COLLOQUIUM MATHEMATICUM

VOL. 121 2010 NO. 1

THE SET FUNCTIONS $\mathcal{T}$ AND $\mathcal{K}$ AND IRREDUCIBLE CONTINUA

BY

LEOBARDO FERNÁNDEZ and SERGIO MACÍAS (México)

Abstract. We study the set functions $\mathcal{T}$ and $\mathcal{K}$ on irreducible continua. We present several properties of these functions when defined on irreducible continua. In particular, we characterize the class of irreducible continua for which these functions are continuous. We also characterize the class of $\mathcal{K}$-symmetric irreducible continua.

1. Introduction. The purpose of this paper is to study the set functions $\mathcal{T}$ and $\mathcal{K}$ defined on irreducible continua. The paper consists of six sections. After the section of definitions and notation, the third section gives general properties of $\mathcal{T}$ and $\mathcal{K}$, for example: (1) If X is an irreducible continuum, then for each subcontinuum A of X, $\mathcal{T}(A) = \mathcal{K}(A)$ (Corollary 3.10); (2) If X is an irreducible continuum, then the image of each closed subset of X under $\mathcal{K}$ is connected (Theorem 3.13). In the fourth section we characterize the class of $\mathcal{K}$-symmetric irreducible continua as those continua which are indecomposable or 2-indecomposable (Theorem 4.6). In the fifth section we present a different proof of a result, with respect to the set function $\mathcal{T}$, by R. W. FitzGerald [6, p. 169], which says that the finest monotone upper semicontinuous decomposition $\mathcal{G}$ of a continuum X of type λ such that each element of $\mathcal{G}$ is nowhere dense and $X/\mathcal{G}$ is an arc can be expressed in terms of the set function $\mathcal{T}$ (Theorem 5.2); we also give the corresponding result for the set function $\mathcal{K}$ (Theorem 5.3). In the sixth section, we characterize the class of irreducible continua for which the set functions $\mathcal{T}$ and $\mathcal{K}$ are continuous as those continua which are continuously irreducible (Theorems 6.6 and 6.7, repectively).

F. B. Jones defined the set functions $\mathcal{T}$ and $\mathcal{K}$ in [12, Theorems 2 and 3] to study aposyndetic continua. In fact, he defined those functions only for points (rather than subsets). Since then many properties related to these functions have been studied. Also, these functions have been applied to study continua. For example, C. L. Hagopian uses the set function $\mathcal{K}$ to study plane continua ([9], [10] and [11]). E. Vought also uses $\mathcal{K}$ to study monotone

2010 *Mathematics Subject Classification*: Primary 54B20.
Key words and phrases: aposyndetic continuum, continuously irreducible continuum, continuum, irreducible continuum, $\mathcal{K}$-symmetric continuum, set function $\mathcal{K}$, set function $\mathcal{T}$, weakly irreducible continuum.

DOI: 10.4064/cm121-1-7

decompositions of continua [25], and continua such that for each pair of its points there exists an irreducible continuum between those two points which is decomposable [26]. Regarding the set function $\mathcal{T}$, for example, D. P. Bellamy and J. J. Charatonik use $\mathcal{T}$ to study contractibility of continua [1]. D. P. Bellamy and C. L. Hagopian use $\mathcal{T}$ to study continua which can be mapped onto their cones [2]. The second named author uses $\mathcal{T}$ to study symmetric products of continua [15].

2. Definitions and notation. Given a subset A of a metric space X with metric d, we denote by $\mathrm{Int}(A)$ the interior of A and by $\mathrm{Cl}(A)$ the closure of A. Also, given a positive number r, we denote by $\mathcal{V}_r(p)$ the open ball of radius r about p, i.e., $\mathcal{V}_r(p) = \{x \in X \mid d(x, p) < r\}$; and for a set A, $\mathcal{V}_r(A)$ denotes the open ball of radius r about A. The set of positive integers is denoted by $\mathbb{N}$.

Given a metric space Z, a *decomposition* of Z is a family $\mathcal{G}$ of nonempty and mutually disjoint subsets of Z such that $\bigcup \mathcal{G} = Z$. A decomposition $\mathcal{G}$ of a metric space Z is said to be *upper semicontinuous* if the quotient map $q\colon Z \twoheadrightarrow Z/\mathcal{G}$ is closed. The decomposition is *continuous* provided that the quotient map q is both closed and open.

A *continuum* is a nonempty compact connected metric space. A continuum X is *aposyndetic* if for each pair of points x_1 and x_2 of X, there exists a subcontinuum W of X such that $x_1 \in \mathrm{Int}_X(W) \subset W \subset X \setminus \{x_2\}$.

Given a continuum X, we define the *hyperspace of nonempty closed subsets of* X to be the set $2^X = \{A \subseteq X \mid A \text{ is nonempty and closed}\}$, topologized with the Hausdorff metric, $\mathcal{H}$, [16]. We also define the *hyperspace of subcontinua of* X to be the set $\mathcal{C}(X) = \{W \in 2^X \mid W \text{ is connected}\}$. On a continuum X, the *set function* $\mathcal{T}$ is defined as follows: for each $A \subseteq X$,

$$\mathcal{T}(A) = \{x \in X \mid \text{if } W \in \mathcal{C}(X) \text{ and } x \in \mathrm{Int}(W), \text{ then } W \cap A \neq \emptyset\},$$

and the *set function* $\mathcal{K}$ is defined as follows: for each $A \subseteq X$,

$$\mathcal{K}(A) = \bigcap \{W \in \mathcal{C}(X) \mid A \subseteq \mathrm{Int}(W)\}.$$

When we need to emphasize that the set functions $\mathcal{T}$ and $\mathcal{K}$ are defined on the continuum X, we write $\mathcal{T}_X$ and $\mathcal{K}_X$, respectively. Let $\mathcal{L} \in \{\mathcal{T}, \mathcal{K}\}$. A continuum X is *$\mathcal{L}$-symmetric* provided that for each pair of closed subsets A and B of X, $A \cap \mathcal{L}(B) = \emptyset$ if and only if $\mathcal{L}(A) \cap B = \emptyset$. A continuum X is *point $\mathcal{L}$-symmetric* if for each pair of points p and q of X, $p \in \mathcal{L}(\{q\})$ if and only if $q \in \mathcal{L}(\{p\})$. Given a continuum X, we say that $\mathcal{L}$ is *continuous* if its restriction to 2^X is continuous, i.e., $\mathcal{L}\colon 2^X \to 2^X$ is continuous.

