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SYMBOLIC EXTENSIONS FOR NONUNIFORMLY ENTROPY EXPANDING MAPS

BY

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Abstract. A nonuniformly entropy expanding map is any $\mathcal{C}^1$ map defined on a compact manifold whose ergodic measures with positive entropy have only nonnegative Lyapunov exponents. We prove that a $\mathcal{C}^r$ nonuniformly entropy expanding map T with $r > 1$ has a symbolic extension and we give an explicit upper bound of the symbolic extension entropy in terms of the positive Lyapunov exponents by following the approach of T. Downarowicz and A. Maass [Invent. Math. 176 (2009)].

1. Introduction. Given a continuous map $T : X \to X$ on a compact metrizable space X one can wonder if this topological dynamical system admits a symbolic extension, i.e. a topological extension which is a subshift over a finite alphabet. The topological symbolic extension entropy $h_{\mathrm{sex}}(T) = \inf\{h_{\mathrm{top}}(Y,S) : (Y,S) \text{ is a symbolic extension of } (X,T)\}$ estimates how the dynamical system (X,T) differs from a symbolic extension from the point of view of entropy. The question of existence of symbolic extensions leads to a deep theory which was developed mainly by M. Boyle and T. Downarowicz, who related the entropy of symbolic extensions to the convergence of the entropy of (X,T) computed at finer and finer scales [4].

Dynamical systems with symbolic extensions have necessarily finite topological entropy, because the topological entropy of a factor is less than or equal to the topological entropy of the extension and the topological entropy of a subshift over a finite alphabet is finite. Joe Auslander asked if the opposite was true: does every finite entropy system have a symbolic extension? M. Boyle answered this question negatively by constructing a zero-dimensional dynamical system with finite topological entropy but without symbolic extensions. Nonetheless it was proved by M. Boyle, D. Fiebig and U. Fiebig [6] that asymptotically h-expansive dynamical systems with finite topological entropy admit principal symbolic extensions, i.e. ones that preserve the entropy of invariant measures. Following Y. Yomdin [23], J. Buzzi showed that $\mathcal{C}^\infty$ maps on a compact manifold are asymptotically h-expansive [10]. In particular such maps admit principal symbolic extensions. Recall

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that uniformly hyperbolic dynamical systems are expansive. It is also known that partially hyperbolic dynamical systems with a central bundle splitting into one-dimensional subbundles [19], [11] are h-expansive. Therefore all these dynamical systems are asymptotically h-expansive and so admit principal symbolic extensions. On the other hand $\mathcal{C}^1$ maps without symbolic extensions have been built in several works by using generic arguments [15], [1] or with an explicit construction [8].

We say that a map $T : M \to M$ defined on a compact manifold is $\mathcal{C}^r$ with $r > 1$ when T is $[r]$ times differentiable (1) and the $[r]$th derivative of T is $r - [r]$-Hölder. T. Downarowicz and A. Maass [14] have recently proved that $\mathcal{C}^r$ maps of the interval $f : [0,1] \to [0,1]$ with $1 < r < +\infty$ have symbolic extensions. More precisely they showed that

$$h_{\text{sex}}(f) \leq \frac{r \log \|f'\|_\infty}{r-1}.$$

The present author [8] built explicit examples proving this upper bound to be sharp. Similar $\mathcal{C}^r$ examples with large symbolic extension entropy have been previously built by T. Downarowicz and S. Newhouse [15] for diffeomorphisms in higher dimensions by using generic arguments on homoclinic tangencies. The present author [7] proved anew that $\mathcal{C}^2$ surface local diffeomorphisms have symbolic extensions. The existence of symbolic extensions for general $\mathcal{C}^r$ maps with $1 < r < +\infty$ is still an open question. The following was conjectured in [15]:

Conjecture 1. *Let $T : M \to M$ be a $\mathcal{C}^r$ map, with $r > 1$, on a compact manifold M of dimension d. Then*

$$h_{\text{sex}}(T) \leq h_{\text{top}}(T) + \frac{dR(T)}{r-1}$$

where $R(T)$ is the dynamical Lipschitz (2) constant of T, that is,

$$R(T) := \lim_{n\to+\infty} \frac{\log^+ \|DT^n\|}{n}.$$

A $\mathcal{C}^1$ map $T : M \to M$ on a compact manifold M will be called *nonuniformly entropy expanding* if any ergodic measure with nonzero entropy has only nonnegative Lyapunov exponents (3). In this paper we prove the conjecture for $\mathcal{C}^r$ nonuniformly entropy expanding maps with $r > 1$ up to a factor d, i.e.

(1) Throughout this paper $[x]$ denotes the integer part of x for all real numbers x.

(2) $R(T)$ does not depend on the Riemannian metric $\| \;\|$ on M.

(3) For usual nonuniformly expanding maps [2] which are well adapted to the study of SRB measures it is required that Lebesgue almost all points have only nonnegative Lyapunov exponents.

$$h_{\text{sex}}(T) \leq h_{\text{top}}(T) + \frac{d^2 R(T)}{r-1}$$

(see Corollary 1).

It follows from Ruelle's inequality that $\mathcal{C}^1$ maps of the interval are nonuniformly entropy expanding. Therefore Theorem 4 generalizes the result of T. Downarowicz and A. Maass [14]. We do not know how large the class of nonuniformly entropy expanding map in higher dimensions is. Does it contain a $\mathcal{C}^r$ open set for some $r \in \mathbb{Z}^+$? Do "Alves–Viana like" maps belong to this class [21]? Anyway, we think the results presented in this paper can be considered as a first step towards the proof of Conjecture 1 (especially the Main Theorem which applies to general $\mathcal{C}^r$ maps).

We now give a class of nontrivial examples of nonuniformly entropy expanding maps. Let $T : N \to N$ be a $\mathcal{C}^r$ isometry on a compact Riemannian manifold N of dimension d. Then any $\mathcal{C}^r$ skew product on $N \times [0,1]$ of the form $T_g(x,y) = (Tx, g(x,y))$ with positive entropy is nonuniformly entropy expanding: any ergodic measure ν has d zero Lyapunov exponents and by Ruelle's inequality the first exponent must be positive when the entropy of ν is positive. Finally observe that the set of $g \in \mathcal{C}^r(N \times [0,1])$ such that T_g has positive entropy contains a $\mathcal{C}^0$ open subset of $\mathcal{C}^r(N \times [0,1])$. Indeed if $f : [0,1] \to [0,1]$ is a $\mathcal{C}^r$ map of the interval with positive entropy then it admits a horseshoe which is persistent under small $\mathcal{C}^0$ perturbations [17]. Therefore there exists a $\mathcal{C}^0$ neighborhood $\mathcal{V}$ of f in $\mathcal{C}([0,1])$ such that if $g(x,\cdot) \in \mathcal{V}$ for all $x \in N$ then $h_{\text{top}}(T_g) > 0$.

2. Preliminaries. In the following we denote by $\mathcal{M}(X,T)$ the set of invariant Borel probability measures of the dynamical system (X,T) and by $\mathcal{M}_e(X,T)$ the subset of ergodic measures. We endow $\mathcal{M}(X,T)$ with the weak star topology. Since X is a compact metric space, this topology is metrizable. We denote by dist a metric on $\mathcal{M}(X,T)$. It is well known that $\mathcal{M}(X,T)$ is a compact convex metric space whose extreme points are exactly the ergodic measures. Moreover if $\mu \in \mathcal{M}(X,T)$ there exists a unique Borel probability measure M_μ on $\mathcal{M}(X,T)$ supported on $\mathcal{M}_e(X,T)$ such that for all Borel sets B we have $\mu(B) = \int \nu(B)\, dM_\mu(\nu)$. This is the so called *ergodic decomposition* of μ. A Borel function $f : \mathcal{M}(X,T) \to \mathbb{R}$ is said to be *harmonic* if $f(\mu) = \int_{\mathcal{M}_e(X,T)} f(\nu)\, dM_\mu(\nu)$ for all $\mu \in \mathcal{M}(X,T)$. It is well known that affine upper semicontinuous functions are harmonic.

If f is a Borel function defined on $\mathcal{M}_e(X,T)$, the *harmonic extension* of f is the function defined on $\mathcal{M}(X,T)$ by

$$\overline{f}(\mu) := \int_{\mathcal{M}_e(X,T)} f(\nu)\, dM_\mu(\nu)$$

It is easily seen that $\overline{f}$ is harmonic.

2.1. Symbolic extension entropy function and entropy structures. A *symbolic extension* of (X,T) is a subshift (Y,S) of a full shift on a finite number of symbols, along with a continuous surjection $\pi : Y \to X$ such that $T\pi = \pi S$. Given a symbolic extension $\pi : (Y,S) \to (X,T)$ we consider the *extension entropy* $h^{\pi}_{\mathrm{ext}} : \mathcal{M}(X,T) \to \mathbb{R}^+$ defined for all $\mu \in \mathcal{M}(X,T)$ by

$$h^{\pi}_{\mathrm{ext}}(\mu) = \sup_{\pi_*\nu=\mu} h(\nu)$$

where π_* is the map induced on measures by π. Then the *symbolic extension entropy* $h_{\mathrm{sex}} : \mathcal{M}(X,T) \to \mathbb{R}^+$ is

$$h_{\mathrm{sex}} = \inf h^{\pi}_{\mathrm{ext}}$$

where the infimum is taken over all the symbolic extensions of (X,T). By convention, if (X,T) does not admit any symbolic extension we simply put $h_{\mathrm{sex}} \equiv +\infty$. Recall that in the Introduction we have defined the topological symbolic extension entropy $h_{\mathrm{sex}}(T)$ as the infimum of the topological entropies of the symbolic extensions of (X,T). We also put $h_{\mathrm{sex}}(T) = +\infty$ when there are no such extensions.

