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SPECTRAL SUBSPACES AND NON-COMMUTATIVE HILBERT TRANSFORMS

BY

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Abstract. Let G be a locally compact abelian group and $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful semifinite normal trace τ. We study Hilbert transforms associated with G-flows on $\mathcal{M}$ and closed semigroups Σ of $\widehat{G}$ satisfying the condition $\Sigma \cup (-\Sigma) = \widehat{G}$. We prove that Hilbert transforms on such closed semigroups satisfy a weak-type estimate and can be extended as linear maps from $L^1(\mathcal{M},\tau)$ into $L^{1,\infty}(\mathcal{M},\tau)$. As an application, we obtain a Matsaev-type result for $p=1$: if x is a quasi-nilpotent compact operator on a Hilbert space and $\mathrm{Im}(x)$ belongs to the trace class then the singular values $\{\mu_n(x)\}_{n=1}^{\infty}$ of x are $O(1/n)$.

1. Introduction. The classical Hardy spaces $H^p(\mathbb{T})$, $1 \le p \le \infty$, and boundedness of Riesz projections have played significant roles in the developments of modern analysis. This theory, which was originally developed for spaces of functions on $\mathbb{T}$, has found many generalizations not only for various function spaces but also from a more abstract operator theory point of view. The basic theme is to decompose a given space into a direct sum of "analytic" and "co-analytic" subspaces, analogous to the decomposition of $L^p(\mathbb{T})$, $1 < p < \infty$, as a direct sum of $H^p(\mathbb{T})$ and its complement.

Let X be a Banach space, G be a locally compact abelian group and $\{U_g\}_{g\in G}$ be a bounded continuous group of linear operators on X. In [1], Arveson introduced a notion of spectrum of any vector in X associated with the group $\{U_g\}_{g\in G}$ (see definition below), which in turn allows one to consider spectral subspaces of X associated with subsets of the dual group $\widehat{G}$ of G. Such spectral subspaces generalize the construction of classical Hardy spaces as subspaces of $L^p(\mathbb{T})$. Motivated by Arveson's spectral analysis of groups of automorphisms on von Neumann algebras, Zsidó [27] studied the existence of projections onto spectral subspaces which generalize the classical Riesz projections. Generalizations of analyticity and Riesz projections have been explored by several authors. In [27], Zsidó established the ex-

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istence of generalized Riesz projections associated with some closed semigroups of $\widehat{G}$ on non-commutative L^p-spaces associated with general semifinite von Neumann algebras for $1 < p < \infty$. The case of symmetric spaces of measurable operators was completely settled by Dodds *et al.* [8]. Berkson *et al.* [4] established an analytic-type decomposition related to compact abelian groups for UMD-spaces. The general case of locally compact groups was considered by Asmar *et al.* [2].

The purpose of the present paper is to examine possible extensions of these results to preduals of von Neumann algebras. Recall that the triangular projection is not bounded in the space of trace class operators [17]; the triangular projection is in fact a non-commutative Riesz projection, so as in the commutative case, Zsidó's result is not valid for $p = 1$. Our main result is a non-commutative weak-type estimate that generalizes the celebrated result of Kolmogorov. We remark that a non-commutative generalization of Kolmogorov's theorem was obtained in [22] in the setting of finite maximal subdiagonal algebras. A similar result was also considered by Dodds *et al.* [9] for triangular truncations relative to a finite family of mutually orthogonal projections.

As an application, Zsidó showed that the theory of generalized Riesz projections onto spectral subspaces of non-commutative L^p-spaces can be used to prove the classical result of Matsaev about compact operators on Hilbert spaces. Combining Zsidó's approach with our main result, we deduce that if x is a quasi-nilpotent operator on a Hilbert space and $\mathrm{Im}(x)$ belongs to the trace class then the singular values of x are $O(1/n)$. This improves an earlier result that states that such an x belongs to the Matsaev ideal $\mathcal{C}_\Omega$.

The paper is organized as follows. We begin by gathering some necessary definitions and present some basic facts concerning spectral subspaces; then we present the main result. The last section is dedicated to the extension of Matsaev's theorem to the case $p = 1$.

2. Non-commutative spaces. We begin by recalling the basic constructions of non-commutative spaces. We denote by $\mathcal{M}$ a semifinite von Neumann algebra on the Hilbert space H, with a fixed faithful and normal semifinite trace τ. The identity in $\mathcal{M}$ is denoted by $\mathbf{1}$, and we denote by $\mathcal{M}^p$ the set of all projections in $\mathcal{M}$. A linear operator $x : \mathrm{dom}(x) \to H$, with domain $\mathrm{dom}(x) \subseteq H$, is called *affiliated with* $\mathcal{M}$ if $ux = xu$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. A closed and densely defined operator x affiliated with $\mathcal{M}$ is called *τ-measurable* if for every $\varepsilon > 0$ there exists $p \in \mathcal{M}^p$ such that $p(H) \subseteq \mathrm{dom}(x)$ and $\tau(\mathbf{1} - p) < \varepsilon$. With the sum and product defined as the respective closures of the algebraic sum and product, $\widetilde{\mathcal{M}}$ is a $*$-algebra. For standard facts concerning von Neumann algebras, we refer to [15] and [24].

We recall the notion of generalized singular value function [11]. Given a self-adjoint operator x in H, we denote by $e^x(\cdot)$ the spectral measure of x. Now assume that $x \in \widetilde{\mathcal{M}}$. Then $e^{|x|}(B) \in \mathcal{M}$ for all Borel sets $B \subseteq \mathbb{R}$, and there exists $s > 0$ such that $\tau(e^{|x|}(s,\infty)) < \infty$. For $x \in \widetilde{\mathcal{M}}$ and $t \geq 0$ we define

$$\mu_t(x) = \inf\{s \geq 0 : \tau(e^{|x|}(s,\infty)) \leq t\}.$$

The function $\mu(x) : [0,\infty) \to [0,\infty]$ is called the *generalized singular value function* (or decreasing rearrangement) of x; note that $\mu_t(x) < \infty$ for all $t > 0$. Suppose that $a > 0$. If we consider $\mathcal{M} = L^\infty([0,a), m)$, where m denotes the Lebesgue measure on the interval $[0,a)$, as an abelian von Neumann algebra acting via multiplication on the Hilbert space $H = L^2([0,a), m)$, with the trace given by integration with respect to m, it is easy to see that $\widetilde{\mathcal{M}}$ consists of all measurable functions on $[0,a)$ which are bounded except on a set of finite measure. Further, if $f \in \widetilde{\mathcal{M}}$, then the generalized singular value function $\mu(f)$ is precisely the classical non-increasing rearrangement of the function $|f|$. On the other hand, if $\mathcal{M}$ is the space of all bounded linear operators in some Hilbert space equipped with the canonical trace tr, then $\widetilde{\mathcal{M}} = \mathcal{M}$ and, if $x \in \mathcal{M}$ is compact, then the generalized singular value function $\mu(x)$ may be identified in a natural manner with the sequence $\{\mu_n(x)\}_{n=0}^\infty$ of singular values of $|x| = \sqrt{x^*x}$, repeated according to multiplicity and arranged in non-increasing order.

By $L^0([0,a), m)$ we denote the space of all $\mathbb{C}$-valued Lebesgue measurable functions on the interval $[0,a)$ (with identification m-a.e.). A Banach space $(E, \|\cdot\|_E)$, where $E \subseteq L^0([0,a), m)$, is called a *rearrangement-invariant Banach function space* on the interval $[0,a)$ if it follows from $f \in E$, $g \in L^0([0,a), m)$ and $\mu(g) \leq \mu(f)$ that $g \in E$ and $\|g\|_E \leq \|f\|_E$. If $(E, \|\cdot\|_E)$ is a rearrangement-invariant Banach function space on $[0,a)$, then E is said to be *symmetric* if $f, g \in E$ and $g \prec\prec f$ imply that $\|g\|_E \leq \|f\|_E$. Here $g \prec\prec f$ denotes submajorization in the sense of Hardy–Littlewood–Pólya:

$$\int_0^t \mu_s(g)\,ds \leq \int_0^t \mu_s(f)\,ds \qquad \text{for all } t > 0.$$

If $(E, \|\cdot\|_E)$ is a rearrangement-invariant, symmetric Banach function space on $[0,a)$, then E will be called *fully symmetric* if $f \in E$, $g \in L^0([0,a), m)$ and $g \prec\prec f$ imply $g \in E$ and $\|g\|_E \leq \|f\|_E$. The general theory of rearrangement-invariant spaces may be found in [18] and [23].