A continuum X is *weakly irreducible* provided that the complement of each finite union of subcontinua of X has a finite number of components. A continuum X is *irreducible between two of its points* if no proper subcontinuum of X contains both points. A continuum is *irreducible* if it is irreducible between two of its points. A continuum X is of *type* λ provided that X is irreducible and each indecomposable subcontinuum of X has empty interior. By [24, Theorem 10, p. 15], a continuum X is of type λ if and only if it admits a finest monotone upper semicontinuous decomposition $\mathcal{G}$ such that each element of $\mathcal{G}$ is nowhere dense and $X/\mathcal{G}$ is an arc. Each element of $\mathcal{G}$ is called a *layer* of X. Following [20], we say that a continuum X of type λ for which $\mathcal{G}$ is continuous is a *continuously irreducible continuum* (see also [19]).

A continuum X is *n-indecomposable* if (1) X is the union of n continua none of which is a subset of the union of the others, and (2) X is not the union of $n+1$ such continua. P. M. Swingle proved that every n-indecomposable continuum is the union of n indecomposable continua none of which is a subset of the union of the others [23, Theorem 2]; and C. E. Burgess showed that these n indecomposable continua are unique [3, Theorem 2].

3. The set functions $\mathcal{T}$ and $\mathcal{K}$. We begin with a result about the set function $\mathcal{K}$:

3.1. THEOREM. *Let X be a continuum. Then $\mathcal{K}(A) = \bigcup\{\mathcal{K}(\{a\}) \mid a \in A\}$ for each $A \in \mathcal{C}(X)$.*

Proof. Since $\bigcup\{\mathcal{K}(\{a\}) \mid a \in A\} \subseteq \mathcal{K}(A)$, we only need to show the opposite inclusion. Let $z \in X \setminus \bigcup\{\mathcal{K}(\{a\}) \mid a \in A\}$. Then $z \in \bigcap\{X \setminus \mathcal{K}(\{a\}) \mid a \in A\}$. Thus, for each $a \in A$, there exists a subcontinuum M_a of X such that $a \in \operatorname{Int}(M_a)$ and $z \notin M_a$. Then $\{\operatorname{Int}(M_a)\}_{a\in A}$ is an open cover of A. Since A is compact, there are $a_1, \ldots, a_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} \operatorname{Int}(M_{a_i})$. Let $M = \bigcup_{i=1}^{n} M_{a_i}$. Then M is a subcontinuum of X such that $A \subseteq \operatorname{Int}(M)$ and $z \notin M$. Hence, $z \in X \setminus \mathcal{K}(A)$. ∎

3.2. REMARK. Note that the result corresponding to Theorem 3.1 for the set function $\mathcal{T}$ is not true. Let X be the suspension over the Cantor set with vertices v_1 and v_2. If A is an arc in X having v_1 and v_2 as its end points, then $\mathcal{T}(A) = X$, meanwhile $\bigcup\{\mathcal{T}(\{a\}) \mid a \in A\} = A$.

The proof of the following theorem may be found in [8, Theorem 160].

3.3. THEOREM. *If X is a point $\mathcal{T}$-symmetric or a point $\mathcal{K}$-symmetric continuum, then $\mathcal{T}(\{x\}) = \mathcal{K}(\{x\})$ for all $x \in X$.*

As a consequence of Theorems 3.1 and 3.3, we have the following theorem.

3.4. THEOREM. *If X is a point $\mathcal{T}$-symmetric continuum, then $\mathcal{K}(A) = \bigcup\{\mathcal{T}(\{a\}) \mid a \in A\}$ for every $A \in \mathcal{C}(X)$.*

3.5. Corollary. *If X is an aposyndetic point $\mathcal{T}$-symmetric continuum, then $\mathcal{K}(A) = A$ for all $A \in \mathcal{C}(X)$.*

The following lemma is easy to establish. It is used in the proof of Theorems 3.13 and 4.6.

3.6. Lemma. *Let X be an indecomposable continuum. Then $\mathcal{K}(A) = X$ for each $A \in 2^X$.*

3.7. Theorem. *Let X be a weakly irreducible continuum. Then $\mathcal{T}(A) \subseteq \mathcal{K}(A)$ for each $A \in 2^X$.*

Proof. Let $A \in 2^X$ and let $x \in X \setminus \mathcal{K}(A)$. Then there exists a subcontinuum M of X such that $A \subseteq \mathrm{Int}(M) \subseteq M$ and $x \notin M$. Since X is weakly irreducible, $X \setminus M$ has finitely many components. Let C be the one containing x. Then $\mathrm{Cl}(C)$ is a subcontinuum of X containing x in its interior [16, 1.6.2] and $\mathrm{Cl}(C) \cap A = \emptyset$. Therefore, $x \in X \setminus \mathcal{T}(A)$. ∎

Since each irreducible continuum is weakly irreducible [16, 1.7.29] we have the following corollary.

3.8. Corollary. *Let X be an irreducible continuum. Then $\mathcal{T}(A) \subseteq \mathcal{K}(A)$ for each $A \in 2^X$.*

The next theorem shows that the reverse inclusion to the one given in Theorem 3.7 is true for subcontinua.