2.2. Newhouse local entropy. Let us first recall some usual notions related to the entropy of dynamical systems (we refer to [22] or [12] for a general introduction to entropy). Consider a continuous map $T : X \to X$ with (X,d) a compact metric space. Let $n \in \mathbb{Z}^+$ and $\delta > 0$. A subset E of X is called (n,δ) *separated* when for all $x, y \in E$ there exists $0 \leq k < n$ such that $d(f^k x, f^k y) > \delta$.

We now recall the "'Newhouse local entropy". We fix some finite open cover $\mathcal{V}$ of X, a point $x \in X$, a number $\delta > 0$, an integer n, and a Borel set $F \subset X$. We will denote by $\mathcal{V}^n$ the open cover consisting of all the open sets of the form $V_0 \cap T^{-1}V_1 \cap \cdots \cap T^{-n+1}V_{n-1}$, where $V_i \in \mathcal{V}$ for each $i = 0, 1, \ldots, n-1$. We define

$$\begin{aligned} H(n,\delta \,|\, F, \mathcal{V}) &:= \log\max\{\sharp E : E \text{ is an } (n,\delta) \text{ separated set} \\ &\qquad\qquad\qquad\qquad\qquad \text{in } F \cap V^n \text{ with } V^n \in \mathcal{V}^n\}, \\ h(\delta \,|\, F, \mathcal{V}) &:= \limsup_{n\to+\infty} \frac{1}{n} H(n,\delta \,|\, F, \mathcal{V}), \\ h(X \,|\, F, \mathcal{V}) &:= \lim_{\delta\to 0} h(\delta \,|\, F, \mathcal{V}). \end{aligned}$$

Then for any ergodic measure ν we put

$$h^{\mathrm{New}}(X \,|\, \nu, \mathcal{V}) := \lim_{\sigma\to 1} \inf_{\nu(F)>\sigma} h(X \,|\, F, \mathcal{V}).$$

Finally we extend the function $h^{\mathrm{New}}(X \,|\, \cdot, \mathcal{V})$ to $\mathcal{M}(X,T)$ by harmonic extension. Given a sequence $(\mathcal{V}_k)_{k\in\mathbb{Z}^+}$ of finite open covers whose diameter is converging to 0 and with $\mathcal{V}_{k+1}$ finer than $\mathcal{V}_k$ for all $k \in \mathbb{Z}^+$, we consider the sequence $\mathcal{H}^{\mathrm{New}} = (h^{\mathrm{New}}_k)_{k\in\mathbb{Z}^+} := (h - h^{\mathrm{New}}(X \,|\, \cdot, \mathcal{V}_k))_{k\in\mathbb{Z}^+}$. T. Downarowicz proved that this sequence defines an entropy structure ([4]) [13] for homeomorphisms and the present author [9] extended that result to the noninvertible case.

2.3. Estimate theorem. One of the main tools introduced in [14] is the so-called Estimate Theorem. We can roughly summarize its statement as follows: in order to estimate the symbolic extension entropy function one only needs to bound the local entropy of an ergodic measure near an invariant measure by the difference of the values of some convex upper semicontinuous function on $\mathcal{M}(X,T)$ at these two measures.

THEOREM 1 (Downarowicz–Maass [14], [12]). *Let (X,T) be a dynamical system with finite topological entropy and fix some $r>1$. Let g be an upper semicontinuous convex positive function on $\mathcal{M}(X,T)$ such that for every $\gamma>0$ and $\mu\in\mathcal{M}(X,T)$ there exist $\tau_\mu>0$ and a finite open cover $\mathcal{V}_\mu>0$ such that for every ergodic measure ν with $\operatorname{dist}(\nu,\mu)<\tau_\mu$ we have*

$$h^{\mathrm{New}}(M\,|\,\nu,\mathcal{V}_\mu)\le g(\mu)-g(\nu)+\gamma. \tag{1}$$

Then there exists a symbolic extension $\pi:(Y,S)\to(X,T)$ satisfying $h^{\pi}_{\mathrm{ext}}=\overline{g}$. In particular $h_{\mathrm{sex}}\le h+\overline{g}$.

We will apply this theorem to smooth dynamical systems where the map g is related to the Lyapunov exponents of invariant measures.

2.4. Ruelle's inequality. Given a compact Riemannian manifold $(M,\|\,\|)$ of dimension d and an integer $k\le d$, we consider the vector bundle Λ^kT^*M over M whose fiber at $x\in M$ is the space of k-forms w_x on the tangent space T_xM. It inherits a norm from the Riemannian structure of M as follows: $\|w_x\|=\sup|w_x(e_1,\dots,e_k)|$ where the supremum is taken over all the orthonormal families $(e_1,\dots,e_k)\in(T_xM)^k$. A $\mathcal{C}^1$ map T on M induces naturally a map $DT^{\wedge k}$ on Λ^kT^*M defined by $D_xT^{\wedge k}(w_x)(v_1,\dots,v_k)=w_x(D_xTv_1,\dots,D_xTv_k)$ for any $w_x\in\Lambda^kT^*_xM$ and any k-tuple $(v_1,\dots,v_k)\in(T_xM)^k$. The operator norm $\|D_xT^{\wedge k}\|=\sup_{\|w_x\|\le1}\|D_xT^{\wedge k}(w_x)\|$ is simply the supremum of the k-volumes of the ellipsoids $D_xT(D_k)$ over all the k-disks D_k of the tangent space with unit k-volume. For $k=n$ it coincides with the jacobian $\mathrm{Jac}_x(T)$ of T at x. Let $\|D_xT^{\wedge}\|=\max_{k=1,\dots,d}\|D_xT^{\wedge k}\|$. For all $k=1,\dots,d$ the cocycle $(x,n)\mapsto\log\|(D_xT^n)^{\wedge k}\|$ is subadditive so that given an ergodic measure ν one can define the *k-volume growth* of the action of DT on TM for ν as the limit $\lim_n n^{-1}\log\|(D_xT^n)^{\wedge k}\|$ for ν-generic points x. For $k=d$ the cocycle is in fact additive and the d-volume growth of DT coincides with $\int\log\mathrm{Jac}_x(T)\,d\nu(x)$. The k-volume growth of DT is related to the Lyapunov exponents as follows:

([4]) In particular $h^{\mathrm{New}}(X\,|\,\cdot,\mathcal{V}_k)$ converges pointwise to zero when k goes to infinity.

THEOREM 2 (Oseledets [18], [20]). *Let $T : M \to M$ be a $\mathcal{C}^1$ map defined on a compact Riemannian manifold $(M, \| \ \|)$ of dimension d. Let ν be an ergodic measure and $+\infty > \chi_1(\nu) \geq \cdots \geq \chi_d(\nu) \geq -\infty$ its Lyapunov exponents. Then for ν-almost all x,*

$$\lim_{n\to+\infty} \frac{1}{n} \log \|(D_x T^n)^{\wedge k}\| = \sum_{i=1}^{k} \chi_i(\nu) \quad \textit{for every } 0 \leq k \leq d$$

and thus

$$\lim_{n\to+\infty} \frac{1}{n} \log^+ \|(D_x T^n)^{\wedge}\| = \sum_{i=1}^{d} \chi_i^+(\nu).$$

Observe that in particular $\int \log \mathrm{Jac}_x(T)\, d\nu(x) = \sum_{i=1}^{d} \chi_i(\nu)$ for all ergodic measures ν. The affine function $g : \mathcal{M}(M,T) \to [-\infty, +\infty[$ defined by $g(\mu) = \int \log \mathrm{Jac}_x(T)\, d\mu(x)$ for all invariant measures μ is upper semicontinuous. Therefore ([5]) $g^+ := \max(g, 0)$ is an upper semicontinuous convex function.

In the following we are interested in the entropy of ergodic measures. We recall Ruelle's inequality which states that the entropy is bounded from above by the "maximal volume growth of DT":

THEOREM 3 (Ruelle's inequality). *Let $T : M \to M$ be a $\mathcal{C}^1$ map on a compact manifold M of dimension d. Then for all ergodic measures ν,*

$$h(\nu) \leq \sum_{i=1}^{d} \chi_i^+(\nu).$$

3. Statements. We first state our Main Theorem which holds for general $\mathcal{C}^r$ maps with $r > 1$:

MAIN THEOREM. *Let $T : M \to M$ be a $\mathcal{C}^r$ map, with $r > 1$, on a compact manifold of dimension d. Let μ be an invariant measure and fix some $\gamma > 0$. Then there exist $\tau_\mu > 0$ and a finite open cover $\mathcal{V}_\mu > 0$ such that for every ergodic measure ν with $\mathrm{dist}(\nu, \mu) < \tau_\mu$ we have*

$$h^{\mathrm{New}}(M \mid \nu, \mathcal{V}_\mu) \leq \frac{d(g^+(\mu) - g^+(\nu))}{r-1} - \sum_{i=1}^{d} \chi_i^-(\nu) + \gamma \tag{2}$$

where $g^+(\xi) = \max(\int \log \mathrm{Jac}_x(T)\, d\xi(x), 0)$ for all invariant measures ξ.

If we assume moreover that T is nonuniformly entropy expanding then the conclusion of the Main Theorem can be rewritten as

$$h^{\mathrm{New}}(M \mid \nu, \mathcal{V}_\mu) \leq \frac{d(\sum_{i=1}^{d} \chi_i^+(\mu) - \sum_{i=1}^{d} \chi_i^+(\nu))}{r-1} + \gamma.$$

([5]) In the following we use the notations $a^+ = \max(a, 0)$ and $a^- = \min(a, 0)$.

