,$$

where the last equality holds since $\phi^{-i}B \supseteq B$ for all i. As $\phi^{-k}M \supseteq \psi^{-k}M$ for all $k \geq 0$, it follows that $|D_n(\psi, M)| \geq |B/H|$, where $H = (N \cap \phi^{-1}N \cap \cdots \cap \phi^{-n+1}N) \cap B$. But $B + (N \cap \phi^{-1}N \cap \cdots \cap \phi^{-n+1}N) = G$ implies $B/H \cong C_n(\phi, N)$, and the desired inequality follows.

Since M was arbitrary, and $\Psi : \mathcal{C}(G) \to \mathcal{C}(B)$ is one-to-one, we easily conclude that $\mathrm{ent}^*(\psi) \geq \mathrm{ent}^*(\phi)$. The converse inequality follows from Proposition 2.2, hence $\mathrm{ent}^*(\psi) = \mathrm{ent}^*(\phi)$. ∎

EXAMPLE 2.5. The assumption that G/B is divisible in the preceding proposition cannot be avoided. For instance, take groups B, G_1, and endomorphisms $\psi \in \mathrm{End}(B)$ with $\mathrm{ent}^*(\psi) = 0$ and $\phi_1 \in \mathrm{End}(G_1)$ with $\mathrm{ent}^*(\phi_1) = \infty$. We consider the group $G = B \oplus G_1$ and its endomorphism $\phi = \psi \oplus \phi_1$. Note that B is pure in G, but G/B is not divisible, since $\mathrm{ent}^*(G_1) = \infty$. Using property (a) of the adjoint entropy, we see that $\infty = \mathrm{ent}^*(\phi) > \mathrm{ent}^*(\phi|_B) = 0$.

Now we can prove the main result of this section.

THEOREM 2.6. *Let G be a reduced p-group such that* $\mathrm{ent}^*(G) = 0$. *Then G is finite.*

Proof. We will show that $\mathrm{ent}^*(G) = \infty$ whenever G is an infinite p-group. Let B be a basic subgroup of G. If B is bounded, then $G = B$ and therefore G is an infinite direct sum of cyclic p-groups. In this case we know, by Proposition 1.5, that $\mathrm{ent}^*(G) = \infty$. So we assume that B is unbounded. We will construct $\phi \in \mathrm{End}(G)$ such that $\mathrm{ent}^*(\phi) = \infty$. The argument is based on a result by Szele [S] that yields a nonzero $\phi \in \mathrm{End}(G)$ such that $\phi(G) \subseteq B$. For $g \in G$, we denote by $e(g)$ the exponent of g, that is, $\mathbb{Z}g \cong \mathbb{Z}/p^{e(g)}\mathbb{Z}$. We may decompose B as a direct sum $B = B' \oplus B''$, where $B' = \bigoplus_{n>0} \mathbb{Z}b_n$ and $e(b_{n+1}) \geq 2e(b_n)$ for all $n > 0$. We define $\psi \in \mathrm{End}(B)$ by extending the assignments

$$b_1 \mapsto 0; \qquad b_{n+1} \mapsto b_n; \qquad a \mapsto 0,\ \forall a \in B''.$$

Note that $\mathrm{ent}^*(\psi) = \infty$. In fact, it is easily seen that the induced endomorphism $\bar{\psi} \in \mathrm{End}(B/pB)$ is not algebraic over $\mathbb{F}_p$, hence Theorem 1.1 applies (for a more general argument see Proposition 6.2 in [DGS]).

We need the following fact, which can be proved using standard arguments of Abelian p-group theory. For every $k > 0$, let $B_k = \bigoplus_{1\leq n\leq k} \mathbb{Z}b_n$; then there exists a direct decomposition $G = B_k \oplus G_k$, where $G_k \supseteq \bigoplus_{j\geq k+1} \mathbb{Z}b_j \oplus B''$.

Now we define a map $\phi : G \to G$ in the following way. For $0 \neq g \in G$, let $m > 0$ be such that $e(b_m) \geq e(g)$. Since $G = B_m \oplus G_m$, we may write $g = b + c$, where $b \in B_m$ and $c \in G_m$. Then we set $\phi(g) = \psi(b)$.

For the sake of completeness, we verify that $\phi(g)$ is well-defined, not depending on the choice of m (cf. Szele [S]). Write $g = b' + c'$ with $b' \in B_k$ and $c' \in G_k$, where we may assume that $k > m$. Then we have $b' = b + b''$ for a suitable $b'' \in \bigoplus_{m<n\leq k} \mathbb{Z}b_n$. In order to verify that ϕ is well-defined, it suffices to show that $\psi(b'') = 0$. We may assume that $b'' \neq 0$. Note that all the exponents $e(b), e(b'), e(b'')$ are $\leq e(g)$. We have $b'' = \sum_{i=m+1}^{k} \alpha_i b_i$ for suitable $\alpha_i \in \mathbb{Z}$. Since, for $m < i \leq k$, we have $e(b_i) > 2e(b_m) \geq 2e(g)$ and $e(b'') \leq e(g)$, necessarily α_i is divisible by p^{e_i} for a suitable e_i satisfying $2e_i \geq e(b_i)$. It follows that $\psi(\alpha_i b_i) = \alpha_i b_{i-1} = 0$, since $e(b_i) \geq 2e(b_{i-1})$. Hence $\psi(b'') = 0$, as desired.

