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LIFTINGS OF 1-FORMS TO $(J^rT^*)^*$

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Abstract. Let J^rT^*M be the r-jet prolongation of the cotangent bundle of an n-dimensional manifold M and let $(J^rT^*M)^*$ be the dual vector bundle. For natural numbers r and n, a complete classification of all linear natural operators lifting 1-forms from M to 1-forms on $(J^rT^*M)^*$ is given.

0. Let J^rT^*M be the r-jet prolongation of the cotangent bundle of an n-manifold M and let $(J^rT^*M)^*$ be the dual vector bundle. In this note we prove that for natural numbers r and n, the vector space over $\mathbb{R}$ of all natural operators lifting 1-forms on M into 1-forms on $(J^rT^*M)^*$ is 2-dimensional if $r \geq 2$ or $n = 1$ and 3-dimensional if $r = 1$ and $n \geq 2$. We construct explicitly a basis of this vector space.

Various linear natural operators lifting 1-forms are used practically in all papers in which problems of prolongation of geometric structures are studied. That is why classifications of natural operators lifting forms to some natural bundles have been studied (see [1]–[3], [5]–[9], etc.).

Throughout, the usual coordinates on $\mathbb{R}^n$ are denoted by $x^1, \ldots, x^n$, and $\partial_i = \partial/\partial x^i$, $i = 1, \ldots, n$.

All manifolds and maps are assumed to be of class C^∞.

1. The *r-jet prolongation* J^rT^*M of the cotangent bundle T^*M of an n-manifold M is the vector bundle of all r-jets of 1-forms on M, i.e. $J^rT^*M = \{j^r_x(\omega) \mid \omega$ is a 1-form on M, $x \in M\}$. It is a vector bundle over M with respect to the source projection $j^r_x(\omega) \mapsto x$.

Let $T^{[r]}M = (J^rT^*M)^*$ be the dual vector bundle and $\pi : T^{[r]}M \to M$ be its projection. Every embedding $\varphi : M \to N$ of n-manifolds induces a vector bundle mapping $T^{[r]}\varphi : T^{[r]}M \to T^{[r]}N$ covering φ given by

$$\langle T^{[r]}\varphi(\Theta), j^r_{\varphi(x)}(\omega)\rangle = \langle \Theta, j^r_x(\varphi^*\omega)\rangle, \quad \omega \in \Omega^1(N),\ \Theta \in T^{[r]}_xM,\ x \in M.$$

Then $T^{[r]}$ is a vector natural bundle over n-manifolds.

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Let F be a natural bundle over n-manifolds. *A linear natural operator* $A : T^* \rightsquigarrow T^*F$ is a system of $\mathbb{R}$-linear maps $A : \Omega^1(M) \to \Omega^1(FM)$ for every n-manifold M such that $A(\varphi^*\omega) = (F\varphi)^*(A(\omega))$ for every embedding $\varphi : M \to N$ of n-manifolds (cf. [4]).

Example 1. Let F be a natural bundle over n-manifolds. The vertical lift of a 1-form $\omega : TM \to \mathbb{R}$ to FM is the 1-form $\omega^{\mathrm{V}} = \omega \circ T\pi : TFM \to \mathbb{R}$, where $\pi : FM \to M$ is the bundle projection. The family $A^{\mathrm{V}} : T^* \rightsquigarrow T^*F$ given by $A^{\mathrm{V}}(\omega) = \omega^{\mathrm{V}}$ is a linear natural operator.

Example 2. For every $\omega \in \Omega^1(M)$ we define $\omega^{[r]} : T^{[r]}M \to \mathbb{R}$ by $\omega^{[r]}(\Theta) = \langle \Theta, j^r_x(\omega)\rangle$, $\Theta \in T^{[r]}_x M$, $x \in M$. The family $A^{[r]} : T^* \rightsquigarrow T^*T^{[r]}$ given by $A^{[r]}(\omega) = d(\omega^{[r]})$ is a linear natural operator.