Indeed if ν is an ergodic measure with nonzero entropy then all its Lyapunov exponents are by assumption nonnegative. Therefore $g(\nu) = \int \log \mathrm{Jac}_x(T)\, d\nu(x) = \sum_{i=1}^d \chi_i(\nu) = \sum_{i=1}^d \chi_i^+(\nu) = g^+(\nu)$. Moreover $g^+(\mu) \le \sum_{i=1}^d \chi_i^+(\mu)$ for all invariant measures μ.

The Estimate Theorem (Theorem 1) implies:

THEOREM 4. *Let $T: M \to M$ be a $\mathcal{C}^r$ nonuniformly entropy expanding map, with $r > 1$, defined on a compact manifold M of dimension d. Then there exists a symbolic extension $\pi : (Y, S) \to (X, T)$ such that*

$$h_{\mathrm{ext}}^{\pi} = h + \frac{d}{r-1}\overline{\sum_{i=1}^{d} \chi_i^+}.$$

In particular,

$$h_{\mathrm{sex}} \le h + \frac{d}{r-1}\overline{\sum_{i=1}^{d} \chi_i^+}.$$

Then the usual variational principle for the entropy and the obvious inequality $\chi_1(\nu) \le R(T)$ for all ergodic measures ν yield (6):

COROLLARY 1. *Let $T : M \to M$ be a $\mathcal{C}^r$ nonuniformly entropy expanding map, with $r > 1$, defined on a compact manifold M of dimension d. Then there exists a symbolic extension (Y, S) of (X, T) such that*

$$h_{\mathrm{top}}(S) \le h_{\mathrm{top}}(T) + \frac{d^2 R(T)}{r-1}.$$

In particular,

$$h_{\mathrm{sex}}(T) \le h_{\mathrm{top}}(T) + \frac{d^2 R(T)}{r-1}.$$

Let $T : M \to M$ be a $\mathcal{C}^1$ map on a compact manifold M. An *n-invertible branch* is any set $A_n \subset M$ such that for each $0 \le k < n$ the set $T^k A_n$ is open and the map $T|_{T^k A_n}$ is a diffeomorphism onto $T^{k+1} A_n$. Any connected component of the set $\{x : \mathrm{Jac}_x(T^n) \neq 0\}$ of noncritical points of T^n is an n-invertible branch. Such n-invertible branches will be called *maximal*. In dimension one, n-invertible branches coincide with the branches of monotonicity of T^n.

The proof of the Main Theorem goes as follows:

- we first prove a Ruelle inequality which bounds from above the entropy in the invertible branches by the sum of the negative Lyapunov exponents;

(6) In fact it is easily seen that the following variational principle holds: $\sup_{\nu \in \mathcal{M}_e(M,T)} \chi_1^+(\nu) = R(T)$.

- then, given a $\mathcal{C}^r$ map, we count the number of invertible branches with a large jacobian;
- finally we bound the Newhouse local entropy of ergodic measures as in [14].

4. Inverse Ruelle inequality. To estimate the entropy of a $\mathcal{C}^1$ map in the invertible branches we introduce the following quantity. We fix a number $\delta > 0$, an integer n, an n-invertible branch A_n and a Borel set $F \subset M$. We define

$$\begin{aligned} H^{\mathrm{inv}}(n, \delta \mid F, A_n) &:= \log \max\{\sharp E : E \text{ is } (n,\delta) \text{ separated in } F \cap A_n\}, \\ H^{\mathrm{inv}}(n, \delta \mid F) &:= \sup_{A_n} H^{\mathrm{inv}}(n, \delta \mid F, A_n), \\ h^{\mathrm{inv}}(\delta \mid F) &:= \limsup_{n \to +\infty} \frac{1}{n} H^{\mathrm{inv}}(n, \delta \mid F), \\ h^{\mathrm{inv}}(M \mid F) &:= \lim_{\delta \to 0} h^{\mathrm{inv}}(\delta \mid F). \end{aligned}$$

Then for any ergodic measure ν we put

$$h^{\mathrm{inv}}(\nu) := \lim_{\sigma \to 1} \inf_{\nu(F) > \sigma} h^{\mathrm{inv}}(M \mid F).$$

We prove in this section the following "inverse Ruelle inequality" of independent interest:

THEOREM 5. *Let $T : M \to M$ be a $\mathcal{C}^{1+\eta}$ map with $\eta > 0$. Then for all ergodic measures ν,*

$$h^{\mathrm{inv}}(\nu) \leq -\sum_{i=1}^{d} \chi_i^-(\nu).$$

Clearly $h^{\mathrm{inv}}(\nu)$ is less than or equal to the usual Kolmogorov–Sinai entropy $h_T(\nu)$. When T is a diffeomorphism, $h^{\mathrm{inv}}(\nu)$ is equal to $h_T(\nu)$. It is well known that $h_T(\nu) = h_{T^{-1}}(\nu)$ and thus Theorem 5 follows in this case from the usual Ruelle inequality. When T is a local diffeomorphism, $h^{\mathrm{inv}}(\nu)$ is greater than or equal to $h_k^{\mathrm{New}}(\nu)$ for large k. In the one-dimensional case it follows easily from the total order on $\mathbb{R}$ that the cardinality of an (n, δ) separated set lying in a given monotone branch of T^n is bounded from above by n/δ and so $h^{\mathrm{inv}}(\nu)$ is zero.

To prove the usual Ruelle inequality one relates the maximal volume growth of DT, which is equal to the sum of the positive Lyapunov exponents, to the maximal volume growth of T, that is, the maximal volume growth of disks of the Riemannian manifold $(M, \| \ \|)$. It is then convenient to work with the exponential map of M to make the connection between these two quantities. We recall the basic properties of the exponential map which we use in the present paper. We refer to [16] for a definition and further developments. We denote by ∂ the distance induced on M by the Riemannian structure and by $\exp_x : T_xM \to M$ the exponential map at $x \in M$. The derivative of $\exp_x$ at the origin of T_xM is the identity map so that by the Inverse Function Theorem the restriction of $\exp_x$ to the ball of the tangent space at x centered at the origin with radius r is a diffeomorphism onto its image for small $r > 0$. The radius of injectivity $R_{\rm inj}$ of the compact Riemannian manifold M is the largest $r > 0$ such that the previous property holds for all $x \in M$. Furthermore the exponential map $\exp_x$ maps bijectively the ball of the tangent space at x centered at the origin with radius $r < R_{\rm inj}$, denoted by $B_x(0,r) := \{v \in T_xM : \|v\|_x < r\}$, onto the ball of M centered at x with radius $r > 0$, denoted by $B(x,r) := \{y \in M : \partial(x,y) < r\}$. The global exponential map $\exp : TM \to M$ defined by $\exp(x,v) = \exp_x(v)$ is $\mathcal{C}^1$. In particular there exists $R < R_{\rm inj}$ such that $\|D_y \exp_x\| < 2$ and $\|D_z(\exp_x^{-1})\| < 2$ for all $y \in B_x(0,R)$, all $z \in B(x,R)$ and all $x \in M$.

We now introduce some geometrical tools which will be useful in the proof of our Ruelle inequality. For each $\alpha > 0$ let M_α be a subset of M which meets any ball of radius $\alpha/16$. One can assume that $\alpha^d \sharp M_\alpha$ is bounded above by a constant $C(M)$ depending only on M. For example consider a finite atlas $\mathcal{A} = \{\Phi_1, \dots, \Phi_K\}$ such that the local charts $\Phi_i :]0,1[^d \to M$ satisfy $\|D\Phi_i\| \le 1$ for all $i = 1, \dots, K$. For all $\beta > 0$ let $L_\beta = \{k\beta \in]0,1[^d : k \in \mathbb{Z}^+\}$. Then the set $M_\alpha = \bigcup_{i=1,\dots,K} \Phi_i(L_{\alpha/16\sqrt{d}})$ meets any ball of radius $\alpha/16$. For each subset S of M let $\mathrm{Cov}(\alpha, S)$ be a subset of M_α with minimal cardinality such that the balls of radius $\alpha/2$ centered at the points of $\mathrm{Cov}(\alpha, S)$ cover S. Let $x \in M$ and $E \subset T_xM$ be an ellipsoid centered at the origin of T_xM. We denote by $\|E\|^{\wedge k}$ the supremum of the k-volumes of $E \cap V$ over all the vector subspaces V of T_xM of dimension k. Let $\|E\|^\wedge = \max_{k=1,\dots,d} \|E\|^{\wedge k}$. With these notations we have $\|D_yT(B_y(0,1))\|^\wedge = \|D_yT\|^\wedge$ for all $y \in M$.

LEMMA 1. *Let $R > \alpha_1, \alpha_2 > 0$. Let $x \in M$ and let $E \subset T_xM$ be an ellipsoid centered at the origin of T_xM such that $\alpha_1 E \subset B_x(0, R/2)$. Then*

$$\sharp\,\mathrm{Cov}(\alpha_2, \exp_x(\alpha_1 E)) \le P([\|E\|^\wedge] + 1)(\max(\alpha_1, \alpha_2)/\alpha_2)^d$$

with a constant P depending only on d.