It is now clear that ϕ is an endomorphism of G. Since $\mathrm{ent}^*(\psi) = \infty$, B is pure in G and G/B is divisible, either from Proposition 2.2 or from Proposition 2.4 we derive $\mathrm{ent}^*(\phi) = \infty$, and the desired conclusion follows. ∎

The following corollary of the preceding theorem is easily verified, using property (a) of the adjoint entropy.

COROLLARY 2.7. *Let G be a reduced torsion group. Then* $\mathrm{ent}^*(G) = 0$ *if and only if each p-primary component of G is finite.*

The following fact on endomorphism rings appears to have gone unnoticed until now.

COROLLARY 2.8. *Let G be a reduced p-group such that* $\mathrm{End}(G)$ *is integral over J_p. Then G is finite.*

Proof. If $\mathrm{End}(G)$ is integral over J_p, then for every $\phi \in \mathrm{End}(G)$, the induced linear transformation $\bar{\phi} : G/pG \to G/pG$ is algebraic over $\mathbb{F}_p$. By Theorem 1.1, this suffices to ensure that $\mathrm{ent}^*(G) = 0$. Then G is finite, by Theorem 2.6. ∎

REMARK 2.9. In [DGSZ] one finds many examples of (infinite) p-groups G such that $\mathrm{ent}(G) = 0$. Typically, they are constructed as follows. One starts with a complete J_p-algebra A, which is integral over J_p and fulfills some technical requirements (automatically satisfied when A is torsionfree of finite rank). Then deep realization theorems due to Corner [C2] ensure the existence of a p-group G such that $\mathrm{End}(G) = A \oplus E_s(G)$, where $E_s(G)$ is the ideal of *small endomorphisms* of G (see [DGSZ] for definitions and other details). Actually, we have no control on $E_s(G)$, but, in view of [DGSZ, Theorem 5.2], or [SZ1, Proposition 3.4], we know that the small endomorphisms are *pointwise integral* (see [DGSZ] for this notion). Then, since A is integral over J_p, one gets $\mathrm{ent}(G) = 0$ ([DGSZ, Corollary 5.11]). However, the preceding corollary provides some new information on small endomorphisms. Namely, as soon as G is an infinite p-group and $\mathrm{End}(G) = A \oplus E_s(G)$, where A is integral over J_p, then there exist some small endomorphisms of G which are pointwise integral, but not integral.

3. Adjoint entropy for torsionfree groups. Let A be a commutative torsionfree $\mathbb{Z}$-algebra; we say that $s \in A$ is *algebraic* over $\mathbb{Q}$ if there exists a nonzero polynomial $f \in \mathbb{Q}[X]$ such that $f(s) = 0$ (of course, $f(s)$ just lies in $A \otimes \mathbb{Q}$, in general).

The next two propositions follow from Theorem 1.1.

PROPOSITION 3.1. *If G is a torsionfree Abelian group of finite rank, then* $\mathrm{ent}^*(G) = 0$.

Proof. Every $\phi \in \mathrm{End}(G)$ may be extended to a linear transformation of $\mathbb{Q}^m$, where m is the rank of G. It follows that ϕ is algebraic over $\mathbb{Q}$, hence ϕ annihilates a nonzero polynomial $f = \sum_{i=0}^{n} a_i X^i \in \mathbb{Z}[X]$, where, since G is torsionfree, we may assume that $a_0, a_1, \ldots, a_n$ are coprime integers. Note that, for each prime p, this polynomial, once reduced modulo p, remains nontrivial, since the a_i are coprime. We readily conclude that the induced endomorphism $\bar{\phi} \in \mathrm{End}(G/pG)$ is algebraic over $\mathbb{F}_p$, for all prime numbers p. Then Theorem 1.1 applies, yielding $\mathrm{ent}^*(\phi) = 0$. Since ϕ was arbitrary, we conclude that $\mathrm{ent}^*(G) = 0$. ∎

We recall that the preceding fact is also shown in Example 3.8(b) of [DGS].

PROPOSITION 3.2. *If G is a torsionfree Abelian group with commutative endomorphism ring* $\mathrm{End}(G) = \mathbb{Z}[s_i : i \in \lambda]$, *where the s_i are algebraic over $\mathbb{Q}$, then* $\mathrm{ent}^*(G) = 0$. *If, on the other hand, G is such that* $\mathrm{End}(G) = \mathbb{Z}[t]$, *with t transcendental over $\mathbb{Q}$, then* $\mathrm{ent}^*(G) = \infty$.

Proof. In the first case of our statement, every $\phi \in \mathrm{End}(G)$ is algebraic over $\mathbb{Q}$, hence, arguing as in the proof of Proposition 3.1, we see that the induced endomorphism $\bar{\phi} \in \mathrm{End}(G/pG)$ is algebraic over $\mathbb{F}_p$ for all prime numbers p. Then $\mathrm{ent}^*(\phi) = 0$, by Theorem 1.1.