3.9. Theorem. *Let X be a weakly irreducible continuum. Then $\mathcal{T}(A) = \mathcal{K}(A)$ for each $A \in \mathcal{C}(X)$.*

Proof. Let $A \in \mathcal{C}(X)$ and let $x \in X \setminus \mathcal{T}(A)$. Then there exists a subcontinuum W of X such that $x \in \mathrm{Int}(W) \subseteq W \subseteq X \setminus A$. Since X is weakly irreducible and W is a subcontinuum of X, $X \setminus W$ has finitely many components. Let M be the one containing A. Since M is connected and open in X, $\mathrm{Cl}(M)$ is a subcontinuum of X such that $A \subseteq M \subseteq \mathrm{Cl}(M)$ and $x \notin \mathrm{Cl}(M)$. Thus, $x \in X \setminus \mathcal{K}(A)$. This implies that $\mathcal{K}(A) \subseteq \mathcal{T}(A)$. The other inclusion follows from Theorem 3.7. ∎

3.10. Corollary. *If X is an irreducible continuum, then $\mathcal{T}(A) = \mathcal{K}(A)$ for each $A \in \mathcal{C}(X)$.*

3.11. Remark. Let us observe that Corollary 3.10 is an extension of [18, Theorem 3.8] for the class of irreducible continua, because we do not require the continuity of $\mathcal{T}$. Also note that each irreducible continuum is point $\mathcal{T}$-symmetric [16, 3.1.37].

Next, we prove that the image under $\mathcal{K}$ of any closed subset of an irreducible continuum X is connected. First, we prove the following:

3.12. Lemma. *Let X be a decomposable continuum which is irreducible between p and q. Let $P^* = \{x \in X \mid X$ is irreducible between x and $q\}$ and*

$Q^* = \{x \in X \mid X$ is irreducible between p and $x\}$. Then, for all $p' \in P^*$ and $q' \in Q^*$, X is irreducible between p' and q'.

Proof. Let $p' \in P^*$ and let $q' \in Q^*$. Then X is irreducible between p' and q, and between p and q'. By [14, Lemma, p. 196], X is irreducible either between p' and q', or between p' and p. Since X is decomposable, $P^* \cap Q^* = \emptyset$, because if $z \in P^* \cap Q^*$, then there are three points of X, p, q and z, such that X is irreducible between any two of them, and thus, by [22, Corollary 11.20], X is indecomposable, which is a contradiction. Hence, $P^* \subseteq X \setminus Q^*$. Since $X \setminus Q^*$ is the composant of p [22, Theorem 11.4], X is not irreducible between p and p'. Therefore, X is irreducible between p' and q'. ∎

3.13. Theorem. *If X is an irreducible continuum, then $\mathcal{K}(A)$ is connected for each $A \in 2^X$.*

Proof. Let X be a continuum which is irreducible between p and q and let $A \in 2^X$. If $\mathcal{K}(A) = X$, then $\mathcal{K}(A)$ is connected. Assume that $\mathcal{K}(A) \neq X$. Thus, by Lemma 3.6, X is decomposable. Let $P^* = \{x \in X \mid X$ is irreducible between x and $q\}$ and $Q^* = \{x \in X \mid X$ is irreducible between p and $x\}$. Note that $p \in P^*$ and $q \in Q^*$. We consider two cases:

Case (i): $A \cap \mathrm{Cl}(P^*) \neq \emptyset$. If we had $A \cap \mathrm{Cl}(Q^*) \neq \emptyset$, then for every subcontinuum M of X such that $A \subseteq \mathrm{Int}(M)$, there would exist $p' \in M \cap P^*$ and $q' \in M \cap Q^*$. Then, by Lemma 3.12, $M = X$, and thus, $\mathcal{K}(A) = X$, contrary to the assumption that $\mathcal{K}(A) \neq X$. Hence, $A \cap \mathrm{Cl}(Q^*) = \emptyset$. Since A is compact, there exists $r > 0$ such that $\mathcal{V}_r(A) \cap \mathrm{Cl}(Q^*) = \emptyset$. Let $N \in \mathbb{N}$ be such that $1/n < r$ for each $n > N$. Given $n > N$, let Q_n be the component of $X \setminus \mathcal{V}_{1/n}(A)$ containing q. By [14, Theorem 3, p. 193], $\mathrm{Cl}(X \setminus Q_n)$ is a subcontinuum of X for each $n > N$. Note that $A \subseteq \mathrm{Int}(\mathrm{Cl}(X \setminus Q_n)) \subseteq \mathrm{Cl}(X \setminus Q_n)$. Let $H = \bigcap_{n>N} \mathrm{Cl}(X \setminus Q_n)$. Then H is a subcontinuum of X [16, 1.7.2]. We claim that $H = \mathcal{K}(A)$. By definition, $\mathcal{K}(A) \subseteq H$. Assume that there exists $z \in H \setminus \mathcal{K}(A)$. Then there exists a subcontinuum W of X such that $A \subseteq \mathrm{Int}(W)$ and $z \notin W$. Since $A \subseteq \mathrm{Int}(W)$, there exists $N_0 \in \mathbb{N}$ such that $\mathcal{V}_{1/n}(A) \subseteq W$; and thus, by [22, Theorem 5.6], $Q_n \cap W \neq \emptyset$ for every $n > N_0$. Since $A \subseteq \mathrm{Int}(W)$ and $A \cap \mathrm{Cl}(P^*) \neq \emptyset$, there exists $p' \in P^*$ such that $p' \in W$. If we had $z \notin Q_m$ for some $m > N_0$, then $W \cup Q_m$ would be a subcontinuum of X such that $p', q \in W \cup Q_m$ but $z \notin W \cup Q_m$, which contradicts the fact that X is irreducible between p' and q. Thus, $z \in Q_n$ for each $n > N_0$. Let $l > N_0$. Then $z \in Q_l$, $p' \in W$, $q \in Q_l$ and $W \cap Q_l \neq \emptyset$. Hence, $X = W \cup Q_l$. This implies that $X \setminus Q_l \subseteq W$. Since W is closed, $\mathrm{Cl}(X \setminus Q_l) \subseteq W$. But $z \in \mathrm{Cl}(X \setminus Q_l)$. Thus, $z \in W$, which is a contradiction.