Proof. Since M_{α_2} meets any ball of radius less than $\alpha_2/16$ and since $\|D_y(\exp_x^{-1})\| < 2$ for all $y \in B(x,R)$ the set $\exp_x^{-1}(M_{\alpha_2})$ meets any ball of radius less than $\alpha_2/8$ which is included in $B_x(0,R)$. The ellipsoid $\alpha_1 E$ can be covered by at most $[\|E\|^\wedge] + 1$ cubes of size α_1 and therefore by at most $([\|E\|^\wedge] + 1)([8\max(\alpha_1,\alpha_2)\sqrt{d}/\alpha_2] + 1)^d$ cubes of size $\alpha_2/8\sqrt{d}$. Such a cube intersecting $\alpha_1 E$ is included in a subball of $B_x(0,R)$ of radius $\alpha_2/8$ and therefore in a subball of $B_x(0,R)$ of radius $\alpha_2/4$ centered at a point of $\exp_x^{-1}(M_{\alpha_2})$. This last subball is mapped by $\exp_x$ into a ball of M of radius

$\alpha_2/2$ centered at a point of M_{α_2} because $\|D_y \exp_x\| < 2$ for all $y \in B_x(0,R)$. We conclude that

$$\mathrm{Cov}(\alpha_2, \exp_x(\alpha_1 E)) \leq ([\|E\|^\wedge] + 1)([8\max(\alpha_1,\alpha_2)\sqrt{d}/\alpha_2] + 1)^d. \ \blacksquare$$

LEMMA 2. *Let ν be an ergodic measure with $\int \log \mathrm{Jac}_x T\, d\nu(x) > -\infty$. Then $(n^{-1}\log^+ \|(D_x T^n)^{-1}\|^\wedge)_{n\in\mathbb{Z}^+}$ converges to $-\sum_{i=1}^d \chi_i^-(\nu)$ for ν-almost all x and $n^{-1}\int \log^+ \|(D_xT^n)^{-1}\|^\wedge\, d\nu(x)$ converges to $-\sum_{i=1}^d \chi_i^-(\nu)$ when n goes to infinity.*

Proof. It is well known that invertible $d\times d$ matrices, endowed with the operator norm $\|\ \|$ induced by the Euclidean norm, satisfy the relations $\|A\|\,\|A^{-1}\| \geq 1$ and $|\det(A)| \leq \|A\|^{d-1}/\|A^{-1}\|$. Therefore

$$\begin{aligned} +\infty &> (d-1)\log\|DT\|_\infty - \int \log \|(D_xT)^{-1}\|\, d\nu(x) \\ &\geq \int \log \mathrm{Jac}_x(T)\, d\nu(x) > -\infty \end{aligned}$$

and so the map $x \mapsto \log\|(D_xT)^{-1}\|$ is ν-integrable. Let $(\overline{M}, \overline{\nu})$ be the natural extension of (M,ν). The invertible cocycle $\overline{x} = (\dots, x_{-1}, x_0, x_1, \dots) \mapsto D_{x_0}T$ over the natural extension is integrable because $\int \log^+ \|D_{x_0}T\|\, d\overline{\nu}(\overline{x}) = \int \log^+\|D_xT\|\, d\nu(x) < +\infty$ and

$$\int \log^+ \|(D_{x_0}T)^{-1}\|\, d\overline{\nu}(\overline{x}) = \int \log^+\|(D_xT)^{-1}\|\, d\nu(x) < +\infty.$$

This cocycle has the same Lyapunov exponents as T, and the sequence

$$\frac{1}{n}\int \log^+ \|(D_{x_{-n}}T^n)^{-1}\|^\wedge\, d\overline{\nu}(\overline{x}) = \frac{1}{n}\int \log^+ \|(D_xT^n)^{-1}\|^\wedge\, d\nu(x)$$

converges to $-\sum_{i=1}^d \chi_i^-(\nu)$ when n goes to infinity according to the cocycle invertible version of Oseledets' Theorem [20]. Now by Kingman's subadditive ergodic theorem applied to the subadditive sequence of integrable functions $x \mapsto \log^+\|(D_xT^n)^{-1}\|^\wedge$ the limit $\lim_{n\to+\infty} n^{-1}\log^+\|(D_xT^n)^{-1}\|^\wedge$ exists for ν-almost all x and coincides with the limit of the integrals $\lim_{n\to+\infty} n^{-1}\int \log^+\|(D_xT^n)^{-1}\|^\wedge d\nu(x)$. $\blacksquare$

It is convenient in the next proofs to use the following terminology:

DEFINITION 1. Let $S \in \mathbb{Z}^+$ and $n \in \mathbb{Z}^+$. We say that a sequence $\mathcal{K}_n := (k_1, \dots, k_n)$ of n positive integers *misses the value* S if $n^{-1}\sum_{i=1}^n k_i \leq S$.

The number of sequences of n positive integers missing the value S is exactly the binomial coefficient $\binom{nS}{n}$. We denote by $H : [0,1] \to \mathbb{R}$ the map defined by $H(t) = -t\log t - (1-t)\log(1-t)$. It is easily seen that

$$\log\binom{nS}{n} \leq nSH(S^{-1}) + 1. \tag{3}$$

For all $\gamma > 0$ we fix $S_\gamma \in \mathbb{Z}^+$ so that $H(S^{-1}) < \gamma$ for all $S \geq S_\gamma$.

We now prove our Ruelle inequality. First let us briefly outline the proof. We will consider some iterate T^N of T such that the negative Lyapunov exponents of ν are almost given by the average of the norm of $(DT^N)^{-1}$ along the orbits of typical points. Then, given an nN-invertible branch A_{nN}, we bound the cardinality of any (n,δ) separated (for T^N) set E in A_{nN} by shadowing the orbits of $T^{nN}E$ under the action of T^{-N}. In the usual Ruelle inequality the orbits under forward iterates are shadowed. Our situation is more difficult because the derivative of T^{-N} is not bounded near the critical values of T^N. However we show that the integrability assumption, $\int \log \operatorname{Jac}_x T \, d\nu(x) > -\infty$, and the Hölder property of the differential allow us to neglect the growth of orbits near the critical values. The proof goes as follows:

- we first define N and exploit our integrability asumption;
- then we bound the volume T^{-N}-growth of balls by distinguishing two cases depending on whether we are far from or close to the critical values of T^N;
- we prove a combinatorial estimate which allows us to consider (n,δ) separated sets E such that the size of $\|DT^{-1}\|$ along the T-orbit of E is fixed;
- finally we detail our shadowing construction.

Proof of Theorem 5. Fix an ergodic measure ν with $\int \log \operatorname{Jac}_x T \, d\nu(x) > -\infty$. Let $\gamma > 0$. By Lemma 2 there exists an integer N and a Borel set G with $\nu(G) > 1 - \gamma/\max(-\sum_{i=1}^d \chi_i^-(\nu), 1)$ such that for all $x \in G$,

$$(4) \qquad -\sum_{i=1}^{d} \chi_i^-(\nu) - \gamma < \frac{1}{N} \log^+ \|(D_x T^N)^{-1}\|^\wedge < -\sum_{i=1}^{d} \chi_i^-(\nu) + \gamma$$

and

$$\frac{1}{N} \int \log^+ \|(D_x T^N)^{-1}\|^\wedge \, d\nu(x) < -\sum_{i=1}^{d} \chi_i^-(\nu) + \gamma.$$

From the above inequalities one deduces easily that

$$\frac{1}{N} \int_{M\setminus G} (\log^+ \|(D_x T^N)^{-1}\|^\wedge + 1) \, d\nu(x) < 4\gamma.$$

Observe also that the set $\operatorname{Crit}(T)$ of critical points has zero ν-measure. Let us denote by $\operatorname{Crit}(T)^\theta = \{y \in M : \partial(y, \operatorname{Crit}(T)) < \theta\}$ the θ-neighborhood of the set of critical points. We also put $\operatorname{Crit}(T)^\theta_N = \bigcup_{k=0}^{N-1} T^{-k} \operatorname{Crit}(T)^\theta$. Since $x \mapsto \sum_{0 \le j < N} \log^+ \|(D_{T^j x} T)^{-1}\|$ is a ν-integrable function there exists $\theta > 0$ such that

$$\int_{\operatorname{Crit}(T)^\theta_N} \sum_{0 \le j < N} (\log^+ \|(D_{T^j x} T)^{-1}\| + 1) \, d\nu(x) < \gamma.$$

Let $\sigma \in]0,1[$. By Birkhoff's ergodic theorem and the previous two inequalities there exists a set F with $\nu(F) > \sigma$ and an integer n_0 such that for all $x \in F$ and $n \geq n_0$,

$$(5)\qquad \frac{1}{nN}\sum_{0\leq l<nN} \mathbb{1}_{\mathrm{Crit}(T)_N^\theta}(T^l x)\sum_{0\leq j<N}(\log^+\|(D_{T^{l+j}x}T)^{-1}\|+1) < \gamma$$

(we write $\mathbb{1}_E$ for the characteristic function of a subset E of M) and

$$\frac{1}{nN^2}\sum_{0\leq l<nN}\mathbb{1}_{M\setminus G}(T^l x)(\log^+\|(D_{T^l x}T^N)^{-1}\|^\wedge + 1) < 4\gamma,$$

in particular there exists some $0 \leq i(x) < N$ such that

$$(6)\qquad \frac{1}{nN}\sum_{0\leq k<N}\mathbb{1}_{M\setminus G}(T^{i(x)+kN}x)(\log^+\|(D_{T^{i(x)+kN}x}T^N)^{-1}\|^\wedge + 1) < 4\gamma.$$

Let n be an integer larger than n_0 and let A_{nN} be an nN-invertible branch. We first control the growth of balls under T^{-1} for pieces of orbits far from the set of critical values. As T is a $\mathcal{C}^1$ map there exists, by an easy continuity argument, a number $0 < \delta < \theta$ such that for all $0 < r < \delta$, all $y \in T^{i+kN}A_{nN} \cap (M \setminus \mathrm{Crit}(T)_N^\theta)$ with $i + (k+1)N < nN$ and all $z \in B(T^N y, r)$,

$$(7)\qquad (T^N|_{T^{i+kN}A_{nN}})^{-1}B(z,r) \subset \exp_y((D_yT^N)^{-1}B_{T^N y}(0,3r)).$$

Observe that δ can be chosen independent of the choice of the invertible branch A_{nN}.