Example 3. By an easy computation in coordinates one can show that there exists a unique linear first order natural operator $B^{(1)} : T^* \rightsquigarrow J^1T^*$ such that $B^{(1)}(fdg)(x_0) = j^1_{x_0}((f - f(x_0))dg - (g - g(x_0))df)$, $f, g : M \to \mathbb{R}$, $x_0 \in M$. For every $\omega \in \Omega^1(M)$ we define $\omega^{(1)} : T^{[1]}M \to \mathbb{R}$ by $\omega^{(1)}(\Theta) = \langle \Theta, B^{(1)}(\omega)(x)\rangle$, $\Theta \in T^{[1]}_x M$, $x \in M$. The family $A^{(1)} : T^* \rightsquigarrow T^*T^{[1]}$ given by $A^{(1)}(\omega) = d(\omega^{(1)})$ is a linear natural operator. (If $n = 1$, $A^{(1)} = 0$.)

The set of all linear natural operators $T^* \rightsquigarrow T^*F$ is a vector space over $\mathbb{R}$ with respect to the obvious operations.

The main result in this note is the following classification theorem.

Theorem 1. *Let r and n be natural numbers.*

(i) *If $r \geq 2$ and $n \geq 2$, then every linear natural operator $A : T^* \rightsquigarrow T^*T^{[r]}$ is a linear combination with real coefficients of $A^{[r]}$ and A^{V}.*

(ii) *If $r = 1$ and $n \geq 2$, then every linear natural operator $A : T^* \rightsquigarrow T^*T^{[1]}$ is a linear combination with real coefficients of $A^{[1]}$, $A^{(1)}$ and A^{V}.*

(iii) *If $n = 1$ then every linear natural operator $A : T^* \rightsquigarrow T^*T^{[r]}$ is a linear combination with real coefficients of $A^{[r]}$ and A^{V}.*

The proof of the theorem will occupy Sections 2–3.

2. First we assume that $n \geq 2$.

In [7], we proved the following reducibility lemma.

Lemma 1. *Let F be a natural bundle over n-manifolds. If $A : T^* \rightsquigarrow T^*F$ is a linear natural operator such that $A(x^2dx^1) = 0$, then $A = 0$.*

By Lemma 1, every linear natural operator $A : T^* \rightsquigarrow T^*T^{[r]}$ is uniquely determined by $A(x^2dx^1)$. So, we shall study $A(x^2dx^1)$.

Set $S = \{(\alpha, j) = (\alpha_1, \ldots, \alpha_n, j) \in (\mathbb{N} \cup \{0\})^n \times \mathbb{N} \mid 0 \leq |\alpha| \leq r$, $j = 1, \ldots, n\}$. On $T^{[r]}\mathbb{R}^n$ we have coordinates $(x^i, X^{(\alpha,j)})$, $i = 1, \ldots, n$,

$(\alpha, j) \in S$, given by

$$x^i(\Theta) = x_0^i, \qquad X^{(\alpha,j)}(\Theta) = \langle \Theta, j^r_{x_0}((x - x_0)^\alpha dx^j)\rangle, \tag{1}$$

where $\Theta \in T^{[r]}_{x_0}\mathbb{R}^n$ and $x_0 = (x_0^1, \ldots, x_0^n) \in \mathbb{R}^n$.

Lemma 2. *Let $A : T^* \rightsquigarrow T^*T^{[r]}$ be a linear natural operator. Then*

$$\begin{aligned} A(x^2dx^1) = {}& \mu_1 x^2 dx^1 + \mu_2 X^{((0),2)}dx^1 + \mu_3 x^1 dx^2 + \mu_4 X^{((0),1)}dx^2 \\ & + \mu_5 x^2 dX^{((0),1)} + \mu_6 X^{((0),2)}dX^{((0),1)} + \mu_7 x^1 dX^{((0),2)} \\ & + \mu_8 X^{((0),1)}dX^{((0),2)} + \mu_9 dX^{(e_2,1)} + \mu_{10} dX^{(e_1,2)} \end{aligned} \tag{2}$$

for some $\mu_1, \ldots, \mu_{10} \in \mathbb{R}$, where $e_i = (0, \ldots, 1, \ldots, 0) \in (\mathbb{N} \cup \{0\})^n$, 1 in the ith position.