In the second case, from $\mathrm{End}(G) = \mathbb{Z}[t]$ with t transcendental, one finds that $\bar{t} \in \mathrm{End}(G/pG)$ is transcendental over $\mathbb{F}_p$ (for any p). In fact, assume for a contradiction that $\bar{t}$ is algebraic. Equivalently, there exists a monic polynomial $f \in \mathbb{Z}[t]$ such that $f(t)(G) \subseteq pG$. Since G is torsionfree, this means that $f(t)/p \in \mathrm{End}(G) = \mathbb{Z}[t]$, impossible. We conclude that $\mathrm{ent}^*(t) = \infty$, again by Theorem 1.1. ∎

COROLLARY 3.3. *There exist indecomposable torsionfree Abelian groups G such that* $\mathrm{End}(G)$ *is countable and* $\mathrm{ent}^*(G) = \infty$. *There exist indecomposable torsionfree Abelian groups G_1 of infinite rank such that* $\mathrm{End}(G_1)$ *is countable and* $\mathrm{ent}^*(G_1) = 0$.

Proof. Let t be transcendental over $\mathbb{Q}$, and let $A = \mathbb{Z}[t]$. Then Corner's Theorem A (see [C1]) ensures the existence of a countable torsionfree group G such that $\mathrm{End}(G) = A$. Thus $\mathrm{ent}^*(G) = \infty$ by the preceding proposition. Let now $\{s_i\}_{i>0}$ be a countable family of elements algebraic over $\mathbb{Q}$ and having unbounded degrees. Let $A_1 = \mathbb{Z}[s_i]_{i>0}$. Again by Corner's Theorem A, there exists G_1 such that $\mathrm{End}(G_1) = A_1$. Then $\mathrm{ent}^*(G_1) = 0$, by Proposition 3.2. Note that G_1 has infinite rank, since otherwise all its endomorphisms would have degrees bounded by the rank of G_1. ∎

In some special cases, we can be more precise, as shown by the next result, whose proof is contained in Examples 3.6 and 3.9 of [DGS].

PROPOSITION 3.4. *Let G be a torsionfree group such that* $\mathrm{End}(G) = A \subseteq \mathbb{Q}$. *Then every subgroup of G of finite index is fully invariant. As a consequence,* $\mathrm{ent}^*(G) = 0$.

The condition that $\mathrm{End}(G)$ is algebraic over $\mathbb{Q}$, as in Proposition 3.2, is not necessary to have zero adjoint entropy.

PROPOSITION 3.5. *There exist (countable) torsionfree Abelian groups G such that* $\mathrm{End}(G)$ *is not algebraic over $\mathbb{Q}$, and* $\mathrm{ent}^*(G) = 0$.

Proof. The easiest example is possibly $G = J_p$. The fact that $\mathrm{ent}^*(J_p) = 0$ is shown in [DGS] (or see the remainder of our proof). Now take any $t \in J_p$

that is transcendental over $\mathbb{Q}$. To conclude, it suffices to regard t as an element of $\mathrm{End}(J_p)$.

If we require that G is countable, we may argue as follows. For every prime p, let r_p be a complex number of degree p over $\mathbb{Q}$. We consider $G = \bigoplus_p \mathbb{Z}_p[r_p]$, where p ranges among all prime numbers. Since $\mathbb{Z}_p[r_p]$ is a finitely generated $\mathbb{Z}_p$-algebra and $q\mathbb{Z}_p[r_p] = \mathbb{Z}_p[r_p]$ for every prime $q \neq p$, we readily see that G/pG is an $\mathbb{F}_p$-vector space of dimension p, for every prime p. Hence, for every $\phi \in \mathrm{End}(G)$, the induced endomorphism $\bar{\phi}$ of G/pG is algebraic over $\mathbb{F}_p$.

It follows that $\mathrm{ent}^*(G) = 0$, by Theorem 1.1. It remains to show that $\mathrm{End}(G)$ is not algebraic over $\mathbb{Q}$. Regard the element $r = (r_p)_p \in \prod_p \mathbb{Z}_p[r_p]$ as an endomorphism of G. Since each component r_p of r has degree p over $\mathbb{Q}$, it is easily seen that r is transcendental over $\mathbb{Q}$. ∎

Remark 3.6. We note that, for $G = \bigoplus_p \mathbb{Z}_p[r_p]$ as in the above proof, we have $\mathrm{End}(G) \supseteq \mathbb{Z}[r]$, where r is transcendental, and, nonetheless, $\mathrm{ent}^*(G) = 0$. Therefore, in the statement of Proposition 3.2, the condition $\mathrm{End}(G) = \mathbb{Z}[t]$ cannot be weakened to $\mathrm{End}(G) \supseteq \mathbb{Z}[t]$.

We recall that an Abelian group G is said to be *superdecomposable* if every nonzero direct summand of G admits a nontrivial decomposition as a direct sum.

Proposition 3.7.

(i) *There exist countable superdecomposable torsionfree Abelian groups G such that* $\mathrm{ent}^*(G) = 0$.
(ii) *There exist countable superdecomposable torsionfree Abelian groups G_1 such that* $\mathrm{ent}^*(G_1) = \infty$.