Now we give satisfactory estimates for pieces of orbits close to the set of critical points. Choose $R < R' < R_{\mathrm{inj}}$ such that $T(B(x,R)) \subset B(Tx,R')$ for all $x \in M$. We consider the local dynamics $\mathcal{T}_x : B_x(0,R) \to B_{Tx}(0,R')$ at x in the local charts defined by the exponential map, i.e. $\mathcal{T}_x := \exp_{Tx}^{-1} \circ T \circ \exp_x$. Fix $y \in T^t A_{nN}$ with $0 \leq t < nN$. Let $0 < Q < R$ be a constant depending only on T such that $\|(D\mathcal{T}_y)|_{B_y(0,Q)}\|_\eta \leq 2\|DT\|_\eta$ and let $h \in T_yM$ with $\|h\| \leq Q$. First notice that

$$\mathcal{T}_y(h) - D_yT(h) = \mathcal{T}_y(h) - D_0\mathcal{T}_y(h) = \int_0^1 (D_{th}\mathcal{T}_y(h) - D_0\mathcal{T}_y(h))\,dt.$$

Then by using the Hölder property of the differential we have

$$\|\mathcal{T}_y(h) - D_yT(h)\| \leq \|(D\mathcal{T}_y)|_{B_y(0,Q)}\|_\eta \|h\|^{1+\eta} \leq 2\|DT\|_\eta\|h\|^{1+\eta}.$$

By assuming $r < a\max(\|(D_yT)^{-1}\|,1)^{-(1+\eta)/\eta}$ with a constant a we have, for all $h \in B_y(0, 2\max(\|(D_yT)^{-1}\|,1)r)$,

$$\begin{aligned}\|\mathcal{T}_y(h) - D_yT(h)\| &< 2\|DT\|_\eta(2\max(\|(D_yT)^{-1}\|,1))^{1+\eta}r^{1+\eta}\\ &< 2\|DT\|_\eta(2\max(\|(D_yT)^{-1}\|,1))^{1+\eta}a^\eta\max(\|(D_yT)^{-1}\|,1)^{-1-\eta}r < r,\end{aligned}$$

where the last inequality follows from an appropriate choice of the constant a.

Moreover obviously $B_{Ty}(0,2r) \subset D_yT(B_y(0,2\max(\|(D_yT)^{-1}\|,1)r))$. Therefore $B_{Ty}(0,r) \subset \mathcal{T}_y(B_y(0,2\max(\|(D_yT)^{-1}\|,1)r))$. Finally by taking the exponential map $\exp_{Ty}$, for all $0 \le r < a\max(\|(D_yT)^{-1}\|,1)^{-(\eta+1)/\eta}$ we have

$$(T|_{T^tA_{nN}})^{-1}B(Ty,r) \subset B(y,2\max(\|(D_yT)^{-1}\|,1)r). \tag{8}$$

We are going to bound $\max\{\sharp E : E \subset F \cap A_{nN}$ and E is (nN,δ) separated$\}$. There exists $\delta' < \delta$ such that $\partial(y,z) < \delta' \Rightarrow \partial(T^ky,T^kz) < \delta$ for $0 \le k \le N$. We also choose $\delta' < a$. Notice that δ' (as δ) does not depend on the invertible branch A_{nN}.

In order to estimate the growth of orbits in $F \cap A_{nN}$ near the critical points, for each $z \in M \setminus \mathrm{Crit}(T)$ we let

$$\psi(z) = [\log_2^+ \|(D_zT)^{-1}\|] + 1.$$

We also consider the following sequences for all $y \in F \cap A_{nN}$:

$$\begin{aligned} J_n(y) &= (\mathbb{1}_{\mathrm{Crit}(T)_N^\theta}(T^{i(y)+kN}y))_{0\le k\le n-2},\\ \widetilde{J}_n(y) &= (\mathbb{1}_{\mathrm{Crit}(T)_N^\theta}(T^{i(y)+kN}y)\psi(T^{i(y)+kN+l}y))_{\substack{0\le k\le n-2\\ 0\le l\le N-1}}. \end{aligned}$$

In order to control the dynamics far from the critical points, for each $z \in M \setminus \mathrm{Crit}(T^N)$ we let

$$\xi(z) = [\log_2^+ \|(D_zT^N)^{-1}\|^\wedge] + 1$$

and we consider the following sequences for all $y \in F \cap A_{nN}$:

$$\begin{aligned} H_n(y) &= (\mathbb{1}_{M\setminus(G\cup\mathrm{Crit}(T)_N^\theta)}(T^{i(y)+kN}y))_{0\le k\le n-2},\\ \widetilde{H}_n(y) &= (\mathbb{1}_{M\setminus(G\cup\mathrm{Crit}(T)_N^\theta)}(T^{i(y)+kN}y)\xi(T^{i(y)+kN}y))_{0\le k\le n-2}. \end{aligned}$$

Now we estimate the cardinality of $\{(\widetilde{H}_n(y),\widetilde{J}_n(y)) : y \in F \cap A_{nN}$ and $i(y) = i$, $H_n(y) = \underline{H}$, $J_n(y) = \underline{J}\}$ for some fixed $n \ge n_0$, $i \in \{0,\dots,N-1\}$, $\underline{H} = (\underline{H}_0,\dots,\underline{H}_{n-2}) \in \{0,1\}^{n-1}$ and $\underline{J} = (\underline{J}_0,\dots,\underline{J}_{n-2}) \in \{0,1\}^{n-1}$. We consider the sequences $\{h_1,\dots,h_{M_{\underline{H}}}\} = \{0 \le j \le n-2 : \underline{H}_j = 1\}$ and $\{j_1,\dots,j_{M_{\underline{J}}}\} = \{0 \le j \le n-2 : \underline{J}_j = 1\}$. For all $y \in F \cap A_{nN}$ with $i(y) = i$, $H_n(y) = \underline{H}$ and $J_n(y) = \underline{J}$ we put

$$\mathcal{H}_n(y) = (\xi(T^{i+h_mN}y))_{1\le m\le M_{\underline{H}}}, \qquad \mathcal{J}_n(y) = (\psi(T^{i+j_mN+l}y))_{\substack{1\le m\le M_{\underline{J}}\\ 0\le l\le N-1}}.$$

The above sequences $\mathcal{H}_n(y)$ and $\mathcal{J}_n(y)$ coincide respectively with $\widetilde{H}_n(y)$ and $\widetilde{J}_n(y)$ once the zeros are removed. Observe that the sequences $\mathcal{H}_n(y)$ and

$\mathcal{J}_n(y)$ miss respectively the values $S_{\underline{H}}$ and $S_{\underline{J}}$ where

$$S_{\underline{H}} := \frac{1}{M_{\underline{H}}} \sup_{y\in F} \sum_{1\leq m\leq M_{\underline{H}}} (\log_2^+ \|(D_{T^{i+h_m N}y}T^N)^{-1}\|^\wedge + 1),$$

$$S_{\underline{J}} := \frac{1}{M_{\underline{J}}N} \sup_{y\in F} \sum_{1\leq m\leq M_{\underline{J}}} \sum_{0\leq l<N} (\log_2^+ \|(D_{T^{i+j_m N+l}y}T)^{-1}\| + 1).$$

By (6) and (5) we have

$$M_{\underline{H}}S_{\underline{H}} = \sup_{y\in F} \sum_{1\leq m\leq M_{\underline{H}}} (\log_2^+ \|(D_{T^{i+h_m N}y}T^N)^{-1}\|^\wedge + 1) < \frac{4\gamma nN}{\log 2},$$

$$M_{\underline{J}}NS_{\underline{J}} = \sup_{y\in F} \sum_{1\leq m\leq M_{\underline{J}}} \sum_{0\leq l<N} (\log_2^+ \|(D_{T^{i+j_m N+l}y}T)^{-1}\| + 1) < \frac{\gamma nN}{\log 2}.$$

By (3) the logarithm of $\sharp\{(\widetilde{H}_n(y), \widetilde{J}_n(y)) : y \in F\cap A_{nN}$ and $i(y) = i$, $H_n(y) = \underline{H}$, $J_n(y) = \underline{J}\}$ is bounded above by $5\gamma nN + 2$ and thus we finally get

$$\log \sharp\{(i(y), \widetilde{H}_n(y), \widetilde{J}_n(y)) : y \in F\cap A_{nN}\} \leq (2\log 2+5\gamma N)n+\log N+2. \tag{9}$$

Fix $0 \leq i < N$, $\widehat{H} = (\widehat{H}_0, \ldots, \widehat{H}_{n-2}) \in {\mathbb{Z}^+}^{n-1}$, $\widehat{J} = (\widehat{J}_0, \ldots, \widehat{J}_{(n-1)N-1}) \in {\mathbb{Z}^+}^{(n-1)N}$. By the combinatorial estimate (9) we only need to bound the cardinality of (n, δ) separated sets ([7]) in

$$F(i, \widehat{H}, \widehat{J}) := F \cap A_{nN} \cap \{y : i(y) = i,\ \widetilde{H}_n(y) = \widehat{H},\ \widetilde{J}_n(y) = \widehat{J}\}.$$

To this end we would like to δ'-shadow the orbits of $T^{nN}F(i, \widehat{H}, \widehat{J})$ under the action of T^{-N} by sequences of points in $M_{\delta'}$, i.e. associate to each $y \in F(i, \widehat{H}, \widehat{J})$ a sequence $y_1, \ldots, y_N \in M_{\delta'}$ such that $d(y_k, T^{kN}y) < \delta'$ for all $k = 1, \ldots, N$. But the volume T^{-1}-growth of balls near the critical values of T is not uniformly bounded and is controlled for balls with radius small compared to the inverse of the norm of DT^{-1} according to (7). Therefore we will also shadow the orbit under the action of T^{-1} when we are close to the critical value of T, and the shadowing scales this time will vary with the norm of DT^{-1}.