Proof. We can write

$$A(x^2dx^1) = \sum_{i=1}^n f_i(x^k, X^{(\beta,l)})dx^i + \sum_{(\alpha,j)\in S} f_{(\alpha,j)}(x^k, X^{(\beta,l)})dX^{(\alpha,j)} \tag{3}$$

for some smooth maps f_i, $f_{(\alpha,j)}$. Using the invariance of A with respect to the homotheties $t\,\mathrm{id}_{\mathbb{R}^n}$ and the linearity of A we obtain

$$t^2 f_{(\alpha,j)}(x^k, X^{(\beta,l)}) = t^{|\alpha|+1} f_{(\alpha,j)}(tx^k, t^{|\beta|+1}X^{(\beta,l)}), \tag{4}$$

$$t^2 f_i(x^k, X^{(\beta,l)}) = t f_i(tx^k, t^{|\beta|+1}X^{(\beta,l)}). \tag{5}$$

From (4) it follows that $f_{(\alpha,j)} = 0$ for $(\alpha, j) \in S$ with $|\alpha| \geq 2$, and $f_{(\alpha,j)} =$ const for $(\alpha, j) \in S$ with $|\alpha| = 1$. Moreover, by the homogeneous function theorem (see [4]), $f_{((0),j)}$ is a linear combination with real coefficients of x^k and $X^{((0),l)}$ for $k, l = 1, \ldots, n$ and it is independent of the $X^{(\beta,l)}$ with $|\beta| \geq 1$.

From (5) and the homogeneous function theorem it follows that f_i is a linear combination with real coefficients of x^k and $X^{((0),l)}$ for $k, l = 1, \ldots, n$ and it is independent of the $X^{(\beta,l)}$ for $|\beta| \geq 1$. Then, by the invariance of A with respect to $(t_1x^1, \ldots, t_nx^n)$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n_+$, we get (2). ∎

We now study $\mu_1, \ldots, \mu_{10}$. We start with the case $r = 1$.

Lemma 3. *If $r = 1$, we have*

$$X^{(e_1,2)} \circ T^{[1]}G = X^{(e_1,2)} + \frac{x^2}{1 - x^1} X^{(e_1,1)}, \tag{6}$$

$$X^{(e_2,1)} \circ T^{[1]}G = X^{(e_2,1)} + \frac{x^2}{1 - x^1} X^{(e_1,1)}, \tag{7}$$

$$X^{((0),1)} \circ T^{[1]}G = (1 - x^1)X^{((0),1)} - X^{(e_1,1)}, \tag{8}$$

$$X^{((0),2)} \circ T^{[1]}G = \frac{1}{(1-x^1)} X^{((0),2)} + \frac{x^2}{(1-x^1)^2} X^{((0),1)} + \frac{1}{(1-x^1)^2} X^{(e_1,2)} + \frac{2x^2}{(1-x^1)^3} X^{(e_1,1)} + \frac{1}{(1-x^1)^2} X^{(e_2,1)} \tag{9}$$

over U, where

$$G = \left(x^1 - \frac{1}{2}(x^1)^2, \frac{x^2}{1-x^1}, x^3, \ldots, x^n\right)$$

is a local diffeomorphism defined on some open neighbourhood U of $0 \in \mathbb{R}^n$.

Proof. We only prove formula (6). The proofs of (7)–(9) are similar.

Consider $x_0 = (x_0^1, \ldots, x_0^n) \in U$ and $\Theta \in T^{[1]}_{x_0}\mathbb{R}^n$. We see that

$$j^1_{x_0}((G^1 - G^1(x_0))dG^2) = j^1_{x_0}\left((x^1 - x_0^1)dx^2 + \frac{x_0^2}{1-x_0^1}(x^1 - x_0^1)dx^1\right), \tag{10}$$

where $G = (G^1, \ldots, G^n)$. (We analyse the suitable partial derivatives at $x = x_0$.)