Given a semifinite von Neumann algebra $(\mathcal{M}, \tau)$ and a fully symmetric Banach function space $(E, \|\cdot\|_E)$ on $([0, \tau(\mathbf{1})), m)$, we define the non-commutative space $E(\mathcal{M}, \tau)$ by setting

$$E(\mathcal{M}, \tau) := \{x \in \widetilde{\mathcal{M}} : \mu(x) \in E\}$$

with

$$\|x\|_{E(\mathcal{M},\tau)} := \|\mu(x)\|_E \quad \text{for } x \in E(\mathcal{M},\tau).$$

Equipped with $\|\cdot\|_{E(\mathcal{M},\tau)}$, the space $E(\mathcal{M},\tau)$ is a Banach space. Moreover, the inclusions

$$L^1(\mathcal{M},\tau) \cap \mathcal{M} \subseteq E(\mathcal{M},\tau) \subseteq L^1(\mathcal{M},\tau) + \mathcal{M}$$

hold with continuous embeddings (here the norms in $L^1(\mathcal{M},\tau) \cap \mathcal{M}$ and $L^1(\mathcal{M},\tau) + \mathcal{M}$ are the usual norms of intersection and sum of Banach spaces). Note that if E is order continuous then $L^1(\mathcal{M},\tau) \cap \mathcal{M}$ is dense in $E(\mathcal{M},\tau)$. We also remark that if $1 \leq p < \infty$ and $E = L^p([0,\tau(\mathbf{1})))$ then $E(\mathcal{M},\tau)$ coincides with the definition of $L^p(\mathcal{M},\tau)$ as in [21] and [25]. In particular, if $\mathcal{M} = B(H)$ with the standard trace then these L^p-spaces are precisely the *Schatten classes* $\mathcal{C}_p$. For additional information on these spaces, we refer to [5], [6] and [7].

We define the non-commutative weak-L^1, $L^{1,\infty}(\mathcal{M},\tau)$, to be the linear subspace of all $x \in \widetilde{\mathcal{M}}$ for which the quasi-norm

$$\|x\|_{1,\infty} := \sup_{t>0} t\mu_t(x) = \sup_{\lambda>0} \lambda\tau(e^{|x|}(\lambda,\infty))$$

is finite. Equipped with the quasi-norm $\|\cdot\|_{1,\infty}$, it is a quasi-Banach space and $\|x\|_{1,\infty} \leq \|x\|_1$ for all $x \in L^1(\mathcal{M},\tau)$.

3. Spectrum and spectral subspaces. Let G be a locally compact abelian group with a fixed Haar measure dg with dual group $\widehat{G}$. Let $(X,\mathcal{F})$ be a *dual pair* of (complex) Banach spaces in the sense of Zsidó ([27]), that is, $(X,\mathcal{F})$ is equipped with a bilinear functional $(x,\phi) \mapsto \langle x,\phi\rangle$, $(x,\phi) \in X \times \mathcal{F}$, such that

(i) $\|x\| = \sup_{\phi\in\mathcal{F},\, \|\phi\|\leq 1} |\langle x,\phi\rangle|$ for all $x \in X$;
(ii) $\|\phi\| = \sup_{x\in X,\, \|x\|\leq 1} |\langle x,\phi\rangle|$ for all $\phi \in \mathcal{F}$;
(iii) the convex hull of every relatively $\mathcal{F}$-compact subset of X is relatively $\mathcal{F}$-compact;
(iv) the convex hull of every relatively X-compact subset of $\mathcal{F}$ is relatively X-compact.

Typical examples of dual pairs are (X,X^*) and (X^*,X) for an arbitrary complex Banach space X. Another example relevant for this paper is $(E(\mathcal{M},\tau), E^\times(\mathcal{M},\tau))$ where $E(\mathcal{M},\tau)$ is a non-commutative space and $E^\times(\mathcal{M},\tau)$ is its Köthe dual (in the sense of [7]).

We denote by $B_{\mathcal{F}}(X)$ the Banach space of all $\mathcal{F}$-continuous linear operators on X. We recall that for any locally compact abelian group G, a subgroup $\{U_g\}_{g\in G}$ of $B_{\mathcal{F}}(X)$ is called an *$\mathcal{F}$-continuous representation* of G if $U_0 = \mathrm{Id}$, $U_{g_1+g_2} = U_{g_1}U_{g_2}$ for all $g_1, g_2 \in G$ and $g \mapsto \langle U_g x,\phi\rangle$ is continuous for every $x \in X$ and $\phi \in \mathcal{F}$.

The representation $\{U_g\}_{g\in G}$ is said to be *bounded* if $\sup\{\|U_g\| : g\in G\} < \infty$. If G is compact, then any $\mathcal{F}$-continuous representation is bounded. If $\{U_g\}_{g\in G}$ is an $\mathcal{F}$-continuous bounded representation of G, $x \in X$ and $f \in L^1(G)$, then we define the *Arveson convolution* $f *_U x$ as follows:

$$f *_U x = \mathcal{F}\text{-}\int_G f(g)U_g(x)\,dg.$$

The *Arveson spectrum* for $x \in X$ is defined by

$$Sp_U(x) = \bigcap\{Z(\widehat{f}\,) : f \in L^1(G) \text{ and } f *_U x = 0\};$$

here $\widehat{f}$ denotes the Fourier transform of f and $Z(\widehat{f}\,) = \{\gamma \in \widehat{G} : \widehat{f}(\gamma) = 0\}$. If $M(G)$ denotes the space of all complex regular Borel measures on G with finite total variation then for every $\mu \in M(G)$, the linear functional

$$U_{\widehat{\mu}} : x \to \mathcal{F}\text{-}\int_G U_g(x)\ d\mu(g)$$

is $\mathcal{F}$-continuous (see [1, Proposition 1.4]).

For any closed subset $F \subseteq \widehat{G}$, we define the corresponding *spectral subspace* X_F^U by setting

$$X_F^U := \{x \in X : \mathrm{Sp}_U(x) \subseteq F\}.$$

The linear subspace X_F^U is closed in X and we remark that if F_1 and F_2 are closed subsets of $\widehat{G}$ and if $F_1 \cap F_2 = \emptyset$ then $X_{F_1}^U \cap X_{F_2}^U = \{0\}$. Moreover,

$$X_F^U = \mathcal{F}\text{-closure}\Big(\bigcup\{X_K^U : K \subseteq F,\ K \text{ compact}\}\Big).$$

From the latter property, one can generalize the definition of spectral subspaces to any subset S of $\widehat{G}$ by setting

$$X_S^U := \mathcal{F}\text{-closure}\Big(\bigcup\{X_K^U : K \subseteq S,\ K \text{ compact}\}\Big).$$

A useful restatement of the definition of spectral subspaces for the case of closed subsets F of $\widehat{G}$ was noted in [27]:

$$\begin{aligned} X_F^U &= \{x \in X : \text{if } \mu \in M(G),\ F \cap \mathrm{supp}(\widehat{\mu}) = \emptyset \text{ then } U_{\widehat{\mu}}(x) = 0\} \\ &= \{x \in X : \text{if } f \in L^1(G),\ \mathrm{supp}(\widehat{f}) \text{ is compact}, \\ &\qquad\qquad F \cap \mathrm{supp}(\widehat{f}\,) = \emptyset \text{ then } U_{\widehat{f}}(x) = 0\}. \end{aligned}$$

Definition 3.1. A representation $\{U_g\}_{g\in G}$ of G in $B_{\mathcal{F}}(X)$ is said to have the *weak projection property* on $S \subseteq \widehat{G}$ if for any closed subset $F \subseteq \widehat{G}$,

$$\mathcal{F}\text{-closure}\Big(\bigcup_{\substack{K\subseteq S\cap F\\ K \text{ compact}}} X_K^U + \bigcup_{\substack{K\subseteq(\widehat{G}\setminus S)\cap F\\ K \text{ compact}}} X_K^U\Big) = X_F^U.$$

DEFINITION 3.2. A representation $\{U_g\}_{g\in G}$ of G in $B_{\mathcal{F}}(X)$ is said to have the *projection property* on $S \subset \widehat{G}$ if it has the weak projection property on S and there exists an $\mathcal{F}$-continuous projection P_S^U with range the spectral subspace X_S^U and kernel $X_{\widehat{G}\setminus S}^U$.

In general, this projection P_S^U need not exist. In [2], Asmar *et al.* considered the case of a locally compact abelian group with ordered dual group $\widehat{G}$ and spectral decompositions of UMD-spaces. Their result can be summarized as follows.

THEOREM 3.3 ([2], Theorem 6.3). *Assume that X is a UMD-space and $\widehat{G}$ is ordered. If $\mathcal{P} = \{\gamma \in \widehat{G} : \gamma \geq 0\}$ and F is the interior of $\mathcal{P} \setminus \{0\}$ then any strongly continuous representation $\{U_g\}_{g\in G}$ of G in X has the projection property on F with $\|P_F^U\| \leq c^3 a_X$, where a_X is a constant depending only on X (but not on the particular group G) and $c = \sup\{\|U_g\| : g \in G\}$.*

For extensive discussions on weak projection properties and projection properties, we refer to [27, Sect. 3].

We will now specialize to non-commutative spaces. Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra. A *G-flow on $\mathcal{M}$* is an ultraweakly continuous representation $\{U_g\}_{g\in G}$ of G on $\mathcal{M}$ with a $*$-automorphism of $\mathcal{M}$ which preserves the trace τ. Since U_g's are trace-preserving, it is clear that any G-flow can be extended to a group of trace-preserving isometries on $L^1(\mathcal{M}, \tau)$. By interpolation, any G-flow on $\mathcal{M}$ extends to a group of rearrangement-preserving maps on $L^1(\mathcal{M}, \tau) + \mathcal{M}$. It follows that if E is a symmetric Banach function space on $\mathbb{R}$ then any G-flow on $\mathcal{M}$ extends to a group $U^E = \{U_g^E\}_{g\in G}$ of isometries on $E(\mathcal{M}, \tau)$.