We define the sequence $(\delta_j)_{i\leq j\leq i+(n-1)N}$ of shadowing scales by

$$\delta_i = \delta', \qquad \delta_{i+t} = \min(\delta', a2^{-\frac{\eta+1}{\eta}\widehat{J}_{t-1}}) \qquad \text{for } t = 1, \ldots, (n-1)N - 1$$

and $\delta_{i+(n-1)N} = \delta'$ again. Observe that

$$\frac{\max(\delta_{i+t}, \delta_{i+t+1})}{\delta_{i+t+1}} \leq 2^{\frac{\eta+1}{\eta}\widehat{J}_t} \qquad \text{for all } 0 \leq t < (n-1)N.$$

([7]) We use the notation $\widetilde{J}_n(x) := (\mathbb{1}_{\mathrm{Crit}(T)^\theta_N}(T^{i(x)+[s/N]}x)\psi(T^{i(x)+s}x))_{0\leq s<(n-1)N}$, slightly different than before.

We shadow the orbits of the set $F(i,\widehat{H},\widehat{J})$ by associating to any y in this set a point $d(y)$ in $M_{\delta'}\cap B(y,\delta'/2)$ and a sequence $C_n(y)=(c(T^{i+kN}y))_{0\leq k\leq n-1}$ where $c(T^{i+kN}y)$ is a point in $M_{\delta_{i+kN}}\cap B(T^{i+kN}y,\delta_{i+kN}/2)$ if $k=n-1$ or $T^{i+kN}y\notin \mathrm{Crit}(T)^\theta_N$, and $c(T^{i+kN}y)$ is an N-tuple $c(T^{i+kN}y)=(C_0,\dots,C_{N-1})$ with $C_l\in M_{\delta_{i+kN+l}}\cap B(T^{i+kN+l}y,\delta_{i+kN+l}/2)$ otherwise.

If $d(y)=d(z)$ and $C_n(y)=C_n(z)$ then z belongs to the Bowen ball $B(y,nN,\delta)$ and therefore y and z are not (nN,δ) separated. Indeed we have first $\partial(y,z)\leq\partial(y,d(y))+\partial(d(z),z)<\delta'$ and so $\partial(T^ly,T^lz)<\delta$ for all $0\leq l<N$. Secondly if m is an integer with $i<N\leq m<nN$ there exist $0\leq k<n$ and $0\leq l<N$ satisfying $m=i+kN+l$. But $c(T^{i+kN}y)=c(T^{i+kN}z)$ implies that

$$\begin{aligned}\partial(T^{i+kN}y,T^{i+kN}z)&\leq\partial(T^{i+kN}y,c(T^{i+kN}y))+\partial(c(T^{i+kN}z),T^{i+kN}z)\\&<\frac{\delta_{i+kN}+\delta_{i+kN}}{2}\leq\delta'\end{aligned}$$

and hence $\partial(T^my,T^mz)<\delta$.

Now we build the sequences $C_n(y)$ for $y\in F(i,\widehat{H},\widehat{J})$ and we estimate their cardinality. We will use the following claim which follows easily from (7) and (8) and Lemma 1:

Claim. *There exists a constant P depending only on d such that for all $0\leq k<n-1$ and $0\leq l<N$ and for all $y\in F(i,\widehat{H},\widehat{J})$:*

$$\begin{aligned}&\sharp\,\mathrm{Cov}(\delta_{i+kN},(T^N|_{T^{i+kN}A_{nN}})^{-1}B(z,\delta_{i+(k+1)N}/2))\\&\qquad\qquad\leq P\left(\frac{\delta_{i+(k+1)N}}{\delta_{i+kN}}\right)^d e^{N(-\sum_{l=1}^d\chi_l^-(\nu)+\gamma)}\end{aligned}$$

for all $z\in B(T^{i+(k+1)N}y,\delta_{i+(k+1)N}/2)$ with $T^{i+kN}y\in G\cap(M\setminus\mathrm{Crit}(T)^\theta_N)$;

$$\sharp\,\mathrm{Cov}(\delta_{i+kN},(T^N|_{T^{i+kN}A_{nN}})^{-1}B(z,\delta_{i+(k+1)N}/2))\leq P\left(\frac{\delta_{i+(k+1)N}}{\delta_{i+kN}}\right)^d 2^{\widehat{H}_k}$$

for all $z\in B(T^{i+(k+1)N}y,\delta_{i+(k+1)N}/2)$ with $T^{i+kN}y\in M\setminus(G\cup\mathrm{Crit}(T)^\theta_N)$;

$$\begin{aligned}&\sharp\,\mathrm{Cov}(\delta_{i+kN+l},(T|_{T^{i+kN+l}A_{nN}})^{-1}B(z,\delta_{i+kN+l+1}/2))\\&\qquad\qquad\leq P\left(\frac{\max(\delta_{i+kN+l},\delta_{i+kN+l+1})}{\delta_{i+kN+l}}2^{\widehat{J}_{kN+l}}\right)^d\end{aligned}$$

for all $z\in B(T^{i+kN+l+1}y,\delta_{i+kN+l+1}/2)$ with $T^{i+kN}y\in\mathrm{Crit}(T)^\theta_N$.

Notice that $\delta_{i+(k+1)N}=\delta'\geq\delta_{i+kN}$ in the first two cases of the Claim since $\widehat{J}_{(k+1)N-1}=0$ when $T^{i+kN}y\notin\mathrm{Crit}(T)^\theta_N$ for some $y\in F(i,\widehat{H},\widehat{J})$.

By decreasing induction on k we define $c(T^{i+kN}y)$ for all $y\in F(i,\widehat{H},\widehat{J})$ and show that

$$\sharp\{(c(T^{i+lN}y))_{k\leq l\leq n-1} : y\in F(i,\widehat{H},\widehat{J})\} \\ \leq \frac{C(M)}{\delta_{i+kN}^d}\cdot P^{n-k}e^{(n-k)N(-\sum_{l=1}^d \chi_l^-(\nu)+\gamma)} \\ \times 2^{\sum_{t=k}^{n-2}\widehat{H}_t}\cdot 2^{(\sum_{t=kN}^{(n-1)N-1}\widehat{J}_t)(d\frac{2\eta+1}{\eta}+\log_2 P)}. \tag{10}$$

First for all $y\in F(i,\widehat{H},\widehat{J})$ we put $c(T^{i+(n-1)N}y)=z$ where z is chosen in $\mathrm{Cov}(\delta_{i+(n-1)N},M)\cap B(T^{i+(n-1)N}y,\delta_{i+(n-1)N}/2)$. Then inequality (10) for $k=n-1$ follows from the Claim. Assume we have already defined $c(T^{i+(k+1)N}y)$ and that (10) holds for $k+1$. We distinguish two cases:

- $\widehat{J}_{kN}=0$, i.e. $T^{i+kN}y$ is far from the critical set of T^N; then we choose
$$c(T^{i+kN}y)\in \mathrm{Cov}(\delta_{i+kN},(T^N|_{T^{i+kN}A_{nN}})^{-1}B(c(T^{i+(k+1)N}y),\delta_{i+(k+1)N}/2)) \\ \cap B(T^{i+kN}y,\delta_{i+kN}/2).$$
- $\widehat{J}_{kN}\neq 0$, i.e. $T^{i+kN}y$ is close to the critical set of T^N; then we define $c(T^{i+kN}y)=(C_0,\dots,C_{N-1})$ with, for all $0\leq l\leq N-1$,
$$C_l\in \mathrm{Cov}(\delta_{i+kN+l},(T|_{T^{i+kN+l}A_{nN}})^{-1}B(C_{l+1},\delta_{i+kN+l+1}/2)) \\ \cap B(T^{i+kN+l}y,\delta_{i+kN+l}/2)$$
and with the convention $C_N=c(T^{i+(k+1)N}y)$.

