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UNBOUNDED HARMONIC FUNCTIONS ON HOMOGENEOUS MANIFOLDS OF NEGATIVE CURVATURE

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Abstract. We study unbounded harmonic functions for a second order differential operator on a homogeneous manifold of negative curvature which is a semidirect product of a nilpotent Lie group N and $A = \mathbb{R}^+$. We prove that if F is harmonic and satisfies some growth condition then F has an asymptotic expansion as $a \to 0$ with coefficients from $\mathcal{D}'(N)$. Then we single out a set of at most two of these coefficients which determine F.

Then using asymptotic expansions we are able to prove some theorems answering partially the following question. *Is a given harmonic function the Poisson integral of "something" from the boundary N?*

1. Introduction. In this paper we consider unbounded harmonic functions for a second order differential operator on a connected, simply connected homogeneous manifold of negative curvature. Such a manifold is a semidirect product $G = N \times_{\mathrm{s}} A$, where N is a nilpotent Lie group and $A = \mathbb{R}^+$ normalizes N (see [7]).

In general, we use upper case Roman letters to denote Lie groups and the corresponding upper case script letter to denote the Lie algebra, which allows us to write

$$\mathcal{G} = \mathcal{N} \times_{\mathrm{s}} \mathbb{R}.$$

Then the negative curvature assumption also implies that $H = (0,1) \in \mathcal{G}$ may be chosen so that the eigenvalues d_j of $\mathrm{ad}(H)$ on $\mathcal{N}$ all have positive real parts which satisfy

$$\operatorname{Re} d_j \geq 3.$$

Let the general element of $G = N \times_{\mathrm{s}} \mathbb{R}^+$ be denoted by (x,a). On G we consider the second order left-invariant operator

$$\mathcal{L}^\gamma = \sum_j (X_j^a)^2 + X^a + a^2\partial_a^2 + (1-\gamma)a\partial_a,$$

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where $X, X_1, \ldots, X_m$ are left-invariant vector fields on N, $X_1, \ldots, X_m$ generate $\mathcal{N}$ as a Lie algebra, and for $Y \in \mathcal{N}$,

$$Y^a = \mathrm{Ad}(\exp(\log a)H)(Y).$$

We say that a function F on $N \times_{\mathrm{s}} A$ is $\mathcal{L}^\gamma$*-harmonic* if $\mathcal{L}^\gamma F = 0$.

There is a vast body of literature on $\mathcal{L}^\gamma$-harmonic functions, much of it devoted to the study of *bounded* harmonic functions in the $\gamma > 0$ case by means of their boundary values on N as $a \to 0$. For $\gamma > 0$ it certainly is of interest to study unbounded harmonic functions as well. For $\gamma = 0$ it is essential, as there are no non-constant bounded harmonic functions in this case. This work is devoted to the unbounded case.

It turns out that in the unbounded case, to uniquely determine an $\mathcal{L}^\gamma$-harmonic function F, we need to know not only its boundary values, but those of a function that is analogous to a normal derivative of F. For example, consider the Laplace operator Δ on the upper half plane $H^+ = \{x + ia \mid a > 0\}$. The typical harmonic function is $F(z) = \mathrm{Re}(f(z))$, where f is holomorphic on H^+. Assume that f is entire and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

We assume (without loss of generality) that a_0 is real. Then, for $x \in \mathbb{R}$,

$$F(x) = \sum_{n=0}^{\infty} \mathrm{Re}(a_n) x^n,$$

which determines only the real parts of the a_n. However, from the Cauchy–Riemann equations, the normal derivative of F is

$$\frac{dF}{da}(x) = \mathrm{Re}(if'(x)) = -\sum_{n=0}^{\infty} n\,\mathrm{Im}(a_n) x^{n-1},$$

which determines the imaginary parts of the a_n.

The general theory is similar. Let F be $\mathcal{L}^\gamma$-harmonic. We identify locally integrable functions on N with elements of $\mathcal{D}'(N)$ (the space of distributions on N) via integration against them with respect to the Haar measure dx on N. We say that F has *moderate growth* as $a \to 0$ if there is an $r \in \mathbb{R}$ such that for all compact $K \subset N$ and $a \in (0, 1]$,

$$\int_K |F(x, a)|\, dx \le C_K a^{-r} \tag{1.1}$$

for some constant C_K. It is clear that (1.1) holds for uniformly bounded solutions (with $r = 0$), although it is, in general, considerably weaker. The following is our first main result.

THEOREM 1.1. *Suppose that F is $\mathcal{L}^\gamma$-harmonic and satisfies* (1.1). *Then the following limits exist in $\mathcal{D}'(N)$*:

$$\lim_{a\to 0^+} F(\cdot,a) =: F_0(\cdot), \qquad 1\ge\gamma>0,$$
$$\lim_{a\to 0^+} a^{-\gamma}(F(\cdot,a)-F_0(\cdot)) =: G_0(\cdot), \qquad 1\ge\gamma>0,$$

and

$$\lim_{a\to 0^+} (\ln a)^{-1} F(\cdot,a) =: F_0^1(\cdot), \qquad \gamma=0,$$
$$\lim_{a\to 0^+} (F(\cdot,a)-(\ln a)F_0^1(\cdot)) =: F_0^0(\cdot), \qquad \gamma=0.$$

Furthermore, if $F_0=G_0=0$ $(1\ge\gamma>0)$ or $F_0^0=F_0^1=0$ $(\gamma=0)$, then F vanishes identically.

One remarkable consequence of this result is that if the growth of $\widetilde{F}(a)=F(\cdot,a)$ as $a\to 0^+$ is known to be of order at most a^{-r}, then the growth is in fact at most of order a^0 for $\gamma>0$, and at most logarithmic for $\gamma=0$.

The proof of Theorem 1.1 is based on an asymptotic expansion for F. Specifically, we show that if F is $\mathcal{L}^\gamma$-harmonic and satisfies (1.1) then F has an asymptotic expansion as $t\to-\infty$,

$$F(\cdot,e^t)\sim\begin{cases}\displaystyle\sum_{\alpha\in I} e^{\alpha t}F_\alpha(t)+e^{\gamma t}\sum_{\alpha\in I} e^{\alpha t}G_\alpha(t), & 1\ge\gamma>0,\\ \displaystyle\sum_{\alpha\in I} e^{\alpha t}F_\alpha(t), & \gamma=0,\end{cases}$$

where $F_\alpha(t)$ and $G_\alpha(t)$ are polynomials on $\mathbb{R}$ with coefficients in $\mathcal{D}'(N)$ and

$$I=\Big\{\sum_j d_jk_j \,\Big|\, k_j\in\mathbb{Z},\ k_j\ge 0\Big\}.$$

We refer the reader to Section 2 for the sense in which these expansions converge. For $\gamma>0$, $F_0(t)$ and $G_0(t)$ are independent of t while for $\gamma=0$, $F_0(t)=F_0^0+tF_0^1$ where $F_0^i\in\mathcal{D}'(N)$. Both this result and its proof were motivated by results of van den Ban and Schlichtkrull [1], although our proof is actually considerably different from theirs.