Note that U^{L^1} is an $\mathcal{M}$-continuous representation of G in $L^1(\mathcal{M}, \tau)$. This follows by observing that if $x \in L^1(\mathcal{M}, \tau)$ and $y \in \mathcal{M}$ then for every $g \in G$,

$$\tau(U_g^{L^1}(x)y) = \tau(xU_{-g}(y)).$$

Similarly, $U^{L^1\cap L^\infty}$ is an $L^1(\mathcal{M}, \tau) + \mathcal{M}$-continuous representation of G in $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$. In general, if E is a separable symmetric Banach function space on $\mathbb{R}^+$ then $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ is dense in $E(\mathcal{M}, \tau)$, so the extension U^E of the G-flow U is uniquely determined by its restriction to $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ and therefore U^E is an $\mathcal{F}$-continuous representation of G on $E(\mathcal{M}, \tau)$ for $\mathcal{F} = E(\mathcal{M}, \tau)^* = E^*(\mathcal{M}, \tau)$.

Unless there is a need for distinction, we will simply denote U^E by U.

We remark that if Σ is a closed semigroup of $\widehat{G}$ then the spectral subspace $\mathcal{M}_\Sigma^U$ is a closed subalgebra of $\mathcal{M}$. More generally, if E, F and V are symmetric spaces on $\mathbb{R}^+$ with $E.F \subseteq V$ and Σ is a closed semigroup of $\widehat{G}$ then $E(\mathcal{M}, \tau)_\Sigma^U . F(\mathcal{M}, \tau)_\Sigma^U \subseteq V(\mathcal{M}, \tau)_\Sigma^U$. These facts can be deduced for instance from [27, Theorem 2.1, Corollary 2.3].

The setting of symmetric spaces of measurable operators was considered by Dodds *et al.* [8] who proved the following result:

THEOREM 3.4. *If E is a separable symmetric Banach function space on $\mathbb{R}^+$, then the following statements are equivalent*:

(i) *E has non-trivial Boyd indices*;

(ii) *there exists a constant $c(E)$ which depends only on E such that for every semifinite von Neumann algebra $(\mathcal{M},\tau)$, for every G-flow $\{U_g\}_{g\in G}$ on $(\mathcal{M},\tau)$ and for all closed semigroups $\Sigma \subseteq \widehat{G}$ such that $\Sigma \cup (-\Sigma)$ is a group, $\{U_g^E\}_{g\in G}$ has the projection property on Σ with $\|P_\Sigma^{U^E}\| \le c(E)$.*

The projections whose existence is guaranteed by Theorems 3.3 and 3.4 are referred to as the *generalized Riesz projections* determined by the G-flow U and the closed semigroup Σ.

The particular case $E = L^p(\mathbb{R}^+)$ with $1 < p < \infty$ is due to Zsidó [27]. Since reflexive $L^p(\mathcal{M},\tau)$'s are UMD-spaces, Theorems 3.3 and 3.4 can be viewed as generalizations of Zsidó's result.

To illustrate these ideas, let $s \mapsto U_s$ be the representation of the group $\mathbb{T}$ on the abelian von Neumann algebra $L^\infty(\mathbb{T})$ given by $U_s f(\theta) = f(\theta - s)$ for all $f \in L^\infty(\mathbb{T})$. Then $\widehat{\mathbb{T}} = \mathbb{Z}$ and if we consider the closed subset $\mathbb{Z}^+$ of $\widehat{\mathbb{T}}$ then for every $1 \le p \le \infty$, the spectral subspace $L^p(\mathbb{T})^U_{\mathbb{Z}^+}$ is the usual Hardy space $H^p(\mathbb{T})$. For another example, recall that a *nest* $\mathcal{N}$ is a set of projections of $B(H)$, totally ordered, that is closed in the strong operator topology and contains 0 and $\mathbf{1}$. The associated nest algebra is $\operatorname{alg}\mathcal{N} = \{a \in B(H) : (\mathbf{1} - p)ap = 0 \text{ for all } p \in \mathcal{N}\}$. Loebl and Muhly [19] proved that for every nest subalgebra of $\mathcal{M}$, there exists an associated inner action $\alpha = \{\alpha_t\}_{t\in\mathbb{R}}$ of the group $\mathbb{R}$ on $\mathcal{M}$ such that $\mathcal{M} \cap \operatorname{alg}\mathcal{N} = \{x \in \mathcal{M} : \operatorname{Sp}_\alpha(x) \subseteq [0,\infty)\} = \mathcal{M}^\alpha_{[0,\infty)}$. The subalgebra $\mathcal{M} \cap \operatorname{alg}\mathcal{N}$ is called the *analytic subalgebra $H^\infty(\alpha)$ associated with α*. Another (equivalent) way to define $H^\infty(\alpha)$ is as the set of all $x \in \mathcal{M}$ for which $t \mapsto \langle \varphi, \alpha_t(x)\rangle \in H^\infty(\mathbb{R})$ for every $\varphi \in \mathcal{M}_*$.

The main goal of this paper is to study the case of $L^1(\mathcal{M},\tau)$. First, we will gather some basic results concerning properties of G-flows related to closed groups of $\widehat{G}$. These results will be crucial for the definition of Hilbert transform and will be used repeatedly in what follows.

PROPOSITION 3.5. *Let $\{U_g\}_{g\in G}$ be a G-flow on $\mathcal{M}$ and Λ be a closed subgroup of $\widehat{G}$. If $\{U_g\}_{g\in G}$ has the projection property on Λ then $\{U_g^{L^1}\}_{g\in G}$ has the projection property. Moreover, $P_\Lambda^U(x) = P_\Lambda^{U^{L^1}}(x) \in L^1(\mathcal{M},\tau) \cap \mathcal{M}$ for every $x \in L^1(\mathcal{M},\tau) \cap \mathcal{M}$.*

Proof. Assume that $\{U_g\}_{g\in G}$ has the projection property for Λ. One can deduce from [27, Theorem 3.16] that $\{U_g^{L^1}\}_{g\in G}$ has the projection property on Λ with $P_\Lambda^U = (P_\Lambda^{U^{L^1}})^*$.

Fix $x \in L^1(\mathcal{M},\tau)\cap\mathcal{M}$ and let $\{\mu_i\}_{i\in I}$ be a net in $M(G)$ such that $\sup_i\|\mu_i\| < \infty$, $\widehat{\mu}_i = 1$ on Λ for all $i \in I$ and $\mu_i * f \to 0$ in $L^1(G)$ for all $f \in L^1(G)$ with $\operatorname{supp}(\widehat{f})$ compact and $\Lambda\cap\operatorname{supp}(\widehat{f}) = \emptyset$. From [27, Corollary 3.17],

$$\lim_i \|U_{\widehat{\mu}_i}^{L^1}(x) - P_\Lambda^{U^{L^1}}(x)\|_p = 0.$$

Recall that $U_{\widehat{\mu}_i}^{L^1}(x) = \int_G U_g(x)\,d\mu_i(g) \in L^1(\mathcal{M},\tau)$ where the integral is the Bochner integral. Since x belongs to $L^1(\mathcal{M},\tau)\cap\mathcal{M}$ so does $U_g(x)$ and considering the representation $\{U_g^{L^1\cap L^\infty}\}_{g\in G}$ with the dual pair $(L^1(\mathcal{M},\tau)\cap\mathcal{M}, L^1(\mathcal{M},\tau)+\mathcal{M})$, we have

$$\langle U_{\widehat{\mu}_i}^{L^1\cap L^\infty}(x), z\rangle = \int_G \langle U_g(x), z\rangle\,d\mu_i(g)$$

for all $z \in L^1(\mathcal{M},\tau)+\mathcal{M}$. Hence

$$\begin{aligned}\|U_{\widehat{\mu}_i}^{L^1\cap L^\infty}(x)\|_{L^1(\mathcal{M},\tau)\cap\mathcal{M}} &\le \int_G \|U_g(x)\|_{L^1(\mathcal{M},\tau)\cap\mathcal{M}}\,d|\mu_i|(g)\\ &\le \sup_{g\in G}\|U_g\|\cdot\|x\|_{L^1(\mathcal{M},\tau)\cap\mathcal{M}}\sup_i\|\mu_i\|.\end{aligned}$$

This shows that $\{U_{\widehat{\mu}_i}^{L^1\cap L^\infty}(x)\}_{i\in I}$ is a bounded net in $L^1(\mathcal{M},\tau)\cap\mathcal{M}$. Let $\theta : L^1(\mathcal{M},\tau)\cap\mathcal{M} \to L^1(\mathcal{M},\tau)$ be the natural inclusion. It is clear that θ is $\sigma(L^1(\mathcal{M},\tau)\cap\mathcal{M}, L^1(\mathcal{M},\tau)+\mathcal{M})$-to-weak continuous and therefore $U_{\widehat{\mu}_i}^{L^1}\circ\theta$ are $\sigma(L^1(\mathcal{M},\tau)\cap\mathcal{M}, L^1(\mathcal{M},\tau)+\mathcal{M})$-continuous maps and for every $z \in L^1(\mathcal{M},\tau)\cap\mathcal{M}$ and $i \in I$,

$$\langle U_{\widehat{\mu}_i}^{L^1}(\theta(x)), z\rangle = \langle U_{\widehat{\mu}_i}^{L^1\cap L^\infty}(x), z\rangle.$$

This shows that $U_{\widehat{\mu}_i}^{L^1}\circ\theta(x) = U_{\widehat{\mu}_i}^{L^1\cap L^\infty}(x)$ for every $i \in I$.