Using (10) we get

$$\begin{aligned} X^{(e_1,2)} \circ T^{[1]}G(\Theta) &= \langle T^{[1]}G(\Theta), j^1_{G(x_0)}((x^1 - G^1(x_0))dx^2)\rangle \\ &= \langle \Theta, j^1_{x_0}((G^1 - G^1(x_0))dG^2)\rangle \\ &= \left(X^{(e_1,2)} + \frac{x^2}{1-x^1} X^{(e_1,1)}\right)(\Theta). \ \blacksquare \end{aligned}$$

LEMMA 4. *Let A be as in Lemma 2. If $r = 1$, then $\mu_3 = \mu_2 = \mu_6 = \mu_8 = \mu_7 = 0$, $-\mu_4 + \mu_9 + \mu_{10} = 0$ and $\mu_4 - \mu_5 = 0$.*

Proof. Since

$$G^{-1} = \left(x^1 - \frac{1}{2}(x^1)^2, \frac{x^2}{1-x^1}, x^3, \ldots, x^n\right)^{-1}$$

preserves the germ at 0 of x^2dx^1, it also preserves the germ at 0 of $\mathcal{O}^*(A(x^2dx^1))$, where $\mathcal{O} : \mathbb{R}^n \to T^{[1]}\mathbb{R}^n$ is the zero section. Hence G^{-1} preserves the germ at 0 of $\mu_1 x^2 dx^1 + \mu_3 x^1 dx^2$, i.e. for the germs we have the equality

$$\begin{aligned} &\mu_1 x^2 dx^1 + \mu_3 x^1 dx^2 \\ &\quad = \mu_1 x^2 dx^1 + \mu_3\left(x^1 - \frac{1}{2}(x^1)^2\right)\left(\frac{1}{1-x^1}dx^2 + \frac{x^2}{(1-x^1)^2}dx^1\right). \end{aligned}$$

So $\mu_3 = 0$.

Then using (2) and (6)–(9), we see that the equivariance of $A(x^2dx^1)$ at $x=0$ with respect to G^{-1} is equivalent to the following equality at $x=0$:

$$\begin{aligned}
&\mu_2 X^{((0),2)}dx^1+\mu_4 X^{((0),1)}dx^2+\mu_6 X^{((0),2)}dX^{((0),1)}\\
&+\mu_8 X^{((0),1)}dX^{((0),2)}+\mu_9 dX^{(e_2,1)}+\mu_{10}dX^{(e_1,2)}\\
&\ =\mu_2(X^{((0),2)}+X^{(e_1,2)}+X^{(e_2,1)})dx^1+\mu_4(X^{((0),1)}-X^{(e_1,1)})dx^2\\
&\qquad+\mu_6(X^{((0),2)}+X^{(e_1,2)}+X^{(e_2,1)})(-X^{((0),1)}dx^1+dX^{((0),1)}-dX^{(e_1,1)})\\
&\qquad+\mu_8(X^{((0),1)}-X^{(e_1,1)})(X^{((0),2)}dx^1+dX^{((0),2)}+X^{((0),1)}dx^2\\
&\qquad+dX^{(e_1,2)}+2X^{(e_1,2)}dx^1+2X^{(e_1,1)}dx^2+dX^{(e_2,1)}+2X^{(e_2,1)}dx^1)\\
&\qquad+\mu_9(dX^{(e_2,1)}+X^{(e_1,1)}dx^2)+\mu_{10}(dX^{(e_1,2)}+X^{(e_1,1)}dx^2).
\end{aligned}$$

Analysing the coefficients of $X^{(e_1,2)}dx^1$ we get $\mu_2=0$. Then analysing the coefficients of $X^{(e_1,1)}dx^2$ we obtain $-\mu_4+\mu_9+\mu_{10}=0$. Next, analysing the coefficients of $X^{(e_1,2)}dX^{((0),1)}$ we have $\mu_6=0$. Finally, considering the coefficients of $X^{(e_1,1)}dX^{((0),2)}$ we derive $\mu_8=0$.