The operator $\mathcal{L}^\gamma$ has a Poisson kernel, i.e. there is a positive function P^γ on N such that for all "nice" functions f,

$$\begin{aligned}F(x,a)&=\int_N f(xy^a)P^\gamma(y)\,dy\\ &=\int_N f(y)P^\gamma((x^{-1}y)^{a^{-1}})a^{-Q}\,dy=:P^\gamma(f)\end{aligned}\tag{1.2}$$

satisfies $\mathcal{L}^\gamma F=0$, where dy is the Haar measure on N,

$$Q=\operatorname{Tr}\operatorname{ad}(H)$$

and y^a is the element of N defined by

$$y^a = \exp((\log a)H)y\exp((-\log a)H).$$

For $\gamma > 0$, P^γ is integrable, but for $\gamma = 0$ it is not. In fact, from [4] for every $\gamma \geq 0$ there is a positive constant C_γ such that

$$C_\gamma^{-1}(1+|x|)^{-Q-\gamma} \leq P^\gamma(x) \leq C_\gamma(1+|x|)^{-Q-\gamma}, \tag{1.3}$$

where $|x|$ is a certain homogeneous norm defined from the action of A on N (cf. [5, 4]). Specifically, for $x, y \in N$ and $a \in \mathbb{R}^+$,

(H1) $|xy| \leq C(|x|+|y|)$ for some fixed constant C.
(H2) $|x| = 0$ if and only if $x = e$.
(H3) $|x^a| = a|x|$.
(H4) $|x^{-1}| = |x|$.

It turns out (see [10]) that we can control constants which appear in the proof of (1.3) in [4] and show that we may in fact choose C_γ independent of γ for $0 \leq \gamma \leq 1$, i.e.

$$C^{-1}(1+|x|)^{-Q-\gamma} \leq P^\gamma(x) \leq C(1+|x|)^{-Q-\gamma}, \quad 0 \leq \gamma \leq 1. \tag{1.4}$$

The statements about the integrability of the Poisson kernel follow from (1.3) and the following well known (and easily proved) lemma.

Lemma 1.2 ([6], Proposition 1.15). *Let $S = \{x \in N \mid |x| = 1\}$. Then there is a unique probability measure dS on S such that for all $f \in L^1(N)$,*

$$\int_N f(x)\,dx = \int_0^\infty \int_S f(s^a)a^{Q-1}\,dS\,da.$$

Since P^0 is not integrable, we normalize our Poisson kernel functions so that $P^\gamma(e) = 1$ instead of assuming that their integral is 1. It follows from results of [3] that for $\gamma > 0$, if $f \in L^\infty(N)$, then $F = P^\gamma(f)$ is a uniformly bounded, $\mathcal{L}^\gamma$-harmonic function and

$$\|P^\gamma\|_1^{-1} \lim_{a\to 0} F(x,a) = f(x). \tag{1.5}$$

Hence the inverse of the Poisson transformation on $L^\infty(G)$ is given by $F \mapsto \|P^\gamma\|_1^{-1}F_0$.

Our second main result is a description of the inverse of the Poisson transform for not necessarily bounded f and all $1 \geq \gamma \geq 0$. We prove the following result. The existence and positivity of the limit defining C_0 is a remarkable consequence of our theory.

Theorem 1.3. *Assume $0 \leq \gamma \leq 1$. Then for $f \in L^1((1+|x|)^{-Q-\gamma}dx)$, $F = P^\gamma(f)$ is $\mathcal{L}^\gamma$-harmonic and satisfies* (1.1)*. Furthermore, as distributions* *on* N,

$$f(x) = \begin{cases} \|P^\gamma\|_1^{-1} \lim_{a\to 0^+} P^\gamma(f)(x,a), & 1 \geq \gamma > 0, \\ -C_0^{-1} \lim_{a\to 0^+} (\ln a)^{-1} P^0(f)(x,a), & \gamma = 0, \end{cases}$$

where

$$C_0 = \lim_{\gamma\to 0^+} \gamma \|P^\gamma\|_1.$$

In the $\gamma > 0$ case, all uniformly bounded $\mathcal{L}^\gamma$-harmonic functions F are Poisson integrals. One might hope that if $F_0 \in L^1((1+|x|)^{-Q-\gamma}dx)$, then $F = \|P^\gamma\|_1^{-1} P^\gamma(F_0)$. This, we suspect, is false. The general theory says that we need both F_0 and G_0 to uniquely determine F. Conditions such as uniform boundedness are essentially boundary conditions that force a portion of the asymptotic expansion to vanish. We can prove the following result.

THEOREM 1.4. *An $\mathcal{L}^0$-harmonic function F is the Poisson integral of a compactly supported element of $L^1(N)$ if and only if*

(a) *F satisfies* (1.1),

(b) *there is a compact set $K \subset N$ such that F is uniformly bounded on $G \setminus (K \times (0,1])$,*

(c) *the boundary distribution F_0^1 is a locally integrable function.*

Our results also yield new information concerning the Poisson kernel:

THEOREM 1.5. *In the topology of $C^\infty(N)$,*

$$\lim_{\gamma\to 0^+} P^\gamma(x) = P^0(x).$$

The existence of the following limit yields interesting information on the rate of growth of the Poisson kernel as we approach the boundary.

THEOREM 1.6. *Let $1 \geq \gamma > 0$. There is a function Q^γ which is locally integrable on $N \setminus \{e\}$ such that in the weak topology on measures on $N \setminus \{e\}$,*

$$\lim_{a\to 0} a^{-Q-\gamma} P^\gamma(x^{a^{-1}})dx = Q^\gamma(x)dx, \qquad \gamma > 0,$$

where dx denotes the Haar measure on N.

Let

$$\varrho_\gamma(x) = (Q^\gamma(x))^{-1/(Q+\gamma)}. \tag{1.6}$$

It is easily seen from (1.4) that there is a constant $C \geq 1$ such that

$$C^{-1}|x| \leq \varrho_\gamma(x) \leq C|x|,$$

so that ϱ_γ has properties (H1) and (H2). From the definition, it is obvious that it also satisfies property (H3), so that only (H4) may not be satisfied. Therefore, we call ϱ_γ an *almost homogeneous norm*.

The following says that for $\gamma > 0$, the G_0 term in F's asymptotic expansion is essentially the "fractional Laplacian" of f with respect to ϱ_γ.

THEOREM 1.7. *Let $f \in C_c^\infty(N)$ and set $F = P^\gamma(f)$. Then for $1 \ge \gamma > 0$, the G_0 term in the asymptotic expansion of F is*

$$G_0(x) = \int_N (f(xy) - f(x))\varrho_\gamma(y)^{-Q-\gamma}\, dy =: \Delta^\gamma(f)(x).$$

The mapping of $L^1((1+|x|)^{-Q-\gamma}dx)$ into $\mathcal{D}'(N)$ defined by $f \mapsto G_0$-term of the asymptotic expansion of $P^\gamma(f)$ is a continuous extension of Δ^γ and is still denoted by Δ^γ. The following corollary is an immediate consequence of the uniqueness in Theorem 1.1.