Recall that $\{U_{\widehat{\mu}_i}^{L^1}(x)\}_{i\in I}$ is a bounded net in $L^1(\mathcal{M},\tau)\cap\mathcal{M}$ that converges to $P_\Lambda^{U^{L^1}}(x)$ in $L^1(\mathcal{M},\tau)$ and by semi-embedding, $P_\Lambda^{U^{L^1}}(x) \in L^1(\mathcal{M},\tau)\cap\mathcal{M}$.

A similar argument also proves that if $\gamma : L^1(\mathcal{M},\tau)\cap\mathcal{M} \to \mathcal{M}$ is the natural inclusion then $U_{\widehat{\mu}_i}\circ\gamma(x) = U_{\widehat{\mu}_i}^{L^1\cap L^\infty}(x)$ for every $i \in I$. Since $P_\Lambda^U(x) = \text{weak}^*\text{-}\lim_i U_{\widehat{\mu}_i}(x)$, we conclude that $P_\Lambda^U(x) = P_\Lambda^{U^{L^1}}(x)$ as elements of $L^1(\mathcal{M},\tau)+\mathcal{M}$. ■

It may be of interest to observe that the argument used in the proof above yields the following: if E and F are separable symmetric Banach function spaces on $\mathbb{R}^+$ and a G-flow $\{U_g\}_{g\in G}$ on a semifinite von Neumann algebra $(\mathcal{M},\tau)$ has the projection property on a closed subgroup Λ onto both $E(\mathcal{M},\tau)_\Lambda^{U^E}$ and $F(\mathcal{M},\tau)_\Lambda^{U^F}$, then $P_\Lambda^{U^E}(x) = P_\Lambda^{U^F}(x)$ for all $x \in E(\mathcal{M},\tau) \cap F(\mathcal{M},\tau)$.

Corollary 3.6. *Let $\{U_g\}_{g\in G}$ be a G-flow on $\mathcal{M}$ and Λ be a closed subgroup of $\widehat{G}$. If $\{U_g\}_{g\in G}$ has the projection property on Λ then $\{U_g^{L^1\cap L^\infty}\}_{g\in G}$ has the projection property on Λ.*

Proof. From Proposition 3.5 above, the operator $Q : L^1(\mathcal{M},\tau)\cap\mathcal{M} \to L^1(\mathcal{M},\tau)\cap\mathcal{M}$ defined by $Q(x) = P_\Lambda^U(x)$ is well defined. It is clear that Q is a projection and its range is equal to $(L^1(\mathcal{M},\tau)\cap\mathcal{M})_\Lambda^U$. Similarly, if $R : L^1(\mathcal{M},\tau)+\mathcal{M} \to L^1(\mathcal{M},\tau)+\mathcal{M}$ is defined by $R(x) = P_\Lambda^{U^{L^1}}(x_1)+P_\Lambda^U(x_2)$ for $x_1 \in L^1(\mathcal{M},\tau)$, $x_2 \in \mathcal{M}$ and $x = x_1 + x_2$, then R is a well defined projection with range being a subspace of $(L^1(\mathcal{M},\tau)+\mathcal{M})_\Lambda^U$ and satisfies $\langle R(x), y\rangle = \langle x, Q(y)\rangle$ for all $x \in L^1(\mathcal{M},\tau) + M$ and $y \in L^1(\mathcal{M},\tau)\cap\mathcal{M}$. Note also that $\{U_g^{L^1+L^\infty}\}_{g\in G}$ has the weak projection property.

We claim that $\mathrm{Ker}(R) = (L^1(\mathcal{M},\tau)+\mathcal{M})_{\widehat{G}\setminus\Lambda}^U$. For this, let $x = x_1 + x_2 \in \mathrm{Ker}(R)$. Then $P_\Lambda^{U^{L^1}}(x_1) = -P_\Lambda^U(x_2)$, so $x = (x_1 - P_\Lambda^{U^{L^1}}(x_1)) + (x_2 - P_\Lambda^U(x_2))$. But $x_1 - P_\Lambda^{U^{L^1}}(x_1)$ and $x_2 - P_\Lambda^U(x_2)$ belong to $L^1(\mathcal{M},\tau)_{\widehat{G}\setminus\Lambda}^U$ and $\mathcal{M}_{\widehat{G}\setminus\Lambda}^U$ respectively so $x \in (L^1(\mathcal{M},\tau)+\mathcal{M})_{\widehat{G}\setminus\Lambda}^U$. Conversely, if $x \in (L^1(\mathcal{M},\tau)+\mathcal{M})_{\widehat{G}\setminus\Lambda}^U$ then $\langle x, y\rangle = 0$ for every $y \in (L^1(\mathcal{M},\tau)\cap\mathcal{M})_\Lambda^U$; but since $\mathrm{Ran}(Q) = (L^1(\mathcal{M},\tau)\cap\mathcal{M})_\Lambda^U$, we have $\langle R(x), y\rangle = \langle x, Q(y)\rangle = 0$ for every $y \in L^1(\mathcal{M},\tau)\cap\mathcal{M}$, hence $R(x) = 0$. The claim is verified.

It is now easy to deduce that $\mathrm{Ran}(R) = (L^1(\mathcal{M},\tau)+\mathcal{M})_\Lambda^U$, which shows that $\{U_g^{L^1+L^\infty}\}_{g\in G}$ (and likewise $\{U_g^{L^1\cap L^\infty}\}_{g\in G}$) has the projection property on Λ. ∎

For the sake of convenience, we list the following basic properties:

Lemma 3.7. *Let $\{U_g\}_{g\in G}$ be a G-flow on a semifinite von Neumann algebra $(\mathcal{M},\tau)$ and Λ be a closed subgroup of $\widehat{G}$. If $\{U_g\}_{g\in G}$ has the projection property on Λ then*:

(i) $P_\Lambda^U(\mathbf{1}) = \mathbf{1}$;
(ii) $P_\Lambda^U(x^*) = (P_\Lambda^U(x))^*$ *for all* $x \in \mathcal{M}$;
(iii) $P_\Lambda^U(x) \geq 0$ *whenever* $0 \leq x \in \mathcal{M}$;
(iv) $\tau(P_\Lambda^{U^{L^1}}(x)) = \tau(x)$ *for all* $x \in L^1(\mathcal{M},\tau)$.

For the case of closed semigroups, the next result is an immediate consequence of [27, Theorem 5.1].

Proposition 3.8. *Let* $\{U_g\}_{g\in G}$ *be a* G*-flow on* $\mathcal{M}$ *and* Σ *be a closed semigroup of* $\widehat{G}$. *If* $\Lambda = \Sigma \cap (-\Sigma)$ *is a subgroup and* $\{U_g\}_{g\in G}$ *has the projection property on* Λ *then*:

(i) $P_\Lambda^U(yxz) = yP_\Lambda^U(x)z$ *for all* $x \in \mathcal{M}$ *and* $y, z \in \mathcal{M}_\Lambda^U$;

(ii) $P_\Lambda^U(xy) = P_\Lambda^U(x)P_\Lambda^U(y)$ *for all* $x, y \in \mathcal{M}_\Sigma^U$.

Proposition 3.9. *Let* $\{U_g\}_{g\in G}$ *be a* G*-flow on* $\mathcal{M}$ *and* Σ *be a closed semigroup of* $\widehat{G}$. *Let* $\Lambda = \Sigma \cap (-\Sigma)$ *and assume that* $\Sigma \cup (-\Sigma) = \widehat{G}$. *If* $\{U_g\}_{g\in G}$ *has the projection property on* Λ *then*

$$(L^1(\mathcal{M},\tau)\cap\mathcal{M})^U_{\Sigma\setminus(-\Sigma)} + (L^1(\mathcal{M},\tau)\cap\mathcal{M})^U_{(-\Sigma)\setminus\Sigma} + (L^1(\mathcal{M},\tau)\cap\mathcal{M})^U_\Lambda$$

is $\sigma(L^1(\mathcal{M},\tau)\cap\mathcal{M}, L^1(\mathcal{M},\tau)+\mathcal{M})$*-dense in* $L^1(\mathcal{M},\tau)\cap\mathcal{M}$.