Hence, using (2) and (6)–(9) again, we see that the equivariance of $d(A(x^2dx^1))$ at $x=0$ with respect to G^{-1} is equivalent to the following equality at $x=0$:

$$\begin{aligned}
&\mu_1 dx^2\wedge dx^1+\mu_4 dX^{((0),1)}\wedge dx^2+\mu_5 dx^2\wedge dX^{((0),1)}+\mu_7 dx^1\wedge dX^{((0),2)}\\
&\qquad=\mu_1 dx^2\wedge dx^1+\mu_4(-X^{((0),1)}dx^1+dX^{((0),1)}-dX^{(e_1,1)})\wedge dx^2\\
&\qquad\quad+\mu_5 dx^2\wedge(-X^{((0),1)}dx^1+dX^{((0),1)}-dX^{(e_1,1)})\\
&\qquad\quad+\mu_7 dx^1\wedge(X^{((0),2)}dx^1+dX^{((0),2)}+X^{((0),1)}dx^2+dX^{(e_1,2)}\\
&\qquad\quad+2X^{(e_1,2)}dx^1+2X^{(e_1,1)}dx^2+dX^{(e_2,1)}+2X^{(e_2,1)}dx^1).
\end{aligned}$$

Analysing the coefficients of $dx^1\wedge dX^{(e_1,2)}$ we get $\mu_7=0$. Finally, analysing the coefficients of $X^{((0),1)}dx^1\wedge dx^2$ we obtain $\mu_4-\mu_5=0$. ∎

Now, let $r=2$.

Lemma 5. *If $r=2$, we have*

$$X^{(e_2,1)}\circ T^{[2]}G=X^{(e_2,1)}+\frac{x^2}{1-x^1}X^{(e_1,1)},\tag{11}$$

$$\begin{aligned}X^{(e_1,2)}\circ T^{[2]}G=X^{(e_1,2)}&+\frac{1}{2}\frac{1}{1-x^1}X^{((2,0,\dots,0),2)}+\frac{x^2}{1-x^1}X^{(e_1,1)}\\
&+\frac{3}{2}\frac{x^2}{(1-x^1)^2}X^{((2,0,\dots,0),1)}+\frac{1}{1-x^1}X^{((1,1,0,\dots,0),1)},\end{aligned}\tag{12}$$

$$X^{((0),1)}\circ T^{[2]}G=(1-x^1)X^{((0),1)}-X^{(e_1,1)},\tag{13}$$

over U, *where*

$$G = \left(x^1 - \frac{1}{2}(x^1)^2, \frac{x^2}{1-x^1}, x^3, \ldots, x^n\right)$$

is as in Lemma 3.

Proof. The proof is similar to that of Lemma 3. ∎

LEMMA 6. *Let A be as in Lemma 2. If $r = 2$, then $\mu_3 = \mu_2 = \mu_6 = \mu_8 = \mu_7 = \mu_{10} = 0$, $-\mu_4 + \mu_9 = 0$ and $\mu_4 - \mu_5 = 0$.*

Proof. We have the natural inclusion $I^{[1,2]} : T^{[1]}M \to T^{[2]}M$ for any n-manifold M. This inclusion is dual to the jet projection $J^2T^*M \to J^1T^*M$. Using $I^{[1,2]}$ we pull-back A. In this way we obtain a linear natural operator $(I^{[1,2]})^*A : T^* \rightsquigarrow T^*T^{[1]}$. Applying Lemma 4 to $(I^{[1,2]})^*A$ we obtain $\mu_3 = \mu_2 = \mu_6 = \mu_8 = \mu_7 = 0$, $-\mu_4 + \mu_9 + \mu_{10} = 0$ and $\mu_4 - \mu_5 = 0$.

It remains to prove that $\mu_{10} = 0$. Using (2) and (11)–(13), we see that the equivariance of $A(x^2dx^1)$ at $x = 0$ with respect to G^{-1} is equivalent to the following equality at $x = 0$:

$$\begin{aligned}
&\mu_4 X^{((0),1)}dx^2 + \mu_9 dX^{(e_2,1)} + \mu_{10} dX^{(e_1,2)} \\
&\quad = \mu_4(X^{((0),1)} - X^{(e_1,1)})dx^2 + \mu_9(dX^{(e_2,1)} + X^{(e_1,1)}dx^2) \\
&\qquad + \mu_{10}\Big(dX^{(e_1,2)} + \frac{1}{2}X^{((2,0,\ldots,0),2)}dx^1 + \frac{1}{2}dX^{((2,0,\ldots,0),2)} + X^{(e_1,1)}dx^2 \\
&\qquad + \frac{3}{2}X^{((2,0,\ldots,0),1)}dx^2 + X^{((1,1,0,\ldots,0),1)}dx^1 + dX^{((1,1,0,\ldots,0),1)}\Big).
\end{aligned}$$