COROLLARY 1.8. *Let F be $\mathcal{L}^\gamma$-harmonic where $1 \ge \gamma > 0$ and have moderate growth. Suppose that $f = \|P^\gamma\|_1^{-1} F_0$ belongs to $L^1((1+|x|)^{-Q-\gamma}dx)$. Then $F = P^\gamma(f)$ if and only if $G_0 = \Delta^\gamma(f)$.*

The structure of the paper is as follows. In Section 2 we study asymptotic expansions mentioned above. The proofs and results are inspired by those in [1], although our proofs are quite different from theirs and somewhat less involved.

In Section 3 we make use of quite general results of Section 2 in our setting. In particular, we prove Theorem 1.1.

Finally, in Section 4 we prove our main results announced in the Introduction: Theorems 1.3–1.7.

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2. Asymptotic expansions. The material here is based on, and somewhat repetitive of, Section 1 of [9]. We include it because (a) at the time of writing, [9] was not in print, (b) our differential operator is somewhat different than that considered in [9], and (c) we require Proposition 2.8, which does not appear in [9], and whose proof requires repetition of the proof of Theorem 2.5.

Let $\mathcal{V}$ be a locally convex complete and reflexive topological linear space over $\mathbb{C}$ and let $\mathcal{C} = C^\infty((-\infty, 0], \mathcal{V})$, given the topology of uniform convergence on compact subsets of functions and their derivatives. For $r \in \mathbb{R}$, let $\mathcal{C}_r^o$ be the set of $F \in \mathcal{C}$ such that $\{e^{-rt}F(t) \mid t \in (-\infty, 0]\}$ is bounded in $\mathcal{V}$. Let $\|\cdot\|_m$, $m \in \Lambda$, where Λ is an index set, be a family of continuous seminorms on $\mathcal{V}$ that defines its topology. We equip $\mathcal{C}_r^o$ with the topology defined by the seminorms

$$\|F\|_{r,m} = \sup_{t\in(-\infty,0]} e^{-rt}\|F(t)\|_m, \qquad \|F\|_{k,n,m} = \sup_{-k\le t\le 0} \|F^{(n)}(t)\|_m,$$

where $k \in \mathbb{N}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We let

$$\mathcal{C}_r = \bigcap_{s<r} \mathcal{C}_s^o$$

given the inverse limit topology. The space $\mathcal{C}_r$ is used since, unlike $\mathcal{C}_r^o$, it is closed under multiplication by polynomials.

Let F and G belong to $\mathcal{C}$. We write

$$F \sim_r G$$

if $F - G \in \mathcal{C}_r$. Note that $F \sim_r G$ implies that $F \sim_u G$ for all $u < r$. (Remember that the functions from $\mathcal{C}$ are defined on $(-\infty, 0]$.)

Let $I \subset \mathbb{C}$ be finite. An *exponential polynomial with exponents from* I is a sum

$$F(t) = \sum_{\alpha \in I} e^{\alpha t} F_\alpha(t), \tag{2.1}$$

where for $\alpha \in I$ we have $F_\alpha \in \mathcal{V}[t]$, $t \in (-\infty, 0]$, i.e.,

$$F_\alpha(t) = \sum_{k=0}^{n_\alpha} F_\alpha^k t^k,$$

where $F_\alpha^k \in \mathcal{V}$ and $n_\alpha \in \mathbb{N}_0$.

By a *formal exponential series with exponents in* I, where $I \subset \mathbb{C}$, we mean a formal sum as in (2.1) where now I might be infinite.

Let $F \in \mathcal{C}$ and let $\check{F}$ be an exponential series. We write

$$F \sim \check{F} = \sum_{\alpha \in I} e^{\alpha t} F_\alpha(t), \tag{2.2}$$

if

(i) for all $r \in \mathbb{R}$, there is a finite subset $I(r) \subset I$ such that $F \sim_r F_r$, where

$$F_r(t) = \sum_{\alpha \in I(r)} e^{\alpha t} F_\alpha(t) \tag{2.3}$$

(ii) $I = \bigcup_r I(r)$.

In this case, we say that $\check{F}$ is an *asymptotic expansion for* F.

Remark. In formula (2.3), any term corresponding to an index α with $\operatorname{Re}\alpha \geq r$ belongs to $\mathcal{C}_r$ and may be omitted. Thus, we may, and will, take $I(r)$ to be contained in the set of $\alpha \in I$ where $\operatorname{Re}\alpha < r$.

We note the following lemma, which is a simple consequence of Lemma 3.3 of [1].

Lemma 2.1. *If the function from* (2.1) *belongs to* $\mathcal{C}_r$, *then* $F_\alpha(t) = 0$ *for all* $\operatorname{Re}\alpha < r$ *and all* $t \in \mathbb{R}$.

LEMMA 2.2. *Let $I(r)$ be chosen as in the preceding remark. Suppose also that all the $F_\alpha(t)$ for $\alpha \in I$ are non-zero. Then $I(r) = \{\alpha \in I \mid \operatorname{Re}\alpha < r\}$. In particular, the set of such α is finite.*

Proof. Let $r < s$. Then $F \sim_r F_r$ and $F \sim_r F_s$. Hence $D_r = F_r - F_s \in \mathcal{C}_r$. Then D_r is an exponential polynomial with index set

$$(I(r) \cup I(s)) \setminus (I(r) \cap I(s)).$$

Lemma 2.1 shows that this set is disjoint from $\operatorname{Re}\alpha < r$, implying that it is disjoint from $I(r)$. Hence $I(r) \subset I(s)$. It also follows that $I(s) \setminus I(r)$ is disjoint from $\{\operatorname{Re}\alpha < r\}$. Hence $\{\operatorname{Re}\alpha < r\} \cap I \subset I(r)$, which proves our lemma. ∎

COROLLARY 2.3. *Let $F \in \mathcal{C}$. Suppose that for each $r \in \mathbb{R}$, there is an exponential polynomial P^r such that $F \sim_r P^r$. Then there is an exponential series $\check{F}$ such that $F \sim \check{F}$.*

Proof. For each $r \in \mathbb{R}$ let

$$P^r(t) = \sum_{\alpha \in S(r)} e^{\alpha t} P^r_\alpha(t),$$

where $S(r)$ is a finite subset of $\mathbb{C}$. We may assume that $P^r_\alpha(t) \neq 0$ for all $\alpha \in S(r)$. Note that if $\alpha \in S(r)$ and $\operatorname{Re}\alpha \geq r$, then the corresponding term in the above sum belongs to $\mathcal{C}_r$ and can be dropped from the sum. Thus, we may assume that for all $\alpha \in S(r)$, $\operatorname{Re}\alpha < r$.

We claim now that if $r < s$, then

$$P^s(t) = P^r(t) + \sum_{S(s)\setminus S(r)} e^{\alpha t} P^s_\alpha(t).$$

In fact, $F \sim_r P^r$ and $F \sim_r P^s$ imply that $P^r \sim_r P^s$. Our claim follows from Lemma 2.1. Since none of the P^r_α is zero, we also have $I(r) \subset I(s)$ for $r < s$.

Our corollary now follows: we let I be the union of the $S(r)$ and let

$$\check{F}_\alpha(t) = P^r_\alpha(t),$$

where $\alpha \in S(r)$. The previous remarks show that this is independent of the choice of r. ∎

The following is left to the reader. The minimum exists due to Lemma 2.2.