Proof. Note that the closed semigroup Σ is polyhedral (in the sense of [27], p. 232). Since $\{U_g^{L^1\cap L^\infty}\}_{g\in G}$ has the projection property on Λ and $\widehat{G} = \Sigma \cup (-\Sigma)$, we have $\Lambda \subset \Sigma \subset \widehat{G}$ and the boundary of Σ is in Λ so $\{U_g^{L^1\cap L^\infty}\}_{g\in G}$ has the weak projection property on Λ. From [27, Lemma 3.8], $(L^1(\mathcal{M},\tau)\cap\mathcal{M})^U_\Sigma + (L^1(\mathcal{M},\tau)\cap\mathcal{M})^U_{\widehat{G}\setminus\Sigma}$ is $\sigma(L^1(\mathcal{M},\tau)\cap\mathcal{M}, L^1(\mathcal{M},\tau)+\mathcal{M})$-dense in $L^1(\mathcal{M},\tau)\cap\mathcal{M}$. From $(\widehat{G}\setminus\Sigma)\cap(\widehat{G}\setminus\Lambda) = (-\Sigma)\setminus\Sigma$ and the assumption that $\{U_g^{L^1\cap L^\infty}\}_{g\in G}$ has the projection property on Λ, the statement of the proposition follows. ∎

4. Hilbert transforms. Throughout this section, $\{U_g\}_{g\in G}$ is a fixed G-flow on a semifinite von Neumann algebra $(\mathcal{M},\tau)$, Σ is a closed semigroup of $\widehat{G}$ such that $\Lambda = \Sigma\cap(-\Sigma)$ is a group such that $\{U_g\}_{g\in G}$ has the projection property on Λ and $\Sigma\cup(-\Sigma) = \widehat{G}$. The following linear subspace of $L^1(\mathcal{M},\tau)\cap\mathcal{M}$ serves as a starting point of our investigation:

$$\begin{aligned}\mathcal{A} := (L^1(\mathcal{M},\tau)\cap\mathcal{M})^U_\Lambda &+ (L^1(\mathcal{M},\tau)\cap\mathcal{M})^U_{\Sigma\setminus(-\Sigma)}\\ &+ (L^1(\mathcal{M},\tau)\cap\mathcal{M})^U_{(-\Sigma)\setminus\Sigma}.\end{aligned}\tag{4.1}$$

From Proposition 3.9, $\mathcal{A}$ is $\sigma(L^1(\mathcal{M},\tau)\cap\mathcal{M}, L^1(\mathcal{M},\tau)+\mathcal{M})$-dense in $L^1(\mathcal{M},\tau)\cap\mathcal{M}$. Our goal is to define a notion of *conjugation* on $\mathcal{A}$ and then extend it continuously to a more general symmetric space of measurable operators. As in [22], we consider the following definition:

Definition 4.1. Let $a_1 \in (L^1(\mathcal{M},\tau)\cap\mathcal{M})^U_{\Sigma\setminus(-\Sigma)}$, $a_2 \in (L^1(\mathcal{M},\tau)\cap\mathcal{M})^U_{(-\Sigma)\setminus\Sigma}$ and $d \in (L^1(\mathcal{M},\tau)\cap\mathcal{M})^U_\Lambda$ so that $x = d + a_1 + a_2 \in \mathcal{A}$. The *conjugate* of x associated with the semigroup Σ is the operator $\widetilde{x} = ia_2 - ia_1 \in L^1(\mathcal{M},\tau)\cap\mathcal{M}$.

Note that for every $x \in \mathcal{A}$ as in the definition above, $x + i\widetilde{x} = 2a_1 + d$ and therefore $x + i\widetilde{x} \in (L^1(\mathcal{M},\tau)\cap\mathcal{M})^U_\Sigma$.

We let the *Hilbert transform* (conjugation) associated with the closed semigroup Σ of $\widehat{G}$ be the map $\mathcal{H}_\Sigma^U : \mathcal{A} \to L^1(\mathcal{M},\tau) \cap \mathcal{M}$ defined by setting $\mathcal{H}_\Sigma^U(x) = \widetilde{x}$ for all $x \in \mathcal{A}$.

From Theorem 3.4, we can deduce the following result:

THEOREM 4.2. *Let E be a separable symmetric Banach function space on $\mathbb{R}^+$. Assume that E has non-trivial Boyd indices. Then there exists a unique continuous linear extension of $\mathcal{H}_\Sigma^U$ from $E(\mathcal{M},\tau)$ into $E(\mathcal{M},\tau)$. In particular, if $1 < p < \infty$, there exists a unique continuous linear extension of $\mathcal{H}_\Sigma^U$ from $L^p(\mathcal{M},\tau)$ into $L^p(\mathcal{M},\tau)$ with the property that $x + i\mathcal{H}_\Sigma^U(x) \in L^p(\mathcal{M},\tau)_\Sigma^U$ for all $x \in L^p(\mathcal{M},\tau)$. Moreover, there exists an absolute constant C (independent of p) such that*

$$\|\mathcal{H}_\Sigma^U(x)\|_p \le Cpq\|x\|_p \quad \textit{for all } x \in L^p(\mathcal{M},\tau) \textit{ and } 1/p + 1/q = 1.$$

We remark that if E is a separable symmetric Banach function space on $\mathbb{R}^+$ that has non-trivial Boyd indices then $\mathcal{H}_\Sigma^U : E(\mathcal{M},\tau) \to E(\mathcal{M},\tau)$ can be defined as

$$\mathcal{H}_\Sigma^U := iP^{U^E}_{(-\Sigma)\setminus\Sigma} - iP^{U^E}_{\Sigma\setminus(-\Sigma)}, \tag{4.2}$$

where the existence of the projections $P^{U^E}_{(-\Sigma)\setminus\Sigma}$ and $P^{U^E}_{\Sigma\setminus(-\Sigma)}$ is guaranteed by Theorem 3.4 above. It is easy to see that $P^{U^E}_{\Lambda} \circ \mathcal{H}_\Sigma^U = 0$.

We now turn to the discussion of the main result of the paper (Theorem 4.3 below). It is the analogue of Theorem 4.2 for the case $p = 1$ and can be viewed as a non-commutative analogue of Kolmogorov's weak type estimate for harmonic conjugate functions.

THEOREM 4.3. *Let $0 \le x \in L^1(\mathcal{M},\tau) \cap \mathcal{M}$ and $f = x + i\mathcal{H}_\Sigma^U(x)$. Then for every $\lambda > 0$,*

$$\tau(e^{|f|}(\lambda,\infty)) \le 4\frac{\|x\|_1}{\lambda}.$$

Our approach is based on the main idea used in [22, Theorem 2]. For convenience, we collect some properties of $\mathcal{H}_\Sigma^U$.

LEMMA 4.4. *For $x \in L^1(\mathcal{M},\tau) \cap \mathcal{M}$ and $f = x + i\mathcal{H}_\Sigma^U(x)$, the formal series*

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{t^k \varepsilon^k f^k}{k!}$$

is absolutely convergent in $(L^1(\mathcal{M},\tau) + \mathcal{M})_\Sigma^U$ for $t < 1/(eM\|x\|_{L^1(\mathcal{M},\tau)\cap\mathcal{M}})$ where $M = 2C + 2$ and C is the absolute constant from Theorem 4.2 above.

Proof. Note that $f \in L^p(\mathcal{M},\tau)$ for every $1 < p < \infty$ so the series $\sum_{k=2}^{\infty} \varepsilon^k f^k/k!$ is absolutely convergent in $L^1(\mathcal{M},\tau)$ (and therefore in $L^1(\mathcal{M},\tau) + \mathcal{M}$) as in [14, Theorem 3a]. Thus $\mathbf{1} + t\varepsilon f + \sum_{k=2}^{\infty} \varepsilon^k f^k/k!$ is absolutely convergent in $L^1(\mathcal{M},\tau) + \mathcal{M}$. ∎

LEMMA 4.5. *If* $0 \leq x \in L^1(\mathcal{M},\tau) \cap \mathcal{M}$ *and* $0 < \varepsilon < 1$, *let* $f = x + i\mathcal{H}^U_\Sigma(x)$. *Then*:

(i) $\mathbf{1} + \varepsilon f$ *is invertible with* $\|(\mathbf{1} + \varepsilon f)^{-1}\| \leq 1$;
(ii) $f_\varepsilon = f(\mathbf{1} + \varepsilon f)^{-1} \in \mathcal{M}^U_\Sigma$;
(iii) $P^U_\Lambda(f_\varepsilon) = P^U_\Lambda(x) P^U_\Lambda((\mathbf{1} + \varepsilon f)^{-1})$.

Proof. Since f is densely defined and $x \geq 0$, we have $|\langle(\mathbf{1} + \varepsilon f)h, h\rangle| \geq \|h\|^2$ for all $h \in D(f)$. So $\mathbf{1}+\varepsilon f$ has bounded inverse with $\|(\mathbf{1}+\varepsilon f)^{-1}\| \leq 1$.