Analysing the coefficients of $dX^{((1,1,0,\ldots,0),1)}$ we get $\mu_{10} = 0$. ∎

Now, let $r \geq 3$.

LEMMA 7. *Let A be as in Lemma 2. If $r \geq 3$, then $\mu_3 = \mu_2 = \mu_6 = \mu_8 = \mu_7 = \mu_{10} = 0$, $-\mu_4 + \mu_9 = 0$ and $\mu_4 - \mu_5 = 0$.*

Proof. We have the natural inclusion $I^{[2,r]} : T^{[2]}M \to T^{[r]}M$ for any n-manifold M. This inclusion is dual to the jet projection $J^rT^*M \to J^2T^*M$. Using $I^{[2,r]}$ we pull-back A. In this way we obtain a linear natural operator $(I^{[2,r]})^*A : T^* \rightsquigarrow T^*T^{[2]}$. Applying Lemma 6 to $(I^{[2,r]})^*A$ we end the proof. ∎

Proof of Theorem 1 for $n \geq 2$. By Lemmas 1 and 4 we see that the vector space of all natural operators $T^* \rightsquigarrow T^*T^{[1]}$ has dimension ≤ 3 if $r = 1$ and $n \geq 2$. On the other hand, the operators A^{V}, $A^{[1]}$ and $A^{(1)}$ are linearly independent. (For, $d(A^{\mathrm{V}}(x^2dx^1)) = dx^2 \wedge dx^1 \neq 0$, $d(A^{[1]}(x^2dx^1)) = d(A^{(1)}(x^2dx^1)) = 0$, $A^{[1]}(dx^1) = dX^{((0),1)} \neq 0$ and $A^{(1)}(dx^1) = 0$.) These facts complete the proof of Theorem 1 for $n \geq 2$ and $r = 1$.

By Lemmas 1, 6 and 7 we see that the vector space of all natural operators $T^* \rightsquigarrow T^*T^{[r]}$ has dimension ≤ 2 if $r \geq 2$ and $n \geq 2$. On the other hand, the operators A^{V} and $A^{[r]}$ are linearly independent. (For, $d(A^{\mathrm{V}}(x^2dx^1)) \neq 0$ and $d(A^{[r]}(x^2dx^1)) = 0$.) These facts complete the proof of Theorem 1 for $n \geq 2$ and $r \geq 2$. ∎

3. Now, we prove Theorem 1 for $n = 1$.

In [7], we proved the following reducibility lemma.

LEMMA 8. *Let F be a natural bundle over 1-manifolds. If $A : T^* \rightsquigarrow T^*F$ is a linear natural operator such that $A(dx^1) = 0$, then $A = 0$.*

We set $S = \{(\alpha, 1) \mid \alpha = 0, \dots, r\}$. On $T^{[r]}\mathbb{R}^1$ we have the coordinates $(x^1, X^{(\alpha,1)})$, $(\alpha, 1) \in S$, given by

$$x^1(\Theta) = x_0^1, \qquad X^{(\alpha,1)}(\Theta) = \langle \Theta, j^r_{x_0}((x^1 - x^1_0)^\alpha dx^1)\rangle,$$

where $\Theta \in T^{[r]}_{x_0}\mathbb{R}^1$ and $x_0 = x^1_0 \in \mathbb{R}^1$.

LEMMA 9. *Let $A : T^* \rightsquigarrow T^*T^{[r]}$ be a linear natural operator. Then*

$$A(dx^1) = \mu_1 dx^1 + \mu_2 dX^{(0,1)}$$

for some $\mu_1, \mu_2 \in \mathbb{R}$.