Proof. (i) The construction of G is inspired by that of a superdecomposable torsionfree group, made in Corner's paper [C1, Theorem 5.2 and Lemma 5.3]. We make use of an algebra more easily defined than that introduced by Corner. Let E_i, $i > 0$, be indeterminates over $\mathbb{Q}$, and consider the $\mathbb{Z}$-algebra $\mathbb{Z}[E_i : i > 0]$ and its ideal $I = \langle E_i^2 - E_i : i > 0\rangle$. Then the ring $A = \mathbb{Z}[E_i : i > 0]/I$ is generated by the idempotents $e_i = E_i + I$. Let us show that A has the following property:

(a) Every nonzero element f of A can be written as a polynomial in the e_i, $i \in F$, for some finite set F of positive integers, where, in each monomial of f, e_i appears with degree ≤ 1. Moreover, if $j \notin F$, then $0 \neq e_j f \neq f$.

The first assertion is obvious, since $A = \mathbb{Z}[e_i : i > 0]$, and the e_i are idempotent. Let us see that, under the above assumptions, $e_j f \neq f$.

Without loss of generality, we assume that $F = \{1, \ldots, k\}$, and we choose $g \in \mathbb{Z}[E_1, \ldots, E_k]$ such that $g + I = f$ in the simplest way, namely lifting each monomial of f. Assume now, for a contradiction, that $E_j g - g \in I$. In particular $g \in \langle E_i : i > 0\rangle$, hence it cannot have a nonzero constant term. Then, possibly reordering the indices, we may suppose that $aE_1 \cdots E_m$ is a monomial of g, where $0 \neq a \in \mathbb{Z}$ and $m \leq k$. Now all the other monomials of g (if there are any) must contain as a factor some E_i with $m < i \leq k$. Therefore, if we specialize at zero all the E_i with $i > m$ and $i \neq j$, from $(E_j - 1)g \in I$ we get the relation

$$(E_j - 1)E_1 \cdots E_m \in \langle E_1^2 - E_1, \ldots, E_m^2 - E_m, E_j^2 - E_j\rangle.$$

Now, specializing at $E_j = 0$ and $E_2 = \cdots = E_m = 1$, we get $E_1 \in (E_1^2 - E_1)\mathbb{Z}[E_1]$, impossible. In a similar way we verify that $e_j f \neq 0$.

Let us show that A does not contain primitive idempotents. Let $0 \neq e = e^2 \in A$. In view of (a), there exists a finite subset F of $\mathbb{N}$ such that e is a polynomial in the e_i, $i \in F$. We pick e_j such that $j \notin F$, and consider the decomposition $e = e_j e + (1 - e_j)e$. Note that both summands are idempotents, since A is commutative, and $0 \neq e_j e \neq e$, again by (a). We conclude that e is not primitive, as desired.

Using now Corner's Theorem A of [C1], we may find a countable torsion-free group G such that $\mathrm{End}(G) = A$. Then G is superdecomposable, since A has no primitive idempotents.

It remains to show that $\mathrm{ent}^*(G) = 0$. This fact follows from Proposition 3.2, since $A = \mathbb{Z}[e_i : i > 0]$, and each e_i is obviously algebraic over $\mathbb{Q}$.

(ii) Let us consider the ring A as in (i), and let t be an indeterminate over A. Again by Corner's Theorem A, there exists a countable torsionfree group G_1 such that $\mathrm{End}(G_1) = A_1 = A[t]$. It is easy to verify that for any $0 \neq a \in A$ we have $a^2 \neq 0$; from this it follows that if $e \in A[t]$ is idempotent, then $e \in A$. Then an argument analogous to that in (i) shows that G_1 is superdecomposable. Moreover, since $\mathrm{End}(G_1) = A[t]$, and t is transcendental over $\mathbb{Q}$, arguing as in the proof of Proposition 3.2, we see that $\mathrm{ent}^*(t) = \infty$. ∎

In the torsion setting, we have seen in the proof of Theorem 2.6 that the existence of basic subgroups forces a torsion group of zero adjoint entropy to be finite. It is natural to ask whether a similar result holds for torsionfree $\mathbb{Z}_p$-modules, where we have a corresponding notion of basic submodules. The answer is negative, as shown by our next result.

PROPOSITION 3.8. *There exists a torsionfree $\mathbb{Z}_p$-module G with basic submodules of infinite rank such that* $\mathrm{ent}^*(G) = 0$.

Proof. Just replacing $\mathbb{Z}$ with $\mathbb{Z}_p$, the same proof of Proposition 3.7(i) yields a superdecomposable $\mathbb{Z}_p$-module G with $\mathrm{ent}^*(G) = 0$ (Corner's theorems are valid for countable torsionfree $\mathbb{Z}_p$-algebras). It is readily seen that a superdecomposable $\mathbb{Z}_p$-module has infinite basic rank. ∎

We recall that a set $\{G_i : i \in \Lambda\}$ of torsionfree groups is said to be a *rigid system* if $\mathrm{End}(G_i)$ is a subring of $\mathbb{Q}$, for all $i \in \Lambda$, and $\mathrm{Hom}(G_i, G_j) = 0$ for $i \neq j$.

PROPOSITION 3.9. *Let $\{G_i : i \in \Lambda\}$ be a rigid system of torsionfree groups, and $G = \bigoplus_{i\in\Lambda} G_i$. Then* $\mathrm{ent}^*(G) = 0$.