For (ii), we will adjust the proof of [22, Lemma 2] to our setting. Note first that f_ε is bounded. In fact, $\varepsilon f_\varepsilon = \mathbf{1} - (\mathbf{1}+\varepsilon f)^{-1}$. To prove that $f_\varepsilon \in \mathcal{M}^U_\Sigma$, it suffices to show that $(\mathbf{1} + \varepsilon f)^{-1} \in \mathcal{M}^U_\Sigma$. Indeed, since $x \in L^1(\mathcal{M},\tau) \cap \mathcal{M}$, we have $f \in L^p(\mathcal{M},\tau)^U_\Sigma$ for every $1 < p < \infty$, so if $(\mathbf{1} + \varepsilon f)^{-1} \in \mathcal{M}^U_\Sigma$ then $f_\varepsilon \in L^p(\mathcal{M},\tau)^U_\Sigma$ and since it is bounded it belongs to $\mathcal{M}^U_\Sigma$.

Set $A = -\varepsilon f$. There exists a (unique) semigroup $(T_t)_{t>0}$ of contractions such that A is the infinitesimal generator of $(T_t)_{t>0}$ (see for instance [26, pp. 246–249]). It is well known that

$$(\mathbf{1} - A)^{-1}h = \int_0^\infty e^{-t} T_t h\, dt \quad \text{for all } h \in H$$

and

$$T_t h = \lim_{n\to\infty} \exp(tA(\mathbf{1} - n^{-1}A)^{-1})h \quad \text{for all } h \in H.$$

We claim that $T_t \in \mathcal{M}^U_\Sigma$ for every $t > 0$.

Since $(T_t)_{t>0}$ is a semigroup and $\mathcal{M}^U_\Sigma$ is a subalgebra, it is enough to verify this claim for small values of t. Assume $2t \leq 1/(eM\|x\|_{L^1(\mathcal{M},\tau)\cap\mathcal{M}})$. Let $\varphi(\cdot)$ be the formal series defined in Lemma 4.4. We will show that $T_t = \varphi(-t)$. Using the series expansion of the exponential and Lemma 4.4, we get

$$\begin{aligned}
\|\exp(tA(\mathbf{1} - n^{-1}A)^{-1}) - \varphi(-t)\|_{L^1(\mathcal{M},\tau)+\mathcal{M}} &\leq \sum_{k\geq 0} \frac{t^k}{k!} \|(A(\mathbf{1} - n^{-1}A)^{-1})^k - A^k\|_{L^1(\mathcal{M},\tau)+\mathcal{M}} \\
&\leq \sum_{k\geq 0} \left\| \frac{(2t)^k}{k!} \varepsilon^k f^k \right\|_{L^1(\mathcal{M},\tau)+\mathcal{M}} < \infty.
\end{aligned}$$

Fix $k \geq 0$ and set $J_n = (\mathbf{1} - n^{-1}A)^{-1}$ for every $n \geq 1$. Since $\|J_n\|_\mathcal{M} \leq 1$ and $\mathbf{1} - J_n = \varepsilon n^{-1} f(\mathbf{1} + \varepsilon n^{-1} f)^{-1}$ for all $n \geq 1$,

$$\begin{aligned}
\|(AJ_n)^k - A^k\|_{L^1(\mathcal{M},\tau)+\mathcal{M}} &= \|A^k (J_n)^k - A^k\|_{L^1(\mathcal{M},\tau)+\mathcal{M}} \\
&= \left\| A^k (J_n - \mathbf{1}) \Big(\sum_{s=0}^{k-1} (J_n)^s \Big) \right\|_{L^1(\mathcal{M},\tau)+\mathcal{M}}
\end{aligned}$$

$$\leq k\|A^k(J_n - \mathbf{1})\|_{L^1(\mathcal{M},\tau)+\mathcal{M}}$$
$$\leq k\varepsilon n^{-1}\|A^k f\|_{L^1(\mathcal{M},\tau)+\mathcal{M}}$$
$$= k\varepsilon^{k+1}n^{-1}\|f^{k+1}\|_{L^1(\mathcal{M},\tau)+\mathcal{M}}.$$

This shows that $\lim_{n\to\infty}(t^k/k!)\|(A(\mathbf{1}-n^{-1}A)^{-1})^k - A^k\|_{L^1(\mathcal{M},\tau)+\mathcal{M})} = 0$, so by the estimate on the series above,

$$\lim_{n\to\infty}\|\exp(tA(\mathbf{1}-n^{-1}A)^{-1}) - \varphi(-t)\|_{L^1(\mathcal{M},\tau)+\mathcal{M}} = 0,$$

which in turn shows that $T_t = \varphi(-t) \in (L^1(\mathcal{M},\tau)+\mathcal{M})^U_\Sigma$ and since T_t is bounded, the claim follows. We conclude the proof of (ii) by noticing that $\mathcal{M}^U_\Sigma$ is weak*-closed in $\mathcal{M}$ and $(\mathbf{1}-A)^{-1}$ is the weak*-integral of an $\mathcal{M}^U_\Sigma$-valued map so $(\mathbf{1}-A)^{-1} \in \mathcal{M}^U_\Sigma$.

For (iii), note that $\mathcal{A}$ is dense in $L^2(\mathcal{M},\tau)$ so there exists a sequence $\{a_n\}_{n=1}^\infty$ in $\mathcal{A}$ such that $\lim_{n\to\infty}\|a_n - x\|_2 = 0$. Since $\mathcal{H}^U_\Sigma$ acts continuously on $L^2(\mathcal{M},\tau)$, we have $\lim_{n\to\infty}\|\mathcal{H}^U_\Sigma(a_n) - \mathcal{H}^U_\Sigma(x)\|_2 = 0$. Similarly, $\lim_{n\to\infty}\|(a_n + i\mathcal{H}^U_\Sigma(a_n))(\mathbf{1}+\varepsilon f)^{-1} - f_\varepsilon\|_2 = 0$. Since both $a_n + i\mathcal{H}^U_\Sigma(a_n)$ and $(\mathbf{1}+\varepsilon f)^{-1}$ belong to $\mathcal{M}^U_\Sigma$, Proposition 3.8 (ii) and the fact that $P^U_\Lambda \circ \mathcal{H}^U_\Sigma = 0$ imply that $P^U_\Lambda((a_n + i\mathcal{H}^U_\Sigma(a_n))(\mathbf{1}+\varepsilon f)^{-1}) = P^U_\Lambda(a_n)P^U_\Lambda((\mathbf{1}+\varepsilon f)^{-1})$. It is now clear that (iii) follows by taking the limit as n goes to ∞. ∎

Proof of Theorem 4.3. Fix $0 < \varepsilon < 1$ and let $f_\varepsilon \in \mathcal{M}^U_\Sigma$ as in Lemma 4.5. For every $\lambda > 0$, consider the following transformation on $\{w : \mathrm{Re}(w) \geq 0\}$:

$$A_\lambda(z) = 1 + \frac{z-\lambda}{z+\lambda}.$$

Since A_λ is analytic, Proposition 3.8(ii) implies that

$$P^U_\Lambda(A_\lambda(f_\varepsilon)) = A_\lambda(P^U_\Lambda(f_\varepsilon)).$$

As $P^U_\Lambda(f_\varepsilon) = P^U_\Lambda(x)P^U_\Lambda((\mathbf{1}+\varepsilon f)^{-1})$, we have

$$P^U_\Lambda(A_\lambda(f_\varepsilon)) = A_\lambda(P^U_\Lambda(x)P^U_\Lambda((\mathbf{1}+\varepsilon f)^{-1})).$$

If $y = P^U_\Lambda(x)P^U_\Lambda((\mathbf{1}+\varepsilon f)^{-1}$ then

$$P^U_\Lambda(A_\lambda(f_\varepsilon)) = \mathbf{1} + (y - \lambda\mathbf{1})(y+\lambda\mathbf{1})^{-1} = 2y(y+\lambda\mathbf{1})^{-1}.$$

Similarly, $P^U_\Lambda((A_\lambda(f_\varepsilon))^*) = 2(y^* + \lambda\mathbf{1})^{-1}y^*$ and hence

$$P^U_\Lambda(\mathrm{Re}(A_\lambda(f_\varepsilon))) = y(y+\lambda\mathbf{1})^{-1} + (y^*+\lambda\mathbf{1})^{-1}y^*.$$

Taking the traces on both sides, we get

$$\tau(P^U_\Lambda(\mathrm{Re}(A_\lambda(f_\varepsilon)))) = 2\,\mathrm{Re}(\tau(y(y+\lambda\mathbf{1})^{-1})) \leq 2\tau(|y|)\cdot\|(y+\lambda\mathbf{1})^{-1}\|_{\mathcal{M}}.$$

As $\mathrm{Re}(y) = \mathrm{Re}(P^U_\Lambda(f_\varepsilon)) = P^U_\Lambda(\mathrm{Re}(f_\varepsilon)) \geq 0$, we have $\|(y+\lambda\mathbf{1})^{-1}\|_{\mathcal{M}} \leq 1/\lambda$ and since $\tau(|y|) \leq \tau(P^U_\Lambda(x)) = \tau(x)$ (as $0 \leq x \in L^1(\mathcal{M},\tau)\cap\mathcal{M}$), we conclude that

$$\tau(P^U_\Lambda(\mathrm{Re}(A_\lambda(f_\varepsilon)))) \leq 2\|x\|_1/\lambda. \tag{4.3}$$