PROPOSITION 2.4. *Suppose that $F \in \mathcal{C}$ has an asymptotic expansion with exponents from I. Then $F \in \mathcal{C}_r$, where $r = \min\{\operatorname{Re}\alpha \mid \alpha \in I, F_\alpha \neq 0\}$.*

We consider a differential equation on $\mathcal{C}$ of the form

$$F'(t) = (Q_0 + Q(t))F(t), \tag{2.4}$$

where F' is the derivative of F as a $\mathcal{V}$-valued function and

$$Q(t) = \sum_{i=1}^{d} p_i(t) e^{\beta_i t} Q_i,$$

where the p_i are $\mathbb{C}$-valued polynomials,

$$1 \leq \operatorname{Re} \beta_1 \leq \ldots \leq \operatorname{Re} \beta_d \tag{2.5}$$

and the Q_i are continuous linear operators on $\mathcal{V}$. We also assume that Q_0 is *finitely triangularizable*, meaning that:

(a) There is a direct sum decomposition

$$\mathcal{V} = \sum_{i=1}^{q} \mathcal{V}^i, \tag{2.6}$$

where the $\mathcal{V}^i$ are closed subspaces of $\mathcal{V}$ invariant under Q_0.

(b) For each i there is an $\alpha_i \in \mathbb{C}$ and an integer n_i such that

$$(Q_0 - \alpha_i I)^{n_i}|_{\mathcal{V}^i} = 0.$$

(c) $\alpha_i \neq \alpha_j$ for $i \neq j$.

In this case, we may define e^{tQ_0} by

$$e^{tQ_0}|_{\mathcal{V}^i} = e^{\alpha_i t} \sum_{n=0}^{n_i} \frac{(Q_0 - \alpha_i I)^n}{n!}\bigg|_{\mathcal{V}^i}.$$

Let

$$I_0 = \Big\{ \sum_j \beta_j k_j \,\Big|\, k_j \in \mathbb{N}_0 \Big\}.$$

The first main result of this section is the following:

THEOREM 2.5. *Let $F \in \mathcal{C}_r$ satisfy* (2.4). *Then F has an asymptotic expansion with exponents from $I = \{\alpha_i \mid 1 \leq i \leq q\} + I_0$.*

Proof. From Corollary 2.3 it suffices to prove that for all $n \in \mathbb{N}$, there is an exponential polynomial $S_n(t)$ with exponents from I such that

$$F(t) - S_n(t) \in \mathcal{C}_{r+n}.$$

We reason by induction on n. Let

$$P(t) = \sum_i p_i(t) e^{(\beta_i - 1)t} Q_i,$$

so that $Q(t) = e^t P(t)$. Note that $\operatorname{Re} \beta_i - 1 \geq 0$ for all i.

We apply the method of Picard iteration to (2.4). Explicitly, (2.4) implies that

$$F(t) = e^{tQ_0} F(0) - \int_t^0 e^{(t-s)Q_0} e^s P(s) F(s)\, ds. \tag{2.7}$$

Let $B_i = (Q_0 - \alpha_i I)|_{\mathcal{V}^i}$. On $\mathcal{V}^i$,

$$e^{tQ_0} = e^{\alpha_i t} A_i(t),$$

where

$$A_i(t) = e^{tB_i} = \sum_{j=0}^{n_i} B_i^j \frac{t^j}{j!}.$$

The ith component in the decomposition (2.6) of the second term on the right in (2.7) is

$$\begin{aligned}(2.8)\qquad -\int_t^0 e^{(t-s)\alpha_i} e^s A_i(t-s)(P(s)F(s))^i\, ds \\ = \sum_{k=0}^{n_i}\sum_{j=0}^{n_i} t^k e^{\alpha_i t}\int_t^0 s^j e^{(1-\alpha_i)s} C_{k,j}(P(s)F(s))^i\, ds,\end{aligned}$$

where the $C_{k,j}$ are continuous operators on $\mathcal{V}^i$.

Since $s \mapsto P(s)F(s)$ belongs to $\mathcal{C}_r$, it follows that for each $v < r$ and each $m \in \Lambda$, there is a constant $M_{v,m}$ such that

$$(2.9)\qquad \|C_{k,j}(P(s)F(s))^i\|_m \le M_{v,m} e^{vs}$$

for all $s < 0$. Hence, (2.8) is bounded in $\|\cdot\|_m$ by

$$(2.10)\qquad C(|t|^N + 1)(e^{(v+1)t} + e^{t\,\mathrm{Re}\,\alpha_i}),$$

where C and N are positive constants. Moreover, it is easy to see that the $\|\cdot\|_{k,n,m}$ norm of (2.8) is also finite. Therefore, the left side of (2.8) belongs to $\mathcal{C}_{r+1}$ if $\mathrm{Re}\,\alpha_i \ge r+1$.

On the other hand, if $\mathrm{Re}\,\alpha_i < r+1$, then we may express the left side of (2.8) as

$$e^{\alpha_i t} H_i(t) + \int_{-\infty}^t e^{(t-s)\alpha_i} e^s A_i(t-s)(P(s)F(s))^i\, ds,$$

where

$$H_i(t) = -\int_{-\infty}^0 e^{s(-\alpha_i+1)} A_i(t-s)(P(s)F(s))^i\, ds.$$

(Note that the integrals converge in the topology of $\mathcal{V}$ since we may choose $\mathrm{Re}\,\alpha_i - 1 < v$ in (2.9).) The H_i term is an exponential polynomial which becomes part of S_1. It follows from (2.9) that the other term belongs to $\mathcal{C}_{r+1}$. Hence there does indeed exist an exponential polynomial $S_1(t)$ with exponents from I such that $F(t) - S_1(t) \in \mathcal{C}_{r+1}$.

Next suppose by induction that we have proved the existence of $S_n(t)$ for some n. We provisionally define

$$S_{n+1}(t) = e^{tQ_0}F(0) - \int_t^0 e^{(t-s)Q_0} e^s P(s) S_n(s)\, ds. \tag{2.11}$$

LEMMA 2.6. *Each S_n is an exponential polynomial with exponents from I.*

Proof. If p is a polynomial, $k, \alpha \in \mathbb{R}$, then there is a polynomial q such that

$$\int_t^0 e^{(t-s)\alpha} e^{ks} p(s)\, ds = q(0)e^{\alpha t} - q(t)e^{kt}.$$

Hence, the answer is an exponential polynomial with exponents from $\{k, \alpha\}$. Our lemma follows by composing (2.9) with the projection to $\mathcal{V}^i$ and induction. ∎

Now we proceed with the theorem. Note that $F - S_{n+1} = R_{n+1}$, where

$$R_{n+1}(t) = -\int_t^0 e^{(t-s)Q_0} e^s P(s) R_n(s)\, ds.$$

An argument virtually identical to that done above shows that if $\operatorname{Re}\alpha_i \geq r+n+1$, then $R^i_{n+1} \in \mathcal{C}_{r+n+1}$. On the other hand, if $\operatorname{Re}\alpha_i < r+n+1$, then

$$R^i_{n+1}(t) = e^{\alpha_i t} H_i(t) + \int_{-\infty}^{t} e^{(t-s)\alpha_i} e^s A_i(t-s)(P(s)R_n(s))^i\, ds,$$

where

$$H_i(t) = -\int_{-\infty}^{0} e^{s(-\alpha_i+1)} A_i(t-s)(P(s)R_n(s))^i\, ds.$$

The $H_i(t)$ terms become part of S_{n+1}. It follows easily by induction that all the terms in $S_n(t)$ have their exponents from I. This finishes the proof of Theorem 2.5. ∎

The following is a consequence of Proposition 2.4 and Theorem 2.5.