Proof. It is well-known that $\mathrm{End}(G) = \prod_{i\in\Lambda} A_i$, where $A_i = \mathrm{End}(G_i) \subseteq \mathbb{Q}$. Hence every $\phi \in \mathrm{End}(G)$ may be viewed as a string $\phi = (a_i)_{i\in\Lambda}$, where $a_i \in A_i$. Then the induced endomorphism $\bar{\phi} \in \mathrm{End}(G/pG)$ has the form $\bar{\phi} = (\alpha_i)_{i\in\Lambda}$, where $\alpha_i = a_i + pA_i \in \mathbb{Z}/p\mathbb{Z}$. Since $\alpha_i^p = \alpha$ for every $i \in \Lambda$, it follows that $\bar{\phi}^p - \bar{\phi} = 0$, which shows that $\bar{\phi}$ is algebraic over $\mathbb{F}_p$. From Theorem 1.1 we get $\mathrm{ent}^*(G) = 0$. ∎

EXAMPLE 3.10. There exists a countable torsionfree Abelian group G such that $\mathrm{ent}^*(G) = 0$ and $\mathcal{C}(G)$ is uncountable. In fact, take $G = \bigoplus_p \mathbb{Z}[1/p]$, where p ranges over the prime numbers. It is straightforward to verify that $\{\mathbb{Z}[1/p]\}_p$ is a rigid system, hence $\mathrm{ent}^*(G) = 0$ by Proposition 3.9. Moreover, since $1/q \notin p\mathbb{Z}[1/q]$ for every prime number $q \neq p$, it is easy to verify that G/pG is infinite for every p. Then Proposition 1.3 shows that G is not narrow, hence $\mathcal{C}(G)$ is uncountable.

One could ask whether, under the hypotheses of Proposition 3.9, we can even conclude that every $N \in \mathcal{C}(G)$ is fully invariant, thus extending Proposition 3.4. This does not hold, as the following example shows.

EXAMPLE 3.11. Using standard arguments in Abelian group theory, one can easily find two torsionfree groups of rank one, say G_1 and G_2, such that $\mathrm{End}(G_i) = \mathbb{Z}$ $(i = 1, 2)$ and $\mathrm{Hom}(G_i, G_j) = 0$ $(1 \le i, j \le 2,\ i \neq j)$. Let $G = G_1 \oplus G_2$. We look for $N \in \mathcal{C}(G)$ that is not fully invariant. It is easy to verify that N is fully invariant if and only if $N = (N \cap G_1) \oplus (N \cap G_2)$ (note that $N \cap G_i$ is fully invariant in G_i, by Proposition 3.4). As a consequence of [A, Theorem 0.3, p. 3], since the G_i have rank one, we have $|G_i/pG_i| = p$ for every prime p. For $i = 1, 2$, we denote by ϕ_i the isomorphism $G_i/pG_i \to \mathbb{Z}/p\mathbb{Z}$. We consider the map $\psi : G \to \mathbb{Z}/p\mathbb{Z}$, defined by $\psi(g_1 + g_2) = \phi_1(g_1 + pG_1) + \phi_2(g_2 + pG_2)$ $(g_i \in G_i)$. Let $N = \ker(\psi)$. One readily checks that $N \cap G_i = pG_i$. Moreover $N \neq (N \cap G_1) \oplus (N \cap G_2) = pG_1 \oplus pG_2$, since $|G/N| = p$ and $|G/(G_1 \oplus G_2)| = p^2$. We conclude that $N \in \mathcal{C}(G)$ cannot be fully invariant.

The next proposition shows that there is no hope to classify all torsionfree groups of zero adjoint entropy.

PROPOSITION 3.12. *There exist torsionfree Abelian groups G of arbitrarily large cardinality such that* $\mathrm{ent}^*(G) = 0$.

Proof. There are two main ways to find such a G. First, by [EM, Theorem 3.5, p. 458], there exist Abelian groups G of arbitrarily large cardinality such that $\mathrm{End}(G) = \mathbb{Z}$. Second, by [GT, Corollary 14.5.3, p. 577], there exist rigid systems $\{G_i : i \in \Lambda\}$ with Λ arbitrarily large; then we can choose $G = \bigoplus_{i\in\Lambda} G_i$. In both cases $\mathrm{ent}^*(G) = 0$, either by Proposition 3.4, or by Proposition 3.9. ∎

4. Adjoint entropy for mixed groups. It is natural to ask whether to have zero adjoint entropy is a local property. Namely, for an Abelian group G, is it true that $\mathrm{ent}^*(G) = 0$ if and only if $\mathrm{ent}^*(G_p) = 0$ for every prime number p, where $G_p = G \otimes \mathbb{Z}_p$?

In the next result we see that sufficiency holds.

THEOREM 4.1. *Let G be an abelian group such that* $\mathrm{ent}^*(G \otimes \mathbb{Z}_p) = 0$ *for every prime number p. Then* $\mathrm{ent}^*(G) = 0$.