Now we shall estimate $\tau(P^U_\Lambda(\mathrm{Re}(A_\lambda(f_\varepsilon))))$ from below. Note that $(f^*_\varepsilon+\lambda\mathbf{1})^{-1}[2|f_\varepsilon|^2+2\lambda\,\mathrm{Re}(f_\varepsilon)](f_\varepsilon+\lambda\mathbf{1})^{-1} \geq 2(f^*_\varepsilon+\lambda\mathbf{1})^{-1}|f_\varepsilon|^2(f_\varepsilon+\lambda\mathbf{1})^{-1}$ (as $\mathrm{Re}(f_\varepsilon)\geq 0$). Therefore

$$P^U_\Lambda(\mathrm{Re}(A_\lambda(f_\varepsilon))) \geq P^U_\Lambda(2(f^*_\varepsilon+\lambda\mathbf{1})^{-1}|f_\varepsilon|^2(f_\varepsilon+\lambda\mathbf{1})^{-1}).$$

We remark that since the Hilbert transform is bounded in $L^2(\mathcal{M},\tau)$, we have $f\in L^2(\mathcal{M},\tau)$ and therefore $|f_\varepsilon|^2$ and $2(f^*_\varepsilon+\lambda\mathbf{1})^{-1}|f_\varepsilon|^2(f_\varepsilon+\lambda\mathbf{1})^{-1}$ belong to $L^1(\mathcal{M},\tau)$. By Lemma 3.7(iv),

$$\begin{aligned}\tau(P^U_\Lambda(2(f^*_\varepsilon+\lambda\mathbf{1})^{-1}&|f_\varepsilon|^2(f_\varepsilon+\lambda\mathbf{1})^{-1}))\\ &= \tau(2(f^*_\varepsilon+\lambda\mathbf{1})^{-1}|f_\varepsilon|^2(f_\varepsilon+\lambda\mathbf{1})^{-1})\\ &= \tau(2|f_\varepsilon|^2(f_\varepsilon+\lambda\mathbf{1})^{-1}(f^*_\varepsilon+\lambda\mathbf{1})^{-1})\\ &= \tau(2|f_\varepsilon|^2(|f_\varepsilon|^2+2\lambda\,\mathrm{Re}(f_\varepsilon)+\lambda^2\mathbf{1})^{-1})\end{aligned}$$

as $(f_\varepsilon+\lambda\mathbf{1})^{-1}(f^*_\varepsilon+\lambda\mathbf{1})^{-1}=(|f_\varepsilon|^2+2\lambda\,\mathrm{Re}(f_\varepsilon)+\lambda^2\mathbf{1})^{-1}$.

Set $Q=e^{|f_\varepsilon|}(\lambda,\infty)$. The projection Q commutes with $|f_\varepsilon|$ and we have

$$\tau(P^U_\Lambda(\mathrm{Re}(A_\lambda(f_\varepsilon)))) \geq \tau(2Q|f_\varepsilon|^2(|f_\varepsilon|^2+2\lambda\,\mathrm{Re}(f_\varepsilon)+\lambda^2\mathbf{1})^{-1}). \tag{4.4}$$

Define

$$A=|f_\varepsilon|^2+2\lambda\,\mathrm{Re}\,(f_\varepsilon)+\lambda^2\mathbf{1}, \qquad B=|f_\varepsilon|^2+2\lambda|f_\varepsilon|+\lambda^2\mathbf{1}.$$

For every positive operator C that commutes with B, we get $\tau(CA^{-1})\geq \tau(CB^{-1})$ (see [22], Lemmas 5 and 6). In particular, for $C=2Q|f_\varepsilon|^2$,

$$\tau(P^U_\Lambda(\mathrm{Re}(A_\lambda(f_\varepsilon)))) \geq \tau(2Q|f_\varepsilon|^2(|f_\varepsilon|^2+2\lambda|f_\varepsilon|+\lambda^2\mathbf{1})^{-1}). \tag{4.5}$$

Using the spectral decomposition of $|f_\varepsilon|$, we can write

$$2Q|f_\varepsilon|^2(|f_\varepsilon|^2+2\lambda|f_\varepsilon|+\lambda^2\mathbf{1})^{-1} = \int_\lambda^\infty \frac{2t^2}{t^2+2\lambda t+\lambda^2}\,de^{|f_\varepsilon|}_t.$$

Let

$$\psi_\lambda(t)=\frac{2t^2}{t^2+2\lambda t+\lambda^2} \quad \text{for } t\in[\lambda,\infty).$$

As in [22], ψ_λ is increasing on $[\lambda,\infty)$ so $\psi_\lambda(t)\geq\psi_\lambda(\lambda)=1/2$ for $t\geq\lambda$, and therefore

$$2Q|f_\varepsilon|^2(|f_\varepsilon|^2+2\lambda|f_\varepsilon|+\lambda^2\mathbf{1})^{-1}\geq \tfrac{1}{2}Q,$$

so we deduce that

$$\tau(P^U_\Lambda(\mathrm{Re}(A_\lambda(f_\varepsilon)))) \geq \tfrac{1}{2}\tau(Q). \tag{4.6}$$

Combining (4.3) and (4.6), we conclude that

$$\tau(Q)\leq 4\|x\|_1/\lambda.$$

Now taking $\varepsilon\to 0$, note that $f-f_\varepsilon=\varepsilon f^2(\mathbf{1}+\varepsilon f)^{-1}$ so (f_ε) converges to f in measure. We deduce from [11, Lemma 3.4] that $\mu_t(f)\leq$

$\liminf_{n\to\infty} \mu_t(f_{\varepsilon_n})$ for each $t > 0$ and $\varepsilon_n \to 0$. This implies that for every $\lambda > 0$ and every $t > 0$, $\chi_{(\lambda,\infty)}(\mu_t(f)) \le \liminf_{n\to\infty} \chi_{(\lambda,\infty)}(\mu_t(f_{\varepsilon_n}))$. Hence by Fatou's lemma,

$$\begin{aligned}\tau(e^{|f|}(\lambda,\infty)) &= \int_0^1 \chi_{(\lambda,\infty)}(\mu_t(|f|))\,dt \\ &\le \liminf_{n\to\infty} \int_0^1 \chi_{(\lambda,\infty)}(\mu_t(|f_{\varepsilon_n}|))\,dt \\ &= \liminf_{n\to\infty} \tau(e^{|f_{\varepsilon_n}|}(\lambda,\infty)) \le 4\|x\|_1/\lambda. \ \blacksquare\end{aligned}$$

As shown in [22] for the case of finite maximal subdiagonal algebras, an immediate corollary of the preceding theorem is the following result:

Corollary 4.6. *The Hilbert transform extends to be a linear map $\mathcal{H}_\Sigma^U$ from $L^1(\mathcal{M},\tau)$ into $L^{1,\infty}(\mathcal{M},\tau)$ with the following property: for every $x \in L^1(\mathcal{M},\tau)$, the operator $x + i\mathcal{H}_\Sigma^U(x)$ belongs to the closure of $L^1(\mathcal{M},\tau)_\Sigma^U$ in $L^{1,\infty}(\mathcal{M},\tau)$.*

As in the case of functions, when the von Neumann algebra $\mathcal{M}$ is finite, one can deduce the next corollary.

Corollary 4.7. *If $\tau(\mathbf{1}) < \infty$ and $0 < p < 1$, then the Hilbert transform $\mathcal{H}_\Sigma^U$ is a bounded linear map from $L^1(\mathcal{M},\tau)$ into $L^p(\mathcal{M},\tau)$.*

Let $\mathcal{P} = \{p_i\}_{i=1}^N$ be an arbitrary finite family of mutually orthogonal projections from $\mathcal{M}$. The *triangular truncation* projection on $\widetilde{\mathcal{M}}$ with respect to $\mathcal{P}$ is defined by

$$Tx := \sum_{1\le n<m\le N} p_m x p_n, \qquad x \in \widetilde{\mathcal{M}}.$$

Theorem 4.8 ([9], Theorem 1.4). *There exists an absolute constant C such that*

$$\|Tx\|_{1,\infty} \le C\|x\|_1$$

for all $x \in L^1(\mathcal{M},\tau)$.