COROLLARY 2.7. *Let $F \in \mathcal{C}_r$ satisfy* (2.4). *Then $F \in \mathcal{C}_{r_0}$ where r_0 is the minimum of $\{\operatorname{Re}\alpha_i\}$. Furthermore, for all continuous seminorms ϱ_0 on $\mathcal{C}_{r_0}$ there is a continuous seminorm ϱ on $\mathcal{C}_r$, independent of F, such that*

$$\varrho_0(F) \leq \varrho(F).$$

Proof. That $F \in \mathcal{C}_{r_0}$ is automatic from Proposition 2.4. The statement about seminorms follows easily from the inductive procedure used in constructing the asymptotic expansions in Theorem 2.5. ∎

The following result is immediately implied by Corollary 2.7 and the non-inductive part of the proof of Theorem 2.5.

PROPOSITION 2.8. *Suppose that* $\operatorname{Re}\alpha_i < r_0 + 1$ *where* r_0 *is as in Corollary* 2.7. *Then*

$$F \sim_{r_0+1} e^{tQ_0} F(0) - \sum_{i=1}^{d} \int_{-\infty}^{0} p_i(s) e^{s\beta_i} e^{(t-s)Q_0} Q_i F(s)\, ds. \tag{2.12}$$

In particular, *the lowest degree portion of* F*'s asymptotic expansion is obtained by expanding* (2.12) *in the decomposition* (2.6).

The following proposition allows us to differentiate asymptotic expansions term-by-term.

PROPOSITION 2.9. *Suppose that* $F \in \mathcal{C}_r$ *satisfies* (2.4). *Let the resulting asymptotic expansion be as in* (2.2). *Then* $F^{(n)} \in \mathcal{C}_r$ *for all* n *and*

$$F^{(n)}(t) \sim \sum_{\alpha \in I} e^{\alpha t} F_\alpha^n(t), \tag{2.13}$$

where

$$F_\alpha^n(t) = e^{-\alpha t} \frac{d^n}{dt^n} (e^{\alpha t} F_\alpha)(t).$$

Proof. Let $\widetilde{\mathcal{V}}_r$ be the space of all elements $F \in \mathcal{C}_r$ for which $F^{(n)} \in \mathcal{C}_r$ for all $n \in \mathbb{N}$, topologized via the seminorms

$$F \mapsto \|F^{(n)}\|_{s,m},$$

where $m \in \Lambda$, $n \in \mathbb{N}_0$ and $s < r$. It is easily seen that $\widetilde{\mathcal{V}}_r$ is a locally convex and complete topological linear space.

Now, let $F \in \mathcal{C}_r$ satisfy (2.4). Pointwise multiplication by the Q_i and by $e^{\beta_i t}$ define continuous mappings of $\mathcal{C}_r$ into itself. Hence, from (2.4), $F' \in \mathcal{C}_r$. It then follows by differentiation of (2.4) and induction that $F^{(n)} \in \mathcal{C}_r$ for all n. Hence, $F \in \widetilde{\mathcal{V}}_r$.

For $F \in \widetilde{\mathcal{V}}_r$, let $M(F)$ be the mapping of $(-\infty, 0]$ into $\widetilde{\mathcal{V}}_r$ defined by

$$M(F)(t) : s \mapsto F(t+s)$$

for $s \in (-\infty, 0]$. It is easily seen that in fact $M(F) \in \mathcal{C}_r(\widetilde{\mathcal{V}}_r)$. Furthermore, if F satisfies (2.4), then

$$M(F)'(t) = Q_0 M(F)(t) + \sum_{i=1}^{d} \sum_{j=1}^{n_i} e^{\beta_i t} \widetilde{p}_j(t) \widetilde{Q}_{i,j} M(F)(t),$$

where

$$p_i(s+t) = \sum_{j=1}^{n_i} \widetilde{p}_j(t) \widetilde{q}_j(s) \quad \text{and} \quad \widetilde{Q}_{i,j} = e^{\beta_i s} \widetilde{q}_j(s) Q_i.$$

It follows from Theorem 2.5 that $M(F)$ has an asymptotic expansion as a $\widetilde{\mathcal{V}}_r$-valued map. It is easily seen that if F's asymptotic expansion is as in (2.2), then

$$M(F)(t) \sim \sum_{\alpha\in I} M(e^{\alpha\cdot}F_\alpha)(t) = \sum_{\alpha\in I} e^{\alpha t}e^{\alpha\cdot}F_\alpha(t+\cdot).$$

Since d/ds is continuous on $\widetilde{\mathcal{V}}_r$, we have

$$\frac{d^n}{ds^n}M(F)(t) \sim \sum_{\alpha\in I} e^{\alpha t}\frac{d^n}{ds^n}(e^{\alpha s}M(F_\alpha)(t)).$$

Our result follows by letting $t=0$ in the above formula. ∎

It follows from Proposition 2.9 and Lemma 2.1 that we may formally substitute F's asymptotic expansion (2.2) into (2.4) and equate coefficients of $e^{\alpha t}$ for $\alpha \in I$. We find that for $\alpha\in I$,

$$F'_\alpha(t) + \alpha F_\alpha(t) = Q_0F_\alpha(t) + \sum_{i=1}^{d} \sum_{\substack{\beta\in I\\ \beta+\beta_i=\alpha}} p_i(t)Q_iF_\beta(t). \tag{2.14}$$

We put a partial ordering on I by saying that $\gamma \succeq \alpha$ if $\gamma-\alpha\in I_0$.

Let F satisfy (2.4) and let its asymptotic expansion be denoted as in (2.1). We say that $F_\alpha(t)$ is a *leading term* and α a *leading exponent* if α is minimal in I with respect to the property that $F_\alpha(t)\neq 0$.

Proposition 2.10. *The leading exponents are all eigenvalues for Q_0. If α_i is a leading exponent, then the degree of $F_{\alpha_i}(t)$ is at most n_i-1, where n_i is defined in* (b) *after formula* (2.6), *and F_{α_i} has values in $\mathcal{V}^i$. In particular, F_{α_i} is constant if $n_i=1$.*

Proof. Let α be a leading exponent. Then (2.14) implies that

$$F'_\alpha(t) = (Q_0-\alpha I)F_\alpha(t). \tag{2.15}$$

Thus

$$0 = F_\alpha^{(n)}(t) = (Q_0-\alpha I)^nF_\alpha(t).$$

Our proposition follows. ∎

Corollary 2.11. *Assume that there is an eigenvalue α_i such that*

$$\operatorname{Re}\alpha_i < \operatorname{Re}\alpha_j$$

for $\alpha_j\neq\alpha_i$, $F_{\alpha_j}\neq 0$. Let F_{α_i} have degree n. Then

$$\lim_{t\to-\infty} e^{-\alpha_i t}t^{-n}F(t) = F^n_{\alpha_i},$$

where $F^n_{\alpha_i}$ is the coefficient of t^n in $F_{\alpha_i}(t)$.