Case (ii): $A \cap \mathrm{Cl}(P^*) = \emptyset$ *and* $A \cap \mathrm{Cl}(Q^*) = \emptyset$. Since A is compact, there exists $r > 0$ such that $\mathcal{V}_r(A) \cap \mathrm{Cl}(P^*) = \emptyset$ and $\mathcal{V}_r(A) \cap \mathrm{Cl}(Q^*) = \emptyset$. Let $N \in \mathbb{N}$ be such that $1/n < r$ for each $n > N$. Given $n > N$, let P_n be the component of $X \setminus \mathcal{V}_{1/n}(A)$ containing p and let Q_n be the component of $X \setminus \mathcal{V}_{1/n}(A)$ containing q. By [14, Theorem 3, p. 193], for each $n > N$, $\mathrm{Cl}(X \setminus P_n)$ and $\mathrm{Cl}(X \setminus Q_n)$ are subcontinua of X. Also, by [14, Theorem 4, p. 193], $X \setminus (P_n \cup Q_n) = (X \setminus P_n) \cap (X \setminus Q_n)$ is connected. Observe that $A \subseteq (X \setminus P_n) \cap (X \setminus Q_n)$. Let $H = \bigcap_{n>N} \mathrm{Cl}((X \setminus P_n) \cap (X \setminus Q_n))$. Note that, as in Case (i), H is a subcontinuum of X. We claim that $H = \mathcal{K}(A)$. By definition, $\mathcal{K}(A) \subseteq H$. Suppose that there exists $z \in H \setminus \mathcal{K}(A)$. Then there exists a subcontinuum W of X such that $A \subseteq \mathrm{Int}(W)$ and $z \notin W$. Note that $z \in \mathrm{Cl}((X \setminus P_n) \cap (X \setminus Q_n))$ for each $n > N$. Since $A \subseteq \mathrm{Int}(W)$, there exists $N_0 \in \mathbb{N}$ such that $N_0 > N$ and $\mathcal{V}_{1/n}(A) \subseteq W$ for each $n > N_0$. Thus, by [22, Theorem 5.6], $P_n \cap W \neq \emptyset$ and $Q_n \cap W \neq \emptyset$ for every $n > N_0$. Now, $z \in P_n \cup Q_n$ for each $n > N_0$. If we had $z \notin P_m \cup Q_m$ for some $m > N_0$, then $P_m \cup W \cup Q_m$ would be a subcontinuum of X containing p and q and $z \notin P_m \cup W \cup Q_m$, contrary to the fact that X is irreducible between p and q. Thus $z \in P_n \cup Q_n$ for each $n > N_0$. Since X is irreducible, $P_n \cup Q_n$ is not connected and we may assume that $z \in Q_n$ for each $n > N_0$. Let $l > N_0$. Then $z \in Q_l \setminus P_l$. Since $p \in P_l \cup W$ and $q \in Q_l$, we have $P_l \cup W \cup Q_l = X$. Hence, $X \setminus Q_l \subseteq P_l \cup W$; thus, $\mathrm{Cl}(X \setminus Q_l) \subseteq P_l \cup W$. This implies that $\mathrm{Cl}(X \setminus P_l) \cap \mathrm{Cl}(X \setminus Q_l) \subseteq \mathrm{Cl}(X \setminus Q_l) \subseteq P_l \cup W$. Since $z \in \mathrm{Cl}(X \setminus P_l) \cap \mathrm{Cl}(X \setminus Q_l)$, it follows that $z \in P_l \cup W$, which is a contradiction. ■

3.14. Remark. Note that Theorem 3.13 is not true for weakly irreducible continua. To show this, let S^1 be the unit circle. Then S^1 is weakly irreducible and $\mathcal{K}$ is the identity on 2^{S^1} [8, Theorem 26].

To prove Lemma 3.16 below, we need the following definition: Let X be a continuum and let $A \subseteq X$. We say that a point $z \in X$ is a *weak cut point of X that separates A* provided that $z \in M$ for every subcontinuum M of X such that $A \subseteq M$. Let $W_C(A)$ denote the set all such points.

3.15. Lemma. *Let X be a continuum and let $A \in 2^X$. Then $W_C(A) \subseteq \mathcal{K}(A)$.*

Proof. This follows from the above definition. ■

3.16. Lemma. *Let X be an irreducible continuum and let $A \in 2^X$. If W is a subcontinuum of X such that $\mathcal{T}(A) \subseteq \mathrm{Int}(W)$, then $\mathcal{K}(A) \subseteq \mathrm{Int}(W)$.*

Proof. Let X be an irreducible continuum between p and q. Suppose $y \in \mathcal{K}(A) \setminus \mathrm{Int}(W)$. Since $\mathcal{T}(A) \subseteq \mathrm{Int}(W)$, we have $y \notin \mathcal{T}(A)$. Then there exists a subcontinuum M of X such that $y \in \mathrm{Int}(M) \subseteq M \subseteq X \setminus A$. Since X is irreducible, $X \setminus M$ has at most two components. If A were contained in one of them, say C_1, then $\mathrm{Cl}(C_1)$ would be a subcontinuum of X such that $A \subseteq \mathrm{Int}(\mathrm{Cl}(C_1)) \subseteq \mathrm{Cl}(C_1) \subseteq X \setminus \{y\}$, contradicting $y \in \mathcal{K}(A)$. Hence, $X \setminus M = C_1 \cup C_2$ where $A \cap C_i \neq \emptyset$ for $i \in \{1, 2\}$ and we may assume that $p \in C_1$ and $q \in C_2$. Then every point of $\mathrm{Int}(M)$ is a weak cut point of X that separates A. If there existed $z \in \mathrm{Int}(M)$ which is not a weak cut point of X that separates A, there would exist a subcontinuum R of X such that $A \subseteq R$ and $z \notin R$. But this implies that $\mathrm{Cl}(C_1) \cup R \cup \mathrm{Cl}(C_2)$ is a proper subcontinuum of X containing p and q, which is a contradiction. Hence, by Lemma 3.15, $\mathrm{Int}(M) \subseteq W$. Thus, $y \in \mathrm{Int}(W)$, contrary to our assumption. ∎

The next result shows that the set functions $\mathcal{T}$ and $\mathcal{K}$ commute on irreducible continua.