Proof. Pick any $\phi \in \mathrm{End}(G)$; let us see that the induced endomorphism $\bar{\phi} \in \mathrm{End}(G/pG)$ is algebraic over $\mathbb{F}_p$ for all prime numbers p. Then our statement will follow from Theorem 1.1. Let $G_p = G \otimes \mathbb{Z}_p$ and consider $\psi = \phi \otimes 1 \in \mathrm{End}(G_p)$. Then $\mathrm{ent}^*(G_p) = 0$ implies that $\bar{\psi} \in \mathrm{End}(G_p/pG_p)$ is algebraic over $\mathbb{F}_p$. Since G/pG and G_p/pG_p are canonically isomorphic, it is straightforward to show that also $\bar{\phi}$ is algebraic, as desired. ∎

In the next example we see that the above theorem is not reversible. We also see that the hypothesis that $\mathrm{ent}^*(G_p) = 0$ *for every* prime p cannot be relaxed.

EXAMPLE 4.2. (i) There exists a torsionfree group G of zero adjoint entropy such that $\mathrm{ent}^*(G_p) = \infty$, for every prime number p.

Let $G = \bigoplus_p \mathbb{Z}[1/p]$. Since, for p ranging over prime numbers, $\{\mathbb{Z}[1/p]\}_p$ is a rigid system, Proposition 3.9 shows that $\mathrm{ent}^*(G) = 0$. However, since $\mathbb{Z}[1/q] \otimes \mathbb{Z}_p = \mathbb{Z}_p$ for $q \neq p$, and $\mathbb{Z}[1/p] \otimes \mathbb{Z}_p = \mathbb{Q}$, for every prime number p we have $G_p = \bigoplus_{i\in\mathbb{N}} Z_i \oplus \mathbb{Q}$, where $Z_i \cong \mathbb{Z}_p$ for every i. Hence G_p has a direct summand which is an infinite direct sum of copies of $\mathbb{Z}_p$, and therefore $\mathrm{ent}^*(G_p) = \infty$, by Proposition 1.5.

(ii) Let S be any proper subset of the set of prime numbers. Then there exists a torsionfree group G such that $\mathrm{ent}^*(G) = \infty$ and $\mathrm{ent}^*(G_p) = 0$ for all $p \in S$.

Let $\mathbb{Z}_S = \mathbb{Z}[1/p : p \in S]$. We consider $G = \bigoplus_{i\in\mathbb{N}} H_i$, where $H_i \cong \mathbb{Z}_S$ for all i. Then $\mathrm{ent}^*(G) = \infty$, again by Proposition 1.5. Since $\mathbb{Z}_S \otimes \mathbb{Z}_p = \mathbb{Q}$ for every $p \in S$, the group G_p is divisible, hence, trivially, $\mathrm{ent}^*(G_p) = 0$.

Our aim is to prove a counterpart of Theorem 2.6 for the important class of cotorsion groups, characterized as those groups G such that $\mathrm{Ext}(\mathbb{Q}, G) = 0$ (see [F1, Ch. IX]).

Recall that a reduced cotorsion group is isomorphic to $\prod_p G_p$, where each G_p is a cotorsion J_p-module. Moreover, for every prime p, $G_p = A_p \oplus C_p$, where $A_p \cong (\bigoplus J_p)^\wedge$ is the completion of a direct sum of copies of J_p, and C_p is "adjusted cotorsion", i.e., $C_p \cong \mathrm{Ext}(\mathbb{Z}(p^\infty), T)$, where T is a reduced p-group and

- $t(C_p) = T$;
- C_p/T is divisible (and torsionfree);
- every $\phi \in \mathrm{End}(T)$ extends uniquely to an endomorphism $\phi^\bullet \in \mathrm{End}(C_p)$.

In the above notation we have

LEMMA 4.3. *If the p-group T is not bounded, then* $\mathrm{ent}^*(C_p) = \infty$.

Proof. In view of Theorem 2.6, there exists $\phi \in \mathrm{End}(T)$ with $\mathrm{ent}^*(\phi) = \infty$; ϕ extends uniquely to $\phi^\bullet \in \mathrm{End}(C_p)$, where $\mathrm{ent}^*(\phi^\bullet) = \infty$, by Proposition 2.4. ∎

THEOREM 4.4. *Let G be a reduced cotorsion group. Then the following are equivalent:*

(i) $\mathrm{ent}^*(G) = 0$;
(ii) $G = \prod_p G_p$ *where G_p is a finitely generated J_p-module for each prime p;*
(iii) *G is compact in the natural topology;*
(iv) *G is narrow.*

Proof. (ii)⇔(iii) follows from Theorem 1 of Orsatti's paper [O].

(i)⇒(ii). For any prime p, let $G_p = A_p \oplus C_p$ be the direct decomposition of the discussion before Lemma 4.3. The notation used there remains valid in the present proof. Since $G = \prod_p G_p$ for any reduced cotorsion group, from $\mathrm{ent}^*(G) = 0$ we get $\mathrm{ent}^*(G_p) = 0$. In fact, G_p is a direct summand of G for every p, and we make use of property (b). In particular, $\mathrm{ent}^*(A_p) = \mathrm{ent}^*(C_p) = 0$.