We say that an eigenvalue α_i is *redundant* if there is an $\alpha_j\neq\alpha_i$ such that $\alpha_i-\alpha_j\in I_0$. We say that the set of eigenvalues is *non-redundant* if there are no redundant eigenvalues. This implies that there are no non-minimal exponents among the α_i.

PROPOSITION 2.12. *Suppose that the set of eigenvalues is non-redundant. Let* $F \in \mathcal{C}_r$ *satisfy* (2.4). *Then the asymptotic expansion is uniquely determined by the elements* $F_{\alpha_i}(0)$.

Proof. Suppose first that α is a minimal exponent. Then, from (2.15),

$$F_\alpha(t) = e^{(Q_0 - \alpha I)t} F_\alpha(0).$$

It follows that F_α is determined by $F_\alpha(0)$.

If there is an α such that $F_\alpha(t)$ is not determined, then there is a minimal such α. Formula (2.14) shows that

$$F'_\alpha(t)^i + (\alpha - \alpha_i) F_\alpha(t)^i = H(t),$$

where $H(t)$ is a known polynomial function and the superscript denotes the ith component in the decomposition (2.6). If $\alpha = \alpha_i$, then the non-redundancy assumption implies that α is a leading exponent; hence F_α is determined, contrary to the hypothesis. Hence $\alpha - \alpha_i \neq 0$. Our result follows from the observation that the operator $d/dt - (\alpha - \alpha_i)$ is injective on the space of polynomials. ∎

3. Explicit asymptotic expansions. We wish to apply the results from the previous section to the differential equation $\mathcal{L}^\gamma F = 0$ from the Introduction. Let $\mathcal{V}$ be the space of distributions $\mathcal{D}'(N)$ given its usual (strong) topology. We identify elements $F \in C^\infty(G)$ with the C^∞ mapping $\widetilde{F} : \mathbb{R} \to \mathcal{V}$ defined by

$$\langle \varphi, \widetilde{F}(t) \rangle = \int_N \varphi(x) F(x, e^t)\, dn.$$

In this notation, $\mathcal{L}^\gamma$ is representable as a $\mathcal{V}$-valued ordinary differential operator of the form

$$\mathcal{L}^\gamma = \frac{d^2}{dt^2} - \gamma \frac{d}{dt} + \sum_{i=1}^{d} e^{\beta_i t} p_i(t) V_i,$$

where the V_i are first or second order elements in the enveloping algebra $\mathfrak{A}(\mathcal{N})$ of $\mathcal{N}$, the p_i are polynomials on $\mathbb{R}$, and the β_i are either eigenvalues of $\operatorname{ad}(H)$ (if V_i is first order) or a sum of two eigenvalues (if V_i is second order). (The V_i only differentiate the N variables.) Notice that from our normalizations, $\operatorname{Re} \beta_i \geq 3$. We assume that our ordering is such that (2.5) holds.

LEMMA 3.1. *Suppose that* $\widetilde{F}$ *satisfies* $\mathcal{L}^\gamma \widetilde{F} = 0$ *and belongs to* $\mathcal{C}_r(\mathcal{V})$ *for some* r. *Then* $\widetilde{F}' \in \mathcal{C}_s(\mathcal{V})$ *for some* s.

Proof. Let
$$H(t) = e^{-\gamma t}\widetilde{F}'(t).$$
Then
$$H'(t) = e^{-\gamma t}(\widetilde{F}''(t) - \gamma\widetilde{F}'(t)) = -e^{-\gamma t}\sum_{i=1}^{d} e^{\beta_i t}p_i(t)V_i\widetilde{F}.$$
Hence,
$$H(t) = H(0) - \sum_{i=1}^{d}\int_0^t e^{(\beta_i-\gamma)s}p_i(s)V_i\widetilde{F}(s)\,ds.$$
If $\|\cdot\|_m$ is any continuous seminorm on $\mathcal{V}$, and $v < r$, then
$$\|p_i(t)\widetilde{F}(t)\|_m < C_m e^{vt}.$$
Our lemma follows easily from the continuity of the V_i on $\mathcal{V}$. ∎

We next transform the equation $\mathcal{L}^\gamma\widetilde{F} = 0$ into the $\mathcal{V}\times\mathcal{V}$-valued first order equation

$$\frac{dY}{dt} = Q_0 Y + \sum_{i=1}^{d} p_i(t)e^{\beta_i t}Q_i Y \tag{3.1}$$

where
$$Y = \begin{bmatrix}\widetilde{F}\\ \widetilde{F}'\end{bmatrix}, \qquad Q_0 = \begin{bmatrix}0 & 1\\ 0 & \gamma\end{bmatrix}, \qquad Q_i = \begin{bmatrix}0 & 0\\ -V_i & 0\end{bmatrix}.$$

If $\widetilde{F}$ belongs to $\mathcal{C}_r(\mathcal{V})$ for some r, then it follows from Lemma 3.1 that $Y \in \mathcal{C}_s(\mathcal{V}\times\mathcal{V})$ for some s and Theorem 2.5 proves the existence of an asymptotic expansion for Y. Projection onto the first component in $\mathcal{V}\times\mathcal{V}$ shows the existence of an asymptotic expansion for $\widetilde{F}$.

We note that the eigenvalues of Q_0 are γ and 0. For $0 < \gamma \le 1$, the eigenvalues are non-redundant since $\operatorname{Re}\beta_i \ge 3$. Hence, Proposition 2.12 yields the following important corollary.

Corollary 3.2. *Suppose that for some $1 \ge \gamma \ge 0$ we have $\mathcal{L}^\gamma\widetilde{F} = 0$, where $\widetilde{F} \in \mathcal{C}_r(\mathcal{V})$ for some r. Let*
$$I = \Big\{\sum_j \beta_j k_j \,\Big|\, k_j \in \mathbb{N}_0\Big\}.$$
Then $\widetilde{F}$ has an asymptotic expansion
$$\widetilde{F}(t) \sim \begin{cases}\sum_{\alpha\in I} e^{\alpha t}F_\alpha(t) + e^{\gamma t}\sum_{\alpha\in I} e^{\alpha t}G_\alpha(t), & 1 \ge \gamma > 0,\\ \sum_{\alpha\in I} e^{\alpha t}F_\alpha(t), & \gamma = 0,\end{cases}$$
where $F_\alpha(t)$ and $G_\alpha(t)$ are polynomials on $\mathbb{R}$ with coefficients from $\mathcal{V}$. For $\gamma > 0$, $F_0, G_0 \in \mathcal{V}$ while for $\gamma = 0$, $F_0 = F_0^0 + tF_0^1$ where $F_0^i \in \mathcal{V}$.