Notice that

$$\frac{1}{\delta_{i+(k+1)N}}\prod_{l=0}^{N-1}\frac{\max(\delta_{i+kN+l},\delta_{i+kN+l+1})}{\delta_{i+kN+l}} \\ \leq \frac{1}{\delta_{i+kN}}\prod_{l=0}^{N-1}\frac{\max(\delta_{i+kN+l},\delta_{i+kN+l+1})}{\delta_{i+kN+l+1}} \\ \leq \frac{2^{\frac{\eta+1}{\eta}\sum_{l=0}^{N-1}\widehat{J}_{kN+l}}}{\delta_{i+kN}}. \tag{11}$$

By using the Claim and (11) we easily check by decreasing induction on k that (10) holds for all $k=0,\dots,n-1$. Then according to (6) and (5) we get, for all $n\geq n_0$,

$$\log\sharp\{(d(y),(c(T^{i+lN}y))_{k\leq l\leq n-1}) : y\in F(i,\widehat{H},\widehat{J})\} \\ \leq -2d\log\delta'+2\log C(M)+n\log P \\ +nN\Big(-\sum_{l=1}^d\chi_l^-(\nu)+\gamma\Big)+4\gamma nN+\gamma nN\Big(d\frac{2\eta+1}{\eta}+\log_2 P\Big).$$

Then by using the combinatorial estimate (9) we have

$$\begin{aligned}
&\log \max\{\sharp E : E \subset F \cap A_{nN} \text{ and } E \text{ is } (nN,\delta) \text{ separated}\}\\
&\qquad \le \log \sharp\{(i(y), \widetilde{H}_n(y), \widetilde{J}_n(y)) : y \in F \cap A_{nN}\}\\
&\qquad\quad + \sup_{\substack{i,\widehat{H},\widehat{J}\\ F(i,\widehat{H},\widehat{J})\neq\emptyset}} \log \sharp\{(d(y), C_n(y)) : y \in F(i,\widehat{H},\widehat{J})\}\\
&\qquad \le -2d\log\delta' + 2\log C(M) + n(2\log 2 + \log P) + \log N + 2\\
&\qquad\quad + nN\Big(-\sum_{l=1}^{d} \chi_l^-(\nu)\Big) + \gamma nN\bigg(d\frac{2\eta+1}{\eta} + \log_2 P + 10\bigg).
\end{aligned}$$

By taking N and then n_0 large enough, we get, for $n \ge n_0$,

$$\begin{aligned}
&\log \max\{\sharp E : E \subset F \cap A_{nN} \text{ and } E \text{ is } (nN,\delta) \text{ separated}\}\\
&\qquad \le nN\Big(-\sum_{l=1}^{d} \chi_l^-(\nu)\Big) + \gamma nN\bigg(d\frac{2\eta+1}{\eta} + \log_2 P + 11\bigg).
\end{aligned}$$

Finally for general $m \in \mathbb{Z}^+$ observe that if $\sigma > \rho > 0$ are such that $d(x,y) < \rho \Rightarrow d(T^k x, T^k y) < \sigma$ for all $0 \le k < N$, then any (m,σ) separated set in A_m is $([m/N]N,\rho)$ separated in $A_{[m/N]N}$. This easily concludes the proof of Theorem 5. ∎

5. Counting lemma. The following lemma is a generalization in any dimension of Lemma 4.1 of [14]. The proof given below is independent and based on a semi-algebraic approach.

LEMMA 3. *Let $f :]-1,1[^d \to \mathbb{R}$ be a $\mathcal{C}^r$ map with $r > 0$. Then there exists a constant c depending only on r and d such that for every $0 < s < 1$ the number of connected components of the open set $\{x : f(x) \neq 0\}$ on which $|f|$ reaches or exceeds the value s is at most $c\max(\|f\|_r, 1)^{d/r} s^{-d/r}$ where $\|f\|_r$ is the supremum norm $\|D^r f\|_\infty$ of the rth derivative if $r \in \mathbb{Z}^+$ and the $r-[r]$-Hölder norm $\|D^{[r]} f\|_{r-[r]}$ of the $[r]$th derivative* ([8]) *if $r \notin \mathbb{Z}^+$.*

Proof. We cover the unit square $]-1,1[^d$ by

$$\bigg(2\bigg[\bigg(\frac{as}{\max(\|f\|_r,1)}\bigg)^{-1/r}\bigg]+1\bigg)^d \text{ subsquares of size } < \bigg(\frac{as}{\max(\|f\|_r,1)}\bigg)^{1/r}$$

where $a = a(r,d)$ is a constant depending only on r and d which we specify later. Consider one such subsquare S and let P_S be the Lagrange polynomial of order $[r-1]$ at the center x_0 of S. By the Taylor–Lagrange formula we

([8]) By convention the 0th derivative of T is the map T itself.

have, for all $x \in S$,

$$f(x) = P_S(x) + \frac{1}{[r-1]!}\int_0^1 (1-t)^{[r]-1} D^{[r]}_{x_0+t(x-x_0)} f(x-x_0)^{[r]}\, dt$$

where $(x-x_0)^{[r]}$ denotes the vector $(\underbrace{x-x_0,\dots,x-x_0}_{[r]\text{ times}}) \in (\mathbb{R}^d)^{[r]}$. Then

$$|f(x) - P_S(x)| \le \frac{\|D^r f\|_\infty}{r!}\|x-x_0\|^r \quad \text{for } r \in \mathbb{Z}^+,$$

and if $r \notin \mathbb{Z}^+$ we have

$$f(x) - P_S(x) - \frac{D^{[r]}_{x_0} f(x-x_0)^{[r]}}{[r]!} = \frac{1}{[r-1]!}\int_0^1 (1-t)^{[r]-1}(D^{[r]}_{x_0+t(x-x_0)} f - D^{[r]}_{x_0} f)(x-x_0)^{[r]}\, dt$$

and thus

$$\left| f(x) - P_S(x) - \frac{D^{[r]}_{x_0} f(x-x_0)^{[r]}}{[r]!}\right| \le \frac{\|D^{[r]} f\|_{r-[r]}}{[r]!}\|x-x_0\|^r.$$

Put $Q_S = P_S$ if $r \in \mathbb{Z}^+$ and $Q_S := P_S + \frac{D^{[r]} f(x_0)(\cdot - x_0)^{[r]}}{[r]!}$ if $r \notin \mathbb{Z}^+$. Then

$$\|f - Q_S\|_\infty \le \frac{\operatorname{diam}(S)^r \|f\|_r}{[r]!}.$$

Then the constant $a = a(r,d)$ can be chosen so that

$$\|f - Q_S\|_\infty < s/2.$$

By the above inequality any connected component of $\{x : f(x) \neq 0\}$ meeting S and on which $|f|$ reaches or exceeds the value s contains at least one connected component of $\{|Q_S| > s/2\}$. In particular the number of such connected components is bounded by the number of connected components of $\{|Q_S| > s/2\}$. But $\{|Q_S| > s/2\}$ is a semi-algebraic set of $\mathbb{R}^d$ and it is well known [25] that the number of connected components of such sets is bounded by a constant $b = b(r,d)$ depending only on r and d and not on the coefficients of the polynomial Q_S, nor on s (this is obvious for $d = 1$ because this number is bounded from above by the number of roots of the polynomial $Q_S^2 - s^2/4$, which is less than $2r$). We conclude that the number of connected components of the open set $\{x : f(x) \neq 0\}$ on which $|f|$ reaches or exceeds the value s is at most $b(2[(as/\max(\|f\|_r, 1))^{-1/r}] + 1)^d$. ■

6. Proof of the Main Theorem. Let $\gamma > 0$ and $\mu \in \mathcal{M}(M,T)$. If $\int \log \operatorname{Jac}_x(T)\, d\mu(x) < 0$ then by the upper semicontinuity of $g : \xi \mapsto \int \log \operatorname{Jac}_x(T)\, d\xi(x)$ we have $\int \log \operatorname{Jac}_x(T)\, d\nu(x) = \sum_{i=1}^d \chi_i(\nu) < 0$ for ergodic measures ν close to μ. The map T being nonuniformly entropy expanding, this implies that $h(\nu) = 0$ and thus (2) is checked.

We assume now that $g(\mu) = \int \log \mathrm{Jac}_x(T)\, d\mu(x) \geq 0$. In particular the set $\mathrm{Crit}(T)$ of critical points of T has zero μ-measure. Let U be an open neighborhood of $\mathrm{Crit}(T)$ satisfying the following properties:

- $\log \mathrm{Jac}_x(T) < -S_\gamma$ for $x \in U$ (recall $S_\gamma \in \mathbb{Z}^+$ was fixed such that $H(S^{-1}) < \gamma$ for all $S \geq S_\gamma$);
- $\mu(\partial U) = 0$ and $\mu(U) < \gamma$;
- $\int \log \mathrm{Jac}_x(T)\, d\mu(x) \leq \int_{M \setminus \overline{U}} \log \mathrm{Jac}_x(T)\, d\mu(x)$
$\leq \int \log \mathrm{Jac}_x(T)\, d\mu(x) + \gamma$.

We fix a Riemannian structure $\| \ \|$ on the manifold M. We denote by R_{inj} the radius of injectivity and by $\exp_x : T_xM \to M$ the exponential map at $x \in M$. There exist $R < R' < R_{\mathrm{inj}}/\sqrt{d}$ such that $T(B(x,R)) \subset B(Tx, R')$ for all $x \in M$. Let $\mathcal{V}_\mu = (W_1, \ldots, W_p, U_1, \ldots, U_q)$ be a finite open cover of M such that:

- $\mathrm{diam}(W_i) < R, \mathrm{diam}(U_i) < R$;
- $\bigcup_{i=1}^q U_i = U$;
- $T|_{W_i}$ is a diffeomorphism onto its image.

It is well-known that the function $\xi \mapsto \int f(x)\xi(x)$ is upper semicontinuous on $\mathcal{M}(M,T)$ when f is an upper semicontinous function on M. In particular $\xi \mapsto \xi(\overline{U})$ is upper semicontinuous on $\mathcal{M}(M,T)$. The function $\xi \mapsto \int_{M \setminus \overline{U}} \log \mathrm{Jac}_x(T)\, d\xi(x)$ is also upper semicontinuous on $\mathcal{M}(M,T)$ since $x \mapsto \mathbb{1}_{M \setminus \overline{U}}(x) \log \mathrm{Jac}_x(T)$ is upper semicontinuous on M: the function $x \mapsto \log \mathrm{Jac}_x(T)$ is continuous on the closure of $M \setminus \overline{U}$ and negative on its boundary. We choose a parameter $\tau_\mu > 0$ such that for all ergodic measures ν with $\mathrm{dist}(\nu, \mu) < \tau_\mu$ we have

$$\nu(\overline{U}) < \gamma,$$
$$\int_{M \setminus \overline{U}} \log \mathrm{Jac}_x(T)\, d\nu(x) < \int_{M \setminus \overline{U}} \log \mathrm{Jac}_x(T)\, d\mu(x) + \gamma. \tag{12}$$

We fix an ergodic measure ν with $\mathrm{dist}(\mu, \nu) < \tau_\mu$. One can assume $g(\nu) = \int \log \mathrm{Jac}_x(T)\, d\nu(x) = \sum_{i=1}^d \chi_i(\nu) \geq 0$ (if not then $h(\nu) = 0$ as already noticed) and thus $g^+(\nu) = g(\nu)$.