3.17. Theorem. *If X is an irreducible continuum, then $\mathcal{T}(\mathcal{K}(A)) = \mathcal{K}(\mathcal{T}(A))$ for each $A \in 2^X$.*

Proof. Let $A \in 2^X$. First we show that $\mathcal{K}(\mathcal{T}(A)) \subseteq \mathcal{T}(\mathcal{K}(A))$. By Corollary 3.8 we have $\mathcal{T}(A) \subseteq \mathcal{K}(A)$. Hence, $\mathcal{K}(\mathcal{T}(A)) \subseteq \mathcal{K}(\mathcal{K}(A))$. By Theorem 3.13, $\mathcal{K}(A)$ is connected and, by Theorem 3.9, $\mathcal{K}(\mathcal{K}(A)) = \mathcal{T}(\mathcal{K}(A))$. Therefore, $\mathcal{K}(\mathcal{T}(A)) \subseteq \mathcal{T}(\mathcal{K}(A))$.

For the other inclusion, let $x \in X \setminus \mathcal{K}(\mathcal{T}(A))$. Then there exists a subcontinuum W of X such that $\mathcal{T}(A) \subseteq \mathrm{Int}(W)$ and $x \notin W$. By Lemma 3.16, $\mathcal{K}(A) \subseteq \mathrm{Int}(W)$. Since X is irreducible, $X \setminus W$ has at most two components. Let C be the one containing x. Then $\mathrm{Cl}(C)$ is a subcontinuum of X such that $x \in \mathrm{Int}(\mathrm{Cl}(C)) \subseteq \mathrm{Cl}(C) \subseteq X \setminus \mathcal{K}(A)$. Thus, $x \in X \setminus \mathcal{T}(\mathcal{K}(A))$. ∎

4. $\mathcal{K}$-symmetric continua. We aim to characterize irreducible $\mathcal{K}$-symmetric continua (Theorem 4.6).

4.1. Theorem. *Let X be a $\mathcal{K}$-symmetric continuum. If $\mathcal{K}(\{x\}) \neq X$ for each $x \in X$, then X is not irreducible.*

Proof. Suppose X is irreducible between p and q. Let $P^* = \{x \in X \mid X \text{ is irreducible between } x \text{ and } q\}$ and $Q^* = \{x \in X \mid X \text{ is irreducible between } p \text{ and } x\}$. By Lemma 3.12, X is irreducible between p' and q', for all $p' \in P^*$ and $q' \in Q^*$. Since $\mathcal{K}(\{x\}) \neq X$ for any $x \in X$, we deduce that $\mathrm{Cl}(P^*) \cap \mathrm{Cl}(Q^*) = \emptyset$: if $z \in \mathrm{Cl}(P^*) \cap \mathrm{Cl}(Q^*)$, then every continuum W such that $z \in \mathrm{Int}(W)$ must contain some $p' \in P^*$ and some $q' \in Q^*$; thus, $W = X$ and $\mathcal{K}(\{z\}) = X$, which is a contradiction.

Let $y \in X \setminus (\mathrm{Cl}(P^*) \cup \mathrm{Cl}(Q^*))$. Let U be an open set such that $y \in U \subseteq \mathrm{Cl}(U) \subseteq X \setminus (\mathrm{Cl}(P^*) \cup \mathrm{Cl}(Q^*))$. Since X is irreducible, $X \setminus U$ is not connected. Let P and Q be the components of $X \setminus U$ containing p and q respectively. By [14, Theorem 3, p. 210], $P^* \subseteq P$ and $Q^* \subseteq Q$. By [14, Theorem 4, p. 193], $X \setminus (P \cup Q)$ is connected and $U \subseteq X \setminus (P \cup Q)$. Hence, $\mathrm{Cl}(X \setminus (P \cup Q))$ is a subcontinuum of X containing y in its interior.

This implies that $\mathcal{K}(\{y\}) \cap (P^* \cup Q^*) = \emptyset$. However, $\mathcal{K}(\{p, q\}) = X$, i.e., $y \in \mathcal{K}(\{p, q\})$, which contradicts the fact that X is $\mathcal{K}$-symmetric. ■

Note that irreducible continua are $\mathcal{T}$-symmetric [16, 3.1.37]. Rewriting Theorem 4.1, we see that most irreducible continua are not $\mathcal{K}$-symmetric.

4.2. Corollary. *If X is a $\mathcal{K}$-symmetric irreducible continuum, then there exists a point $x \in X$ such that $\mathcal{K}(\{x\}) = X$.*

Note that [4, Theorem 7] may be stated as follows:

4.3. Theorem. *If X is a continuum, then there exists a positive integer n such that X is n-indecomposable if and only if the collection $\mathcal{E} = \{\mathcal{T}(\{x\}) \mid x \in X\}$ is finite.*

4.4. Theorem. *Let X be an irreducible continuum and let $x_0 \in X$. If $\mathcal{T}(\{x_0\}) = X$, then the collection $\mathcal{E} = \{\mathcal{T}(\{x\}) \mid x \in X\}$ is finite.*

Proof. First note that if X is indecomposable, then clearly $\mathcal{E} = \{X\}$ [16, 3.1.34]. Hence, $\mathcal{E}$ is finite.

Suppose X is decomposable and irreducible between the points a and b. Let x_0 be a point of X such that $\mathcal{T}(\{x_0\}) = X$. Since X is irreducible, X is $\mathcal{T}$-symmetric [16, 3.1.37]. Thus, X is point $\mathcal{T}$-symmetric. Since $\mathcal{T}(\{x_0\}) = X$, it follows that $x_0 \in \mathcal{T}(\{a\}) \cap \mathcal{T}(\{b\})$. Hence, by the irreducibility of X, $X = \mathcal{T}(\{a\}) \cup \mathcal{T}(\{b\})$.