We first note that, necessarily, A_p is the completion of a direct sum of finitely many copies of J_p, hence a finitely generated J_p-module. Otherwise, A_p has a basic submodule B which is a direct sum of infinitely many copies of J_p. Then any endomorphism of B with infinite adjoint entropy extends uniquely to an endomorphism of A_p with infinite adjoint entropy (see Proposition 2.4), which, in particular, yields $\mathrm{ent}^*(G_p) = \infty$.

Now we examine C_p. If T is unbounded, then Lemma 4.3 shows that $\mathrm{ent}^*(C_p) = \infty$, impossible. Then T is necessarily bounded, hence $C_p \cong T$ is a direct sum of cyclic groups, so $\mathrm{ent}^*(C_p) = 0$ if and only if it is finite. The desired conclusion follows, since p was arbitrary.

(ii)⇒(iv). If each G_p is a finitely generated J_p-module, it is clear that G/pG is finite for all p, hence G is narrow by Proposition 1.3.

(iv)⇒(i) follows from Proposition 1.4. ∎

We remark that the mixed cotorsion group $G = \prod_p \mathbb{Z}/p\mathbb{Z}$ satisfies $\mathrm{ent}^*(G) = 0$ and is nonsplitting.

Corollary 4.5. *Let G be an Abelian group such that its cotorsion hull $G^\bullet$ satisfies* $\mathrm{ent}^*(G^\bullet) = 0$*. Then G is narrow; consequently,* $\mathrm{ent}^*(G) = 0$.

Proof. The preceding theorem shows that $G^\bullet$ is narrow. Since $G^\bullet/G$ is torsionfree divisible, Proposition 2.1 shows that $\mathcal{C}(G)$ and $\mathcal{C}(G^\bullet)$ have the same cardinality, that is, they are both countable. Hence G is narrow as well. ∎

Our next result shows that the groups G in a vast class of nonsplitting mixed Abelian groups, satisfy $\mathrm{ent}^*(G) = \infty$

Theorem 4.6. *Let G be a mixed reduced $\mathbb{Z}_p$-module satisfying the following conditions:*

(1) *the torsion part T of G is separable (i.e., $p^\omega T = 0$);*
(2) *$G/p^\omega G$ is torsion.*

Then there exists an endomorphism ϕ of G such that $\mathrm{ent}^*(\phi) = \infty$.

Proof. First note that G is necessarily nonsplitting. In fact, in case $G = T \oplus H$ we get $p^\omega G = p^\omega H = 0$, since G is reduced and $p^\omega T = 0$, and then $G/p^\omega G$ cannot be torsion. In particular, T is unbounded.

The proof is divided into two parts. We first assume that G satisfies the further condition

(3) G/T is nonzero divisible.

Let B be a basic subgroup of T. Let ψ be the endomorphism of B, constructed in the proof of Theorem 2.6 such that $\mathrm{ent}^*(\psi) = \infty$. Since $0 = p^\omega T = T \cap p^\omega G$, we have

$$B \leq T \cong (T \oplus p^\omega G)/p^\omega G \leq G/p^\omega G.$$

Let us denote by $\epsilon : B \to G/p^\omega G$ the above embedding, and note that $\epsilon = \pi \cdot i$, where $\pi : G \to G/p^\omega G$ is the canonical surjection and $i : B \to G$ is the inclusion map. Recall that B is pure in T and T/B is divisible; furthermore we readily see that $(T \oplus p^\omega G)/p^\omega G$ is pure in $G/p^\omega G$, and $G/(T \oplus p^\omega G)$, being a factor of G/T, is divisible. Therefore $\epsilon(B)$ is a basic subgroup of $G/p^\omega G$, which is a p-group, by hypothesis (2). As in the proof of Theorem 2.6, we extend ψ to a map $\phi' : G/p^\omega G \to B \le G$. Let $\phi : G \to G$ be the map $\phi' \cdot \pi$. An easy computation shows that the restriction of ϕ to B coincides with ψ, hence from Proposition 2.4 we get $\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\psi) = \infty$. This concludes the first part of the proof.

We now assume that G satisfies (1) and (2), but not necessarily (3). Let D be the divisible hull of G/T, and denote by D' the cokernel of the inclusion of G/T into D. From the exact sequence

$$\mathrm{Ext}^1(D,T) \to \mathrm{Ext}^1(G/T,T) \to \mathrm{Ext}^2(D',T) = 0$$

we derive that the exact sequence $0 \to T \to G \to G/T \to 0$ is the image of an exact sequence

$$0 \to T \to X \to D \to 0.$$

We conclude that we may regard G as a subgroup of X such that $X/G \cong D'$.

Since D is torsionfree, T is the torsion part of X, so (1) and (3) are satisfied by X. Note that X is reduced, since any divisible subgroup E of X intersects T trivially, so embeds into D, and hence it is torsionfree. Moreover, as we will see in a moment, G is pure in X, which easily implies that E trivially intersects G, so it embeds into D'. It follows that E is also torsion, so, in conclusion, $E = 0$. Note that $X/p^\omega X$ is torsion. In fact, let $x \in X \setminus T$; then $p^n x \in G$ for some positive integer n and, since $G/p^\omega G$ is torsion, $p^{n+k}x \in p^\omega G \le p^\omega X$ for some positive integer k. Thus X satisfies condition (2), as well.