Proof. Without loss of generality, we may assume that $\sum_{i=1}^N p_i = \mathbf{1}$. For $t \in \mathbb{T}$, consider the unitary operator $u_t \in \mathcal{M}$ defined by

$$u_t = \sum_{m=1}^N e^{imt} p_m,$$

and the $*$-automorphisms

$$U_t(x) = {u_t}^* x u_t, \qquad x \in \mathcal{M}.$$

Then $\{U_t\}_{t\in\mathbb{T}}$ is a $\mathbb{T}$-flow on $\mathcal{M}$ and $\widehat{\mathbb{T}}=\mathbb{Z}$. If we consider $\Sigma=\mathbb{Z}^+$ then it is easy to verify that

$$L^1(\mathcal{M},\tau)^U_{\Sigma\setminus\{0\}}=\Big\{\sum_{1\le n<m\le N} p_m x p_n : x\in L^1(\mathcal{M},\tau)\Big\},$$

$$L^1(\mathcal{M},\tau)^U_{\{0\}}=\Big\{\sum_{n=1}^{N} p_n x p_n : x\in L^1(\mathcal{M},\tau)\Big\}.$$

Let $x\in L^1(\mathcal{M},\tau)\cap\mathcal{M}$. It is clear that

$$P^U_{\Sigma\setminus\{0\}}(x)=\tfrac{1}{2}(x+i\mathcal{H}^U_{\Sigma}(x)-P^U_{\{0\}}(x))$$

and therefore the conclusion follows directly from Corollary 4.6. ∎

5. Application: Matsaev's Theorem for trace class operators. In this section, we will discuss variants of the well known result of Matsaev [20]: if $1<p<\infty$, there exists a constant $K(p)$ such that if x is a quasi-nilpotent compact operator in a separable Hilbert space such that $\mathrm{Im}(x)$ belongs to the Schatten ideal $\mathcal{C}_p$ then so does $\mathrm{Re}(x)$ and

$$\|\mathrm{Re}(x)\|_p\le K(p)\|\mathrm{Im}(x)\|_p.$$

In [27], Zsidó proved that Matsaev's Theorem follows directly from boundedness of some generalized Riesz projections. This connection was also discovered independently by Asmar *et al.* [2]. Extensions of Matsaev's result have been considered by several authors. For instance, generalizations to non-commutative L^p-spaces were studied in [3], and the case of symmetric spaces of measurable operators was considered in [12] and [8]. Our result in this section treats the case of the trace class ideal.

Let $\mathcal{C}_\Omega$ be the normed ideal of compact operators x such that

$$\|x\|_\Omega=\sup_n\Big\{\sum_{j=1}^{n}(2j-1)^{-1}\Big(\sum_{j=1}^{n}\mu_j(x)\Big)\Big\}<\infty.$$

The ideal $\mathcal{C}_\Omega$ is strictly larger than $\mathcal{C}_1$ and is contained in the intersection of all $\mathcal{C}_p$ for $1<p<\infty$. It is known that if x is quasi-nilpotent and $\mathrm{Im}(x)\in\mathcal{C}_1$ then $x\in\mathcal{C}_\Omega$ (see for instance [13], p. 192). The next theorem is an improvement of this fact.

THEOREM 5.1. *Let x be a quasi-nilpotent compact operator on a separable Hilbert space such that* $\mathrm{Im}(x)$ *belongs to* $\mathcal{C}_1$. *Then the sequence* $\{\mu_n(x)\}_{n=1}^\infty$ *is* $O(1/n)$.

The proof is based on the following subdiagonalization of compact operators:

THEOREM 5.2 ([9], Theorem 5.5). *Let H be a separable complex Hilbert space and $x\in K(H)$. Then there exists a uniformly continuous one-para-*

meter group $\{u_t\}_{t\in\mathbb{R}}$ of unitaries on H such that, denoting by $\{U_t\}_{t\in\mathbb{R}}$ the uniformly continuous group of *-automorphisms of $K(H)$ defined by $U_t(y) = u_t y u_{-t}$, we have:

(i) $K(H)^U_{(\{0\})}$ *is the closed linear span of a sequence of mutually orthogonal projections with one-dimensional range, so it is an abelian C^*-subalgebra of $K(H)$;*

(ii) $\{U_t\}_{t\in\mathbb{R}}$ *has the projection property on* $\{0\}$;

(iii) $x \in K(H)^U_{(-\infty,0]}$.

Proof of Theorem 5.1. Let x be quasi-nilpotent and compact on the Hilbert space H and $\{U_t\}_{t\in\mathbb{R}}$ denote the $\mathbb{R}$-flow on $B(H)$ defined by

$$U_t(y) = u_t y u_{-t}, \qquad y \in B(H),$$

with $x \in K(H)^U_{(-\infty,0]}$ as in Theorem 5.2. Since x is quasi-nilpotent, by [27, Corollary 5.2], so is $P^U_{\{0\}}(x)$ and therefore $P^U_{\{0\}}(x) = 0$. Consequently, $x \in K(H)^U_{(-\infty,0)}$. We conclude that $\mathrm{Im}(x) \in K(H)^U_{\mathbb{R}\setminus\{0\}}$. Since $\mathrm{Im}(x) \in \mathcal{C}_1$, we get $\mathrm{Im}(x) \in (\mathcal{C}_1)^U_{\mathbb{R}\setminus\{0\}} \subset (\mathcal{C}_p)^U_{\mathbb{R}\setminus\{0\}}$ for every $1 < p < \infty$. As in the proof of [27, Theorem 5.7], we remark that $x = 2iP^U_{(-\infty,0]}(\mathrm{Im}(x))$ (where the projection $P^U_{(-\infty,0]}$ is a bounded map in $\mathcal{C}_p$).

If we consider the closed semigroup $\Sigma = [0,\infty)$ of $\mathbb{R}$ then $i\mathcal{H}^U_\Sigma(y) = -P^V_{(-\infty,0)}(y) + P^V_{(0,\infty)}(y)$ for every $y \in \mathcal{C}_p$, which gives

$$P^V_{(-\infty,0]}(y) = \tfrac{1}{2}(y - i\mathcal{H}^U_\Sigma(y) + P^U_{\{0\}}(y)).$$

This shows that

$$x = i(\mathrm{Im}(x) - i\mathcal{H}^U_\Sigma(\mathrm{Im}(x)) + P^U_{\{0\}}(\mathrm{Im}(x)))$$

and

$$\mathrm{Re}(x) = \mathcal{H}^U_\Sigma(\mathrm{Im}(x)).$$

It follows from Corollary 4.6 that there exists an absolute constant K such that $\|x\|_{1,\infty} \le K\|\mathrm{Im}(x)\|_1$ and by the definition of $\|\cdot\|_{1,\infty}$, $\mu_n(x) \le (K\|\mathrm{Im}(x)\|_1)/n$ for all $n \ge 1$. ∎

Let x be a compact operator on a separable Hilbert space H. We denote by $\lambda_n(x)$, $n \ge 1$, the eigenvalues of x repeated according to algebraic multiplicity and arranged in decreasing order of absolute values (this arrangement is not unique). Note that the s-numbers $\mu_n(x) = \lambda_n(|x|)$, $n \ge 1$. For any (two-sided) ideal $\mathcal{I}$ contained in $K(H)$, we recall the *commutator subspace* $\mathrm{Com}(\mathcal{I})$, i.e. the closed linear span of the commutators $[x,y] = xy - yx$ where $x \in \mathcal{I}$ and $y \in B(H)$. A *trace* on $\mathcal{I}$ is a linear functional $\tau : \mathcal{I} \to \mathbb{C}$ that is unitarily invariant, or equivalently, that vanishes on $\mathrm{Com}(\mathcal{I})$ (see [10] and [16] for background).

THEOREM 5.3. *Let x be a quasi-nilpotent compact operator on a separable Hilbert space H such that* $\mathrm{Im}(x) \in \mathcal{C}_1$. *Then*

$$\sup_{n\geq 1}\Big|\sum_{i=1}^{n}\lambda_i(\mathrm{Re}(x))\Big| < \infty.$$

Proof. Let x be a quasi-nilpotent operator in $K(H)$ such that $\mathrm{Im}(x) \in \mathcal{C}_1$. By Theorem 5.1, there exists a constant K such that $\mu_n(x) \leq K/n$ for $n \geq 1$. *A fortiori*, there exists a constant C such that $\mu_n(\mathrm{Re}(x)) \leq C/n$ for $n \geq 1$. Consider the (two-sided) ideal

$$\mathcal{I} = \{a \in K(H) : \{\mu_n(a)\}_{n=1}^{\infty} \text{ is } O(1/n)\}.$$

Clearly, $\mathcal{I}$ is a geometrically stable ideal, i.e. if (s_n) is a decreasing real sequence then $\mathrm{diag}\{s_n\} \in \mathcal{I}$ if and only if $\mathrm{diag}\{(s_1 \dots s_n)^{1/n}\} \in \mathcal{I}$. Suppose that τ is a trace on $\mathcal{I}$. By a result of Dykema and Kalton [10, Corollary 2.4], for any given operator $a \in \mathcal{I}$, the value of $\tau(a)$ depends only on the eigenvalues of a and their algebraic multiplicities. Since x is quasi-nilpotent and $x \in \mathcal{I}$, we have $\tau(x) = 0$ and therefore $\tau(\mathrm{Re}(x)) = 0$. Hence $\mathrm{Re}(x) \in \mathrm{Com}(\mathcal{I})$. We apply a result of Kalton [16, Theorem 3.1] to conclude that the diagonal operator $\mathrm{diag}\left\{\frac{1}{n}(\lambda_1 + \ldots + \lambda_n)\right\}$ is in $\mathcal{I}$. That is, there exists a fixed constant C such that

$$\left|\frac{\lambda_1 + \ldots + \lambda_n}{n}\right| \leq \frac{C}{n},$$

which shows that $|\sum_{i=1}^{n}\lambda_i| \leq C$ for all $n \geq 1$. ∎

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