Now, we show that $\mathrm{Cl}(X \setminus \mathcal{T}(\{b\})) = \mathcal{T}(\{a\})$. Suppose there exists $x \in \mathcal{T}(\{a\}) \setminus \mathrm{Cl}(X \setminus \mathcal{T}(\{b\}))$. Then $\mathrm{Cl}(X \setminus \mathcal{T}(\{b\}))$ is a subcontinuum of X [14, Theorem 3, p. 193] such that $a \in \mathrm{Int}(\mathrm{Cl}(X \setminus \mathcal{T}(\{b\}))) \subset \mathrm{Cl}(X \setminus \mathcal{T}(\{b\})) \subset X \setminus \{x\}$. Hence, $a \notin \mathcal{T}(\{x\})$. Since X is point $\mathcal{T}$-symmetric [16, 3.1.37], we obtain $x \notin \mathcal{T}(\{a\})$, a contradiction. Therefore, $\mathrm{Cl}(X \setminus \mathcal{T}(\{b\})) = \mathcal{T}(\{a\})$. Thus, $\mathcal{T}(\{a\})$ is irreducible between a and each point of $\mathrm{Bd}(\mathcal{T}(\{a\}))$ [14, Theorem 7, p. 194]. Since, with a similar argument, we can show that $x_0 \in \mathrm{Cl}(X \setminus \mathcal{T}(\{a\}))$, it follows that $x_0 \in \mathrm{Bd}(\mathcal{T}(\{a\}))$. Therefore, $\mathcal{T}(\{a\})$ is irreducible between a and x_0. Similarly, $\mathcal{T}(\{b\})$ is irreducible between b and x_0.

Let $x \in X$. If $x \in \mathcal{T}(\{a\}) \cap \mathcal{T}(\{b\})$, then $\mathcal{T}(\{x\}) = X$. Suppose $x \in \mathcal{T}(\{a\}) \setminus \mathcal{T}(\{b\})$. Since X is point $\mathcal{T}$-symmetric [16, 3.1.37], we have $a \in \mathcal{T}(\{x\})$. Hence, $\mathcal{T}(\{x\}) = \mathcal{T}(\{a\})$, since $x \notin \mathcal{T}(\{b\})$. Similarly, if $x \in \mathcal{T}(\{b\}) \setminus \mathcal{T}(\{a\})$, then $\mathcal{T}(\{x\}) = \mathcal{T}(\{b\})$. Thus, $\mathcal{E} = \{X, \mathcal{T}(\{a\}), \mathcal{T}(\{b\})\}$. Therefore, $\mathcal{E}$ has three elements. ■

4.5. Theorem. *Let X be an n-indecomposable continuum. If $n \geq 3$, then either X is not irreducible, or $\mathcal{K}(\{x\}) \neq X$ for any $x \in X$.*

Proof. Since X is an n-indecomposable continuum, $X = \bigcup_{j=1}^{n} Z_j$, where Z_j is an indecomposable subcontinuum of X for $j \in \{1, \ldots, n\}$ and no Z_j is a subset of the union of the other continua. We consider two cases.

CASE (i): $\bigcap_{j=1}^{n} Z_j \neq \emptyset$. Then, since $n \geq 3$, X is not irreducible.

CASE (ii): $\bigcap_{j=1}^{n} Z_j = \emptyset$. Then there exist $j_1, j_2 \in \{1, \ldots, n\}$ such that $Z_{j_1} \cap Z_{j_2} = \emptyset$. Hence, we have two possibilities:

(a) If $x \in Z_{j_1}$, then $Z_{j_2} \setminus \mathcal{K}(\{x\}) \neq \emptyset$ and $\mathcal{K}(\{x\}) \neq X$. Similarly, if $x \in Z_{j_2}$, then $\mathcal{K}(\{x\}) \neq X$.
(b) If $x \in \bigcup\{Z_\ell \mid \ell \in \{1, \ldots, n\} \setminus \{j_1, j_2\}\}$, then $\mathcal{K}(\{x\}) \subset \bigcup\{Z_\ell \mid \ell \in \{1, \ldots, n\} \setminus \{j_1, j_2\}\}$, and $\mathcal{K}(\{x\}) \neq X$. Hence, $\mathcal{K}(\{x\}) \neq X$ for any $x \in X$.

Therefore, either X is not irreducible, or $\mathcal{K}(\{x\}) \neq X$ for any $x \in X$. ■

4.6. THEOREM. *Let X be an irreducible continuum. Then X is $\mathcal{K}$-symmetric if and only if X is indecomposable or 2-indecomposable.*

Proof. Suppose X is an irreducible $\mathcal{K}$-symmetric continuum. Thus, there exists $x_0 \in X$ such that $\mathcal{K}(\{x_0\}) = X$, by Corollary 4.2. It follows that $\mathcal{T}(\{x_0\}) = X$ (Theorem 3.3). Then, by Theorem 4.4, the collection $\mathcal{E} = \{\mathcal{T}(\{x\}) \mid x \in X\}$ is finite. Hence, there exists a positive integer n such that X is n-indecomposable (Theorem 4.3). Then, by Theorem 4.5, $n \leq 2$. Therefore, X is either indecomposable or 2-indecomposable.

Now, if X is indecomposable, then $\mathcal{K}(\{x\}) = X$ for all $x \in X$, by Lemma 3.6. Suppose X is 2-indecomposable. Then there exist two indecomposable subcontinua H and L of X such that $X = H \cup L$, $H \setminus L \neq \emptyset$ and $L \setminus H \neq \emptyset$. Note that, since H and K are indecomposable with union X, for each $A \in 2^X$, $\mathcal{K}(A) \in \{X, H, L\}$. Now it follows that X is $\mathcal{K}$-symmetric. ■

5. Continua of type λ. We present a different proof of a result, with respect to the set function $\mathcal{T}$, by R. W. FitzGerald [6, p. 169], which says that given a continuum of type λ, X, the finest monotone upper semicontinuous decomposition $\mathcal{G}$ of X such that each element of $\mathcal{G}$ is nowhere dense and $X/\mathcal{G}$ is an arc, can be expressed in terms of the set function $\mathcal{T}$ (Theorem 5.2). We also give the corresponding result for the set function $\mathcal{K}$ (Theorem 5.3).

We begin by proving the following easy lemma.