We note the following uniqueness result:

COROLLARY 3.3. *Suppose that $\widetilde{F}^1$ and $\widetilde{F}^2$ both satisfy the hypotheses of Corollary* 3.2. *For $i = 1, 2$, let $F^i_\alpha(t)$ and $G^i_\alpha(t)$ be the corresponding terms of the asymptotic expansions.* (*If $\gamma = 0$, then we set $G^i_\alpha(t) = 0$.*) *Then $\widetilde{F}^1 = \widetilde{F}^2$ if and only if $F^1_0(t) = F^2_0(t)$ and $G^1_0 = G^2_0$.*

Proof. Formula (2.13), with $n = 1$, shows that the leading terms of F determine those of Y where Y is as in 3.1. Thus, from Proposition 2.12, $\widetilde{F}^1$ and $\widetilde{F}^2$ have the same asymptotic expansion. Let

$$F^0(x, a) = \widetilde{F}^1(\ln a)(x) - \widetilde{F}^2(\ln a)(x).$$

Then F^0 vanishes to infinite order at $a = 0$ and satisfies $\mathcal{L}^\gamma F^0 = 0$. We define $F^0(a) = 0$ for $a < 0$.

We would like to apply Theorem 2 of [2] with $\mathcal{P} = \mathcal{L}^\gamma$, $m = k = 2$, $p = 0$, and $t = a$. Comparison with equation (1) in [2] shows that the hypotheses of [2] are met since $\operatorname{Re}\beta_i - 1 \geq 2$ implies that the functions $a_{p,\beta}$ from [2] will be at least C^2 in t. It follows, then, that F^0 is zero on a neighborhood of e in $N \times_{\mathrm{s}} A$. Since $\mathcal{L}^\gamma$ is analytic-hypoelliptic, it follows that F^0 is zero, proving our result. ∎

REMARK. Since the solution is uniquely determined by F_0 and G_0, there is an inverse mapping $(F_0, G_0) \mapsto F$. We can consider this transformation as an *abstract Poisson transformation*.

We may use formula (2.12) from Proposition 2.8 to compute the leading terms in the expansion of Y and therefore for $\widetilde{F}$.

PROPOSITION 3.4. *Let $\widetilde{F}$ satisfy the hypotheses of Corollary* 3.2. *Then for $1 \geq \gamma > 0$,*

$$F_0 = \widetilde{F}(0) - \gamma^{-1}\widetilde{F}'(0) - \gamma^{-1}\sum_{i=1}^{d}\int_{-\infty}^{0} p_i(s)e^{\beta_i s}V_i\widetilde{F}(s)\,ds,$$

$$G_0 = \gamma^{-1}\widetilde{F}'(0) + \gamma^{-1}\sum_{i=1}^{d}\int_{-\infty}^{0} p_i(s)e^{(\beta_i-\gamma)s}V_i\widetilde{F}(s)\,ds.$$

If $\gamma = 0$, we set $F_0(t) = F^0_0 + tF^1_0$ where $F^i_0 \in \mathcal{V}$, and we have

$$F^0_0 = \widetilde{F}(0) - \sum_{i=1}^{d}\int_{-\infty}^{0} sp_i(s)e^{\beta_i s}V_i\widetilde{F}(s)\,ds,$$

$$F^1_0 = \widetilde{F}'(0) + \sum_{i=1}^{d}\int_{-\infty}^{0} p_i(s)e^{\beta_i s}V_i\widetilde{F}(s)\,ds.$$

Proof. It is easily computed that for $\gamma \neq 0$,

$$e^{tQ_0} = \begin{bmatrix} 1 & (e^{t\gamma}-1)/\gamma \\ 0 & e^{t\gamma} \end{bmatrix} = \begin{bmatrix} 1 & -1/\gamma \\ 0 & 0 \end{bmatrix} + e^{t\gamma} \begin{bmatrix} 0 & 1/\gamma \\ 0 & 1 \end{bmatrix}.$$

Moreover, the first component of $e^{(t-s)Q_0} Q_i Y(s)$ is $-\gamma^{-1}(e^{\gamma(t-s)}-1) V_i \widetilde{F}(s)$. The first pair of formulas now follows directly from Proposition 2.8. For $\gamma = 0$,

$$e^{tQ_0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Now the first component of $e^{(t-s)Q_0} Q_i Y$ is $-(t-s) V_i \widetilde{F}(s)$. Our second pair of formulas then follows again from Proposition 2.8. ∎

4. Proofs of the main results. Notice that Theorem 1.1 has actually been proved in the previous section.

The first goal of this section is to prove Theorem 1.5. For $\gamma \geq 0$, let $\{\mu_t^\gamma\}_{t>0}$ denote the (unique) one-parameter strongly continuous semigroup of probability measures on G generated by $\mathcal{L}^\gamma$, i.e. for all $f \in C_c^\infty(G)$,

$$\frac{d}{dt}(f * \mu_t^\gamma) = \mathcal{L}^\gamma (f * \mu_t^\gamma), \qquad \lim_{t\to 0+} \|f * \mu_t^\gamma - f\|_\infty = 0.$$

The μ_t^γ have smooth densities, which we denote by h_t^γ. The following lemma is a simple consequence of the observation that

$$\mathcal{L}^\gamma f = a^{\gamma/2}(\mathcal{L}^0 - \gamma^2/4)(a^{-\gamma/2} f).$$

LEMMA 4.1. *For $\gamma \geq 0$, one has*

$$h_t^\gamma(x,a) = e^{-t\gamma^2/4} a^{\gamma/2} h_t^0(x,a).$$

We identify N with G/A. For probability measures μ and ν on G and N respectively, we define their convolution (which is a measure on N) by

$$(f, \mu * \nu) = \int f(g \cdot x)\, d\mu(g)\, d\nu(x),$$

where "$\cdot$" denotes the action of G on N given by $g \cdot x = (y,a) \cdot x := y x^a$. It is well known that if μ is a probability measure, then convolution with μ defines a continuous operator on $L^2(N)$ with the operator norm at most 1.

It follows from (1.4) that $P^\gamma \in L^2(N)$ for all $\gamma \in [0,1]$. Therefore $\mu_t^\gamma * P^\gamma \in L^2(N)$. It is known (see [3, 4]) that P^γ is uniquely determined by the stipulations that

(a) $P^\gamma(e) = 1$,
(b) $\check{\mu}_t^\gamma * P^\gamma = P^\gamma$.

Proof of Theorem 1.5. It follows from the second form of (1.2) that the function

$$F^\gamma(x,a) = a^{-Q} P^\gamma((x^{-1})^{a^{-1}}) \tag{4.1}$$

is $\mathcal{L}^\gamma$-harmonic. We require an estimate on the derivatives of F^γ. Let Y_i, $i = 1, \ldots, n$, be a basis for $\mathcal{N}$. For each multi-index $I = (i_0, \ldots, i_n)$ of length $n+1$ we define

$$Y^I = H^{i_0} Y_1^{i_1} \ldots Y_n^{i_n}.$$

We apply the Harnack inequality to F^γ concluding that for each compact set $K \subset G$, and for each multi-index I, there is a constant $C_{I,K}$ such that

$$\|Y^I F^\gamma\|_{K,\infty} \le C_{I,K} \|F^\gamma\|_{K,\infty}, \tag{4.2}$$

where $\|\cdot\|_{K,\infty}$ is the sup norm over K. As noted in Theorem III.2.4 of [11], $C_{I,K}$ may be chosen to be independent of γ for $0 \le \gamma \le 1$.