Proof. We use similar methods to those in the proof of Lemma 2. ∎

Proof of Theorem 1 for $n = 1$. By Lemmas 8 and 9 we see that the vector space of all natural operators $T^* \rightsquigarrow T^*T^{[r]}$ has dimension ≤ 2 if $n = 1$. On the other hand, the operators A^{V} and $A^{[r]}$ are linearly independent. These facts complete the proof of Theorem 1 for $n = 1$. ∎

4. As an application of Theorem 1 we get a classification of all linear natural transformations $B : J^rT^* \to J^rT^*$ over n-manifolds.

EXAMPLE 4. For any n-manifold M we have the identity map $\mathrm{id} : J^rT^*M \to J^rT^*M$. Thus we have the identity natural transformation $\mathrm{id} : J^rT^* \to J^rT^*$ over n-manifolds.

EXAMPLE 5. The first order linear natural operator $B^{(1)} : T^* \rightsquigarrow J^1T^*$ from Example 3 defines the corresponding linear natural transformation $B^{(1)} : J^1T^* \to J^1T^*$ over n-manifolds such that $B^{(1)}(j^1_x(\omega)) = B^{(1)}(\omega)(x)$, $\omega \in \Omega^1(M)$, $x \in M$.

COROLLARY 1. *Let r and n be natural numbers.*

(i) *If $r \geq 2$ and $n \geq 2$, then every linear natural transformation $B : J^rT^* \to J^rT^*$ is a constant multiple of the identity natural transformation.*

(ii) *If* $r = 1$ *and* $n \geq 2$, *then every linear natural transformation* $B : J^1T^* \to J^1T^*$ *is a linear combination with real coefficients of* id *and* $B^{(1)}$.

(iii) *If* $n = 1$ *then every linear natural transformation* $B : J^rT^* \to J^rT^*$ *is a constant multiple of the identity natural transformation.*

Proof. We prove only (i). The proofs of (ii) and (iii) are similar.

Let $B : J^rT^* \to J^rT^*$ be a linear natural transformation over n-manifolds. Then we have a linear natural operator $A^{[B]} : T^* \rightsquigarrow T^*T^{[r]}$ defined as follows. Given $\omega \in \Omega^1(M)$ we define $\omega^{[B]} : T^{[r]}M \to \mathbb{R}$ by $\omega^{[B]}(\Theta) = \langle \Theta, B(j^r_x(\omega))\rangle$, $\Theta \in T^{[r]}_x M$, $x \in M$. We put $A^{[B]}(\omega) = d(\omega^{[B]})$.

By Theorem 1(i) there are $a, b \in \mathbb{R}$ such that $d(\omega^{[B]}) = a\omega^{\mathrm{V}} + bd(\omega^{[r]})$ for every $\omega \in \Omega^1(M)$. Taking the differential of both sides we get $ad(\omega^{\mathrm{V}}) = 0$, i.e. $a = 0$ because $d((x^2dx^1)^{\mathrm{V}}) \neq 0$. Hence $\omega^{[B]} = b\omega^{[r]} + C(\omega)$ for some $C(\omega) \in \mathbb{R}$. Evaluating both sides at $\Theta = 0 \in T^{[r]}_x M$ we get $C(\omega) = 0$. Hence $B = b\,\mathrm{id}$. ∎

REMARK. There is a linear first order natural operator $B^{(1)} : T^* \rightsquigarrow J^1T^*$ with $B^{(1)}(fdg)(x_0) = j^1_{x_0}((f - f(x_0))dg - (g - g(x_0))df)$, $f, g : M \to \mathbb{R}$, $x_0 \in M$ (see Example 3). Corollary 1 shows that this construction cannot be generalized to higher orders. Namely, from Corollary 1 it follows that if $r \geq 2$, then there is no linear rth order natural operator $B^{(r)} : T^* \rightsquigarrow J^rT^*$ with $B^{(r)}(fdg)(x_0) = j^r_{x_0}((f - f(x_0))dg - (g - g(x_0))df)$, $f, g : M \to \mathbb{R}$, $x_0 \in M$.

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