5.1. LEMMA. *Let X be a continuum of type λ. Let $\mathcal{G}$ be the finest monotone upper semicontinuous decomposition of X such that each element of $\mathcal{G}$ is nowhere dense and $X/\mathcal{G}$ is an arc, and let $q\colon X \twoheadrightarrow [0,1]$ be the quotient map. If $x \in X$ and $A \subset q^{-1}(q(x))$, then $\mathcal{T}(A) \subset q^{-1}(q(x))$.*

Proof. Let $y \in X \setminus q^{-1}(q(x))$. Then $q(y) \neq q(x)$. Hence, there exists a closed subinterval $[r,t]$ of $[0,1]$ such that $q(y) \in \mathrm{Int}_{[0,1]}([r,t])$ and $q(x) \in [0,1] \setminus [r,t]$. This implies that $q^{-1}(q(x)) \cap q^{-1}([r,t]) = \emptyset$. Since q is monotone, $q^{-1}([r,t])$ is a subcontinuum of X. By construction, $y \in \mathrm{Int}_X(q^{-1}([r,t])$. Therefore, $y \in X \setminus \mathcal{T}(A)$. ■

5.2. Theorem. *If X is a continuum of type λ, then $\{\mathcal{T}^2(\{x\}) \mid x \in X\}$ is the finest monotone upper semicontinuous decomposition $\mathcal{G}$ of X such that each element of $\mathcal{G}$ is nowhere dense and $X/\mathcal{G}$ is an arc.*

Proof. Let $q\colon X \twoheadrightarrow [0,1]$ be the quotient map corresponding to $\mathcal{G}$. Observe that $\mathcal{G} = \{q^{-1}(q(x)) \mid x \in X\}$.

Let $x \in X$. Note that $\mathcal{T}(\{x\}) \subset q^{-1}(q(x))$, by Lemma 5.1. By [24, Theorem 18, p. 26], there exists $z \in q^{-1}(q(x))$ such that $\mathcal{T}(\{z\}) = q^{-1}(q(x))$. Since X is $\mathcal{T}$-symmetric [16, 3.1.37] and $x \in \mathcal{T}(\{z\})$, we have $z \in \mathcal{T}(\{x\})$. Hence, $q^{-1}(q(x)) = \mathcal{T}(\{z\}) \subset \mathcal{T}^2(\{x\}) \subset q^{-1}(q(x))$ (the last inclusion is true by Lemma 5.1). Thus, $\mathcal{T}^2(\{x\}) = q^{-1}(q(x))$. Since x is an arbitrary point of X, we conclude that $\mathcal{G} = \{\mathcal{T}^2(\{x\}) \mid x \in X\}$. ∎

As a consequence of Theorem 5.2 and Corollary 3.10 we have the following:

5.3. Theorem. *If X is a continuum of type λ, then $\{\mathcal{K}^2(\{x\}) \mid x \in X\}$ is the finest monotone upper semicontinuous decomposition $\mathcal{G}$ of X such that each element of $\mathcal{G}$ is nowhere dense and $X/\mathcal{G}$ is an arc.*

6. Continuity of $\mathcal{T}$ and $\mathcal{K}$. We begin noting a couple of theorems on the continuity of $\mathcal{K}$, one on hereditarily unicoherent continua (Theorem 6.1) and the other on irreducible continua (Theorem 6.2). Next, we characterize the class of irreducible continua for which the set functions $\mathcal{T}$ and $\mathcal{K}$ are continuous (Theorems 6.6 and 6.7, repectively).

The proof of the following theorem may be found in [7, Theorem 7].

6.1. Theorem. *Let X be a hereditarily unicoherent continuum. If for each $x \in X$, $\mathcal{K}$ is continuous at $\{x\}$, then $\mathcal{K}$ is continuous.*

Essential to the proof of Theorem 6.1 is the fact that $\mathcal{K}(A)$ is connected for each $A \in 2^X$. Since by Theorem 3.13 we have this property on irreducible continua, the proof of the following theorem is very similar to the proof of Theorem 6.1 and we omit it.

6.2. Theorem. *Let X be an irreducible continuum. If for each $x \in X$, $\mathcal{K}$ is continuous at $\{x\}$, then $\mathcal{K}$ is continuous.*

To prove our theorem about the continuity of the set function $\mathcal{T}$ (Theorem 6.6), we need the following results.

6.3. Lemma. *Let X be a continuum, let $A \in 2^X$ and let $\mathcal{A} = \{\mathcal{T}(\{a\} \mid a \in A\}$. If $\mathcal{T}$ is continuous on singletons, then $\mathcal{A}$ is closed in 2^X.*

Proof. Let $B \in \mathrm{Cl}_{2^X}(\mathcal{A})$. Then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of points of A such that the sequence $\{\mathcal{T}(\{a_n\})\}_{n=1}^{\infty}$ converges to B. Since A is compact, without loss of generality, we assume that $\{a_n\}_{n=1}^{\infty}$ converges to a point

$a \in A$. Since $\mathcal{T}$ is continuous on singletons, we have $B = \lim_{n\to\infty} \mathcal{T}(\{a_n\}) = \mathcal{T}(\{a\})$. Therefore, $B \in \mathcal{A}$ and $\mathcal{A}$ is closed in 2^X. ■

6.4. Theorem. *Let X be a continuum with $\mathcal{T}(A) = \bigcup\{\mathcal{T}(\{a\}) \mid a \in A\}$ for all $A \in 2^X$. If $\mathcal{T}$ is continuous on singletons, then $\mathcal{T}$ is continuous.*

Proof. Let $\varepsilon > 0$, and let $\delta > 0$ be given by the uniform continuity of $\mathcal{T}|_{\mathcal{F}_1(X)}$.