Applying the first part of the proof to X, we find a map $\phi' : X \to B$ such that $\mathrm{ent}^*(\phi') = \infty$. To reach our final conclusion, we want to show that the restriction ϕ of ϕ' to G also satisfies $\mathrm{ent}^*(\phi) = \infty$. Note that X/G is divisible; hence, in order to apply again Proposition 2.4, it suffices to prove that G is pure in X.

Let $x \in X \setminus T$ be such that $p^n x \in G$ for some positive integer n. Then, as above, $p^{n+k}x \in p^\omega G$, so $p^{n+k}x = p^{n+k}g$ for some $g \in G$, hence $p^n x = p^n g + (p^n x - p^n g)$, where $p^n x - p^n g \in T \cap p^n X = p^n T$. But T is pure in X, hence $p^n x - p^n g = p^n t$ for some $t \in T \le G$, so $p^n x \in p^n G$, as desired. ∎

Note that the mixed abelian groups $H(X)$ considered in [KMT, pp. 242–247] (following Franzen–Goldsmith [FG]) satisfy the hypothesis of Theorem 4.6, as soon as one starts with a reduced torsionfree $\mathbb{Z}_p$-module X and a p-group T. In that construction, $X \cong p^\omega H(X)$, and it is proved that $\mathrm{End}(H(X))$ is a split extension of the subring $\mathrm{End}(X)$ by the ideal $\mathrm{Hom}(H(X),T)$. So the endomorphism ϕ of $H(X)$ with $\mathrm{ent}^*(\phi) = \infty$, as constructed in the above proof, belongs to this ideal.

Motivated by Theorem 4.6, we formulate the following

Conjecture. For any reduced mixed group G, the property $\mathrm{ent}^*(G) = 0$ implies that every p-primary component $t_p(G)$ of G is finite.

Note that if the conjecture is true, then $\mathrm{ent}^*(G) = 0$ also implies that the first Ulm subgroup G^1 of G vanishes. In fact, for every prime number p, since $t_p(G)$ is finite, we get $G = t_p(G) \oplus X_p$, where $p^\omega G = p^\omega X_p$. Since the p-torsion of X_p is zero, we deduce that $p^\omega G$ is the maximal p-divisible subgroup of G. It follows that G^1 coincides with the divisible subgroup of G, which is zero.

REFERENCES

[AKM] R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. 114 (1965), 309–319.

[AADGH] M. Akhavin, F. A. Z. Shirazi, D. Dikranjan, A. Giordano Bruno and A. Hosseini, *Algebraic entropy of shift endomorphisms on Abelian groups*, Quaestiones Math. 32 (2009), 1–22.

[A] D. Arnold, *Finite Rank Torsion Free Abelian Groups and Rings*, Lecture Notes in Math. 931, Springer, Berlin, 1982.

[C1] A. L. S. Corner, *Every countable reduced torsionfree ring is an endomorphism ring*, Proc. London Math. Soc. 13 (1963), 687–710.

[C2] —, *On endomorphism rings of primary Abelian groups*, Quart. J. Math. Oxford (2) 20 (1969), 277–296.

[DGS] D. Dikranjan, A. Giordano Bruno and L. Salce, *Adjoint algebraic entropy*, J. Algebra 324 (2010), 442–463.

[DGSZ] D. Dikranjan, B. Goldsmith, L. Salce and P. Zanardo, *Algebraic entropy for Abelian groups*, Trans. Amer. Math. Soc. 361 (2009), 3401–3434.

[EM] P. Eklof and A. Mekler, *Almost Free Modules. Set-Theoretic Methods*, North-Holland, Amsterdam, 2002.

[FG] B. Franzen and B. Goldsmith, *On endomorphism algebras of mixed modules*, J. London Math. Soc. (2) 31 (1985), 468–472.

[F1] L. Fuchs, *Infinite Abelian Groups*, Vol. I, Academic Press, 1970.

[F2] —, *Infinite Abelian Groups*, Vol. II, Academic Press, 1973.

[G] A. Giordano Bruno, *Algebraic entropy of generalized shifts on direct products*, Comm. Algebra, to appear.

[GS] R. Goebel and L. Salce, *Endomorphism rings with different rank-entropy supports*, preprint.

[GT] R. Goebel and J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, de Gruyter Exp. Math. 41, de Gruyter, Berlin, 2006.

[KMT] P. A. Krylov, A. V. Mikhalev and A. A. Tuganbaev, *Endomorphism Rings of Abelian Groups*, Kluwer, 2003.

[O] A. Orsatti, *Una caratterizzazione dei gruppi abeliani compatti o localmente compatti nella topologia naturale*, Rend. Sem. Mat. Univ. Padova 39 (1967), 219–225.

[SZ1] L. Salce and P. Zanardo, *Commutativity modulo small endomorphisms and endomorphisms of zero algebraic entropy*, in: Models, Modules and Abelian Groups, de Gruyter, 2008, 487–498.

[SZ2] L. Salce and P. Zanardo, *A general notion of algebraic entropy and the rank-entropy*, Forum Math. 21 (2009), 579–599.

[S] T. Szele, *On the basic subgroups of Abelian groups*, Acta Math. Acad. Sci. Hungar. 13 (1949), 51–53.

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