We break the integral $\int \log \mathrm{Jac}_x(T)\, d\nu(x)$ into the sum of three integrals: over U, $M \setminus \overline{U}$ and ∂U. Since $\log \mathrm{Jac}_x(T)$ is negative on ∂U, by dropping the last term we can only increase the right hand side. Moving the terms around we get

$$(13)\qquad -\int_U \log \mathrm{Jac}_x(T)\,d\nu(x)$$
$$\le \int_{M\setminus\overline{U}} \log \mathrm{Jac}_x(T)\,d\nu(x) - \int \log \mathrm{Jac}_x(T)\,d\nu(x)$$
$$\le \int_{M\setminus\overline{U}} \log \mathrm{Jac}_x(T)\,d\mu(x) - \int \log \mathrm{Jac}_x(T)\,d\nu(x) + \gamma$$
$$\le \int \log \mathrm{Jac}_x(T)\,d\mu(x) - \int \log \mathrm{Jac}_x(T)\,d\nu(x) + 2\gamma.$$

Let $\sigma \in]0,1[$. Let F be a Borel set of ν-measure larger than σ such that

$$(14)\qquad h^{\mathrm{inv}}(M \mid F) < -\sum_{i=1}^{d} \chi_i^-(\nu) + \gamma.$$

One can also assume by Birkhoff's ergodic theorem that the sequences $(n^{-1}\sum_{k=0}^{n-1} \mathbb{1}_U(T^k x))_n$ and $(n^{-1}\sum_{k=0}^{n-1} \mathbb{1}_U(T^k x) \log \mathrm{Jac}_{T^k x} T)_n$ converge uniformly in $x \in F$ to $\nu(U)$ and $\int_U \log \mathrm{Jac}_x(T)\,d\nu(x)$, respectively.

Let $V^n = \bigcap_{0\le k<n} T^{-k} V_k \in \mathcal{V}_\mu^n$. Consider the sequence $\{i_1,\dots,i_N\} = \{0 \le k < n : \text{there exists } 0 \le l \le q \text{ such that } V_k = U_l\}$. To any maximal n-invertible branch A_n intersecting V^n we associate the sequence $\mathcal{K}(A_n) = (k_1(A_n),\dots,k_N(A_n))$ defined by

$$\forall j = 1,\dots,N, \qquad k_j(A_n) = \Big[\inf_{x\in A_n\cap V^n} -\log \mathrm{Jac}_{T^{i_j}x}(T)\Big] + 1.$$

With these notations note that the $\mathcal{C}^{r-1}$ function defined on M by $x \mapsto \mathrm{Jac}_x(T)$ reaches or exceeds the value $e^{-k_j(A_n)}$ on $T^{i_j}A_n \cap V_{i_j}$. We consider a sequence $\mathcal{K} = (k_1,\dots,k_N)$ of N positive integers. By Lemma 3 applied for $1 \le j \le N$ to the Jacobian of $\exp_{Tx_j}^{-1} \circ T \circ \exp_{x_j}(R\,\cdot\,) :]-1,1[^d \subset T_{x_j}M \to T_{Tx_j}M$ with some fixed $x_j \in V_{i_j}$, the number of maximal n-invertible branches A_n meeting V^n with $\mathcal{K}(A_n) = \mathcal{K}$ is bounded above by $c^N e^{\sum_{j=1}^N dk_j/(r-1)}$ where c depends only on r, d, M and $\max_{s=1,\dots,[r],r} \|T\|_s$. If we assume moreover that A_n meets $F \cap V^n$ then $\mathcal{K}(A_n)$ misses the value

$$S := \sup_{x\in F} \frac{1}{N} \sum_{j=1}^{N} (-\log \mathrm{Jac}_{T^{i_j}x} T + 1) \ge S_\gamma,$$

i.e. $N^{-1}\sum_{j=1}^N k_j(A_n) \le S$. Since the number of sequences of N positive integers missing S is $\binom{NS}{N}$, we deduce by inequality (3) and since we have arranged that $H(S^{-1}) < \gamma$, that the logarithm of the number of maximal n-invertible branches meeting $F \cap V^n$ is bounded above by

$$N \log c + NS\left(\frac{d}{r-1} + \gamma\right) + 1.$$

Observe now that $N \le \sup_{x\in F} \sum_{k=0}^{n-1} \mathbb{1}_U(T^k x)$ and

$$NS = \sup_{x\in F}\sum_{j=1}^{N}(-\log \mathrm{Jac}_{T^{i_j}x} T + 1) \le \sup_{x\in F}\sum_{k=0}^{n-1} \mathbb{1}_U(T^k x)(-\log \mathrm{Jac}_{T^k x} T + 1).$$

Therefore for each $V^n \in \mathcal{V}_\mu^n$ (by A_n we always denote a maximal n-invertible branch) we get

$$\begin{aligned}
(15)\qquad & \log \sharp\{A_n : A_n \cap F \cap V^n \neq \emptyset\} \\
&\le N\log c + NS\left(\frac{d}{r-1} + \gamma\right) + 1 \\
&\le \left(\sup_{x\in F}\sum_{k=0}^{n-1} \mathbb{1}_U(T^k x)(-\log \mathrm{Jac}_{T^k x} T + 1)\right)\left(\frac{d}{r-1}+\gamma\right) \\
&\quad + \left(\sup_{x\in F}\sum_{k=0}^{n-1} \mathbb{1}_U(T^k x)\right)\log c + 1.
\end{aligned}$$

Fix $\delta > 0$. Clearly $E \cap A_n$ is (n,δ) separated in $F\cap A_n$ for any maximal n-invertible branch A_n when E is (n,δ) separated in F. It follows that

$$\begin{aligned}
&\max\{\sharp E : E \text{ is } (n,\delta) \text{ separated in } F\cap V^n \text{ with } V^n \in \mathcal{V}_\mu^n\} \\
&\qquad \le \max_{V^n\in\mathcal{V}_\mu^n} \sharp\{A_n : A_n\cap F\cap V^n \neq \emptyset\} \\
&\qquad\quad \times \sup_{A_n}(\max\{\sharp E : E \text{ is } (n,\delta) \text{ separated in } F\cap A_n\}).
\end{aligned}$$

By taking the logarithmic limit in n and then letting δ go to zero, we get

$$h(M\mid F,\mathcal{V}_\mu) \le \limsup_{n\to+\infty}\frac{1}{n}\log \max_{V^n\in\mathcal{V}_\mu^n}\sharp\{A_n : A_n\cap F\cap V^n\neq\emptyset\} + h^{\mathrm{inv}}(M\mid F).$$

Finally by (14) and (15) and by the uniform convergence on F of the Birkhoff sums we obtain

$$\begin{aligned}
&h(M|F,\mathcal{V}_\mu) \\
&\quad\le \lim_{n\to+\infty}\left(\sup_{x\in F} -\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{1}_U(T^k x)\log \mathrm{Jac}_{T^k x} T\right)\left(\frac{d}{r-1}+\gamma\right) \\
&\qquad + \lim_{n\to+\infty}\left(\sup_{x\in F}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{1}_U(T^k x)\right)\left(\frac{d}{r-1}+\gamma+\log c\right) - \sum_{i=1}^{d}\chi_i^-(\nu)+\gamma \\
&\quad\le -\int_U \log \mathrm{Jac}_x(T)\,d\nu(x)\left(\frac{d}{r-1}+\gamma\right) + \nu(U)\left(\frac{d}{r-1}+\gamma+\log c\right) \\
&\qquad - \sum_{i=1}^{d}\chi_i^-(\nu)+\gamma.
\end{aligned}$$

Since ν has been chosen close to μ, according to (12) and (13) we have (denoting by C the constant $\frac{d}{r-1}+\gamma+\log c+1$)

$$h(M\,|\,F,\mathcal{V}_\mu) \leq \left(\int \log \mathrm{Jac}_x(T)\,d\mu(x) - \int \log \mathrm{Jac}_x(T)\,d\nu(x) + 2\gamma\right)\left(\frac{d}{r-1}+\gamma\right) - \sum_{i=1}^{d} \chi_i^-(\nu) + C\gamma$$

$$\leq (g^+(\mu) - g^+(\nu) + 2\gamma)\left(\frac{d}{r-1}+\gamma\right) - \sum_{i=1}^{d} \chi_i^-(\nu) + C\gamma.$$

Then by taking the infimum over the Borel sets F of ν-measure larger than σ and by letting $\sigma \to 1$ we get

$$h^{\mathrm{New}}(M\,|\,\nu,\mathcal{V}_\mu) \leq (g^+(\mu) - g^+(\nu) + 2\gamma)\left(\frac{d}{r-1}+\gamma\right) - \sum_{i=1}^{d} \chi_i^-(\nu) + C\gamma.$$

This concludes the proof since γ can be chosen arbitrarily small.

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