Let $A, B \in 2^X$ be such that $\mathcal{H}(A,B) < \delta$. Let $\mathcal{A} = \{\mathcal{T}(\{a\}) \mid a \in A\}$ and let $\mathcal{B} = \{\mathcal{T}(\{b\}) \mid b \in B\}$. By Lemma 6.3, $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of 2^X. Let $\mathcal{T}(\{a\}) \in \mathcal{A}$. Since $\mathcal{H}(A,B) < \delta$, there exists $b \in B$ such that $\mathcal{H}(\{a\},\{b\}) < \delta$. Hence, by the choice of δ, $\mathcal{H}(\mathcal{T}(\{a\}),\mathcal{T}(\{b\})) < \varepsilon$. Thus, $\mathcal{A} \subset \mathcal{V}^{\mathcal{H}}_\varepsilon(\mathcal{B})$. Similarly, $\mathcal{B} \subset \mathcal{V}^{\mathcal{H}}_\varepsilon(\mathcal{A})$. Therefore, $\mathcal{H}_2(\mathcal{A},\mathcal{B}) < \varepsilon$, where $\mathcal{H}_2$ is the Hausdorff metric on 2^{2^X} induced by $\mathcal{H}$. Then, by [21, (1.48)], $\mathcal{H}(\bigcup\mathcal{A},\bigcup\mathcal{B}) \le \mathcal{H}_2(\mathcal{A},\mathcal{B}) < \varepsilon$. By hypothesis, $\mathcal{T}(A) = \bigcup\mathcal{A}$ and $\mathcal{T}(B) = \bigcup\mathcal{B}$. Hence, we have proved that if $\mathcal{H}(A,B) < \delta$, then $\mathcal{H}(\mathcal{T}(A),\mathcal{T}(B)) < \varepsilon$. Therefore, $\mathcal{T}$ is continuous. ■

6.5. Corollary. *Let X be a $\mathcal{T}$-additive continuum. If $\mathcal{T}$ is continuous on singletons, then $\mathcal{T}$ is continuous.*

Proof. Since X is $\mathcal{T}$-additive, $\mathcal{T}(A) = \bigcup\{\mathcal{T}(\{a\}) \mid a \in A\}$ for all $A \in 2^X$ [16, 3.1.46]. Now, the corollary follows from Theorem 6.4. ■

Now, we characterize the class of irreducible continua for which $\mathcal{T}$ is continuous.

6.6. Theorem. *Let X be an irreducible continuum. Then $\mathcal{T}$ is continuous for X if and only if X is continuously irreducible.*

Proof. Note that if X is continuously irreducible, then $\mathcal{T}$ is continuous for X [19, Theorem 3.2].

Suppose that X is an irreducible continuum for which $\mathcal{T}$ is continuous. Since X is irreducible, it is $\mathcal{T}$-symmetric [16, 3.1.37]. Hence, it is point $\mathcal{T}$-symmetric. Since $\mathcal{T}$ is continuous, it follows that

$$\mathcal{G} = \{\mathcal{T}(\{x\}) \mid x \in X\}$$

is a monotone continuous decomposition of X such that $X/\mathcal{G}$ is a locally connected continuum, and the elements of $\mathcal{G}$ are nowhere dense [17, Theorem 3.8]. Since X is irreducible and the quotient map $q\colon X \twoheadrightarrow X/\mathcal{G}$ is monotone, $X/\mathcal{G}$ is an irreducible continuum [14, Theorem 3, p. 192]. Hence, $X/\mathcal{G}$ is an arc. Thus, X is a continuum of type λ. Since $\mathcal{T}$ is continuous, X is continuously irreducible [19, Theorem 3.2]. ■

Now, we characterize the class of irreducible continua for which $\mathcal{K}$ is continuous.

6.7. THEOREM. *Let X be an irreducible continuum. Then $\mathcal{K}_X$ is continuous if and only if X is continuously irreducible.*

Proof. Since X is irreducible, it is $\mathcal{T}_X$-symmetric [16, 3.1.37]. Hence, it is $\mathcal{T}_X$-additive [16, 3.144] and point $\mathcal{T}_X$-symmetric. Since X is point $\mathcal{T}_X$-symmetric, $\mathcal{K}_X(\{x\}) = \mathcal{T}_X(\{x\})$, by Theorem 3.3. If $\mathcal{K}_X$ is continuous, then $\mathcal{T}_X$ is continuous on singletons. Thus, by Corollary 6.5, it is continuous. Therefore, by Theorem 6.6, X is continuously irreducible.

Now, suppose X is a continuously irreducible continuum. Let $q\colon X \twoheadrightarrow [0,1]$ be the quotient map. Then $q^{-1}(t)$ is a terminal subcontinuum of X for all $t \in [0,1]$ [19, Lemma 3.3]. Since q is monotone, open and each $q^{-1}(t)$ is a terminal continuum, $\mathcal{K}_X(A) = q^{-1}\mathcal{K}_{[0,1]}q(A)$ for each $A \in 2^X$ [18, Theorem 3.2]. Let $\Im(q)\colon 2^{[0,1]} \to 2^X$ be given by $\Im(q)(B) = q^{-1}(B)$. Since q is continuous and open, we deduce that $\Im(q)$ and 2^q are continuous ([13, Theorem 2, p. 165] and [21, (1.168)], respectively). Since $\mathcal{K}_{[0,1]}$ is continuous [8, Theorem 27], and $\mathcal{K}_X = \Im(q) \circ \mathcal{K}_{[0,1]} \circ 2^q$, it follows that $\mathcal{K}_X$ is continuous. ■

6.8. REMARK. Note that the "if" implication follows from Theorem 6.6, [16, 3.1.37], [19, Lemma 3.3] and [18, Corollary 6.3], but we preferred to give a proof using only the set function $\mathcal{K}$. It would be of interest to find a proof of the reverse implication using only $\mathcal{K}$, i.e., without using the set function $\mathcal{T}$.

Acknowledgements. We thank P. Krupski for pointing out that S. T. Czuba proved the following: *for each subcontinuum A of a continuum X, we have $\mathcal{K}(A) = \{x \in X \mid \mathcal{T}(\{x\}) \cap A \neq \emptyset\}$* [5, Lemma 6], which is equivalent to Theorem 3.1.

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Leobardo Fernández, Sergio Macías
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior
Ciudad Universitaria
México D.F., C.P. 04510, México
E-mail: leobardof@ciencias.unam.mx
macias@servidor.unam.mx

Received 11 November 2009;
revised 7 February 2010 (5271)