From (1.4), $\|F^\gamma\|_{K,\infty}$ is bounded independently of γ. It follows that for every sequence $\gamma_n \in (0,1]$ where $\lim_{n\to\infty} \gamma_n = 0$, there is a subsequence δ_k and an element $F_0 \in C^\infty(G)$ such that

$$F^{\delta_k} \to F_0$$

in the topology of $C^\infty(G)$. Restriction to N shows that

$$P^{\delta_k} \to P_0, \tag{4.3}$$

where $P_0 = F_0|_N$.

If we can show that $P_0 = P^0$, Theorem 1.5 will follow from the comments immediately preceding Theorem 9 in Chapter 2 (p. 74) of [8].

Clearly, $P_0(e) = 1$, so it suffices to show that condition (b) stated after Lemma 4.1 holds for P_0 with $\gamma = 0$. From (1.4), (4.3) holds in $L^2(N)$. Note that

$$\check{\mu}_t^{\delta_n} * P^{\delta_n} - \check{\mu}_t^0 * P_0 = (\check{\mu}_t^{\delta_n} * P^{\delta_n} - \check{\mu}_t^{\delta_n} * P_0) + (\check{\mu}_t^{\delta_n} * P_0 - \check{\mu}_t^0 * P_0).$$

As $n \to \infty$, the term in the first pair of parentheses goes to 0 in $L^2(N)$ because convolution with a probability measure defines an operator of operator norm at most 1, while Lemma 4.1 shows that the term in the second pair of parentheses goes to zero as well. Condition (b) follows from the observation that $\check{\mu}_t^\gamma * P^\gamma = P^\gamma$. ∎

Now, we would like to apply the results of the previous section to the harmonic function (4.1). As before, we set

$$\widetilde{F}^\gamma(t)(x) = F^\gamma(x, e^t).$$

It is clear from (1.4) that there is a constant C such that in $L^2(N)$,

$$\|P^\gamma\|_2 \le C$$

for $0 \leq \gamma \leq 1$. Hence, from formula (4.1), for $t \in (-\infty, 0]$,

$$\|\widetilde{F}^{\gamma}(t)\|_2 \leq Ce^{-Qt}, \tag{4.4}$$

showing that the hypotheses of Corollary 3.2 are met where $\mathcal{V} = \mathcal{D}'(N)$. We denote the resulting asymptotic expansion as in Corollary 3.2, except that, for $\gamma \neq 0$, we append γ as a superscript.

COROLLARY 4.2. *The following limits exist as limits in $\mathcal{V}$ and equal the stated quantity*:

$$\lim_{\gamma \to 0^+} (F_0^{\gamma} + G_0^{\gamma}) = F_0^0, \qquad \lim_{\gamma \to 0^+} \gamma G_0^{\gamma} = F_0^1.$$

Proof. Formally, our corollary is a simple consequence of Proposition 3.4 since from Theorem 1.5, $\widetilde{F}^{\gamma}(t) \to \widetilde{F}^0(t)$ as $\gamma \to 0$. We need only show that we can exchange the integrals and the limits.

Let ϱ be a continuous seminorm on $\mathcal{V}$. From (4.4), $\widetilde{F}^{\gamma} \in \mathcal{C}_{-Q}$. In fact, it follows from (4.2) with $Y^I = H^n$, $n \in \mathbb{N}_0$, that $\gamma \mapsto \widetilde{F}^{\gamma}$ is uniformly bounded as a map of $[0,1]$ into $\mathcal{C}_{-Q}$. Corollary 2.7 (with $r_0 = 0$) then shows that $\varrho(\widetilde{F}^{\gamma}(t))$ is uniformly bounded for $(\gamma, t) \in [0,1] \times (-\infty, 0]$. Hence, $\varrho(V_i\widetilde{F}^{\gamma}(s) - V_i\widetilde{F}^0(s))$ is uniformly bounded. The desired convergence follows from the dominated convergence theorem together with the observation that $\operatorname{Re}\beta_i > 1 \geq \gamma$. ∎

Notice that it follows from Corollary 4.2 that

$$0 = \lim_{\gamma \to 0} \gamma(F_0^{\gamma} + G_0^{\gamma}) = \lim_{\gamma \to 0} \gamma F_0^{\gamma} + \lim_{\gamma \to 0} \gamma G_0^{\gamma} = \lim_{\gamma \to 0} \gamma F_0^{\gamma} + F_0^1.$$

On the other hand, for $\gamma > 0$, Proposition 2.10 shows that F_0^{γ} is independent of t and

$$F_0^{\gamma} = \lim_{t \to -\infty} \widetilde{F}^{\gamma}(t).$$

Then (1.5) shows that

$$F_0^{\gamma} = \|P^{\gamma}\|_1 \delta_e. \tag{4.5}$$

COROLLARY 4.3. $F_0^1 = -\lim_{\gamma \to 0} \gamma \|P^{\gamma}\|_1 \delta_e$.

To prove Theorem 1.3 we will need the following simple lemma:

LEMMA 4.4. *For all $K \subset N$, K compact, there are positive constants C_K and C_K' such that*

$$C_K'(1 + |yx|) \leq 1 + |x| \leq C_K(1 + |yx|) \tag{4.6}$$

for all $x \in N$ and $y \in K$.

Proof. The first inequality is a direct consequence of the triangle inequality and is left to the reader. For the second, we note that from the triangle inequality, there is a constant $C \geq 1$ such that

$$|x| = |y^{-1}(yx)| \leq C(|y| + |yx|).$$

Hence,

$$|yx| \geq C^{-1}|x| - |y|.$$

Let $K \subset N$ be given and let m be the max of $|\cdot|$ on K. From compactness, we may find C_K such that (4.6) holds for $|x| \leq 2Cm$. For $|x| > 2Cm$,

$$1 + |yx| \geq 1 + \frac{|x|}{C} - m \geq 1 + \frac{|x|}{2C} > (2C)^{-1}(1 + |x|).$$

Our lemma follows. ∎

Proof of Theorem 1.3. Let $f \in L^1((1+|x|)^{-Q-\gamma}dx)$. It follows from Lemma 4.4, (1.4) and formula (1.2) that $P^\gamma(f)$ satisfies (1.1) with $r = Q$. Hence $P^\gamma(f)$ has an asymptotic expansion. It follows from Theorem 1.1 that the limits in the statement of Theorem 1.3 compute the lowest order terms in the asymptotic expansion. For $\gamma > 0$, the map $f \mapsto F_0$-term of $P^\gamma(f)$ is continuous from $L^1((1+|x|)^{-Q-\gamma}dx)$ into $\mathcal{D}'(N)$. Formula (4.5) and a density argument show that this mapping is just $\|P^\gamma\|_1 I$. Similar comments prove the $\gamma = 0$ part of Theorem 1.3. ∎

Proof of Theorem 1.4. The next lemma proves one implication of Theorem 1.4.

LEMMA 4.5. *Let $f \in L^\infty(N)$ be compactly supported. Then there is a compact set $K \subset N$ such that $P^0(f)$ is uniformly bounded on $G\backslash(K\times(0,1])$.*

Proof. Let the support of f be contained in $|y| \leq M$. From (1.2),

$$\begin{aligned}|F(xa)| &\leq \int_{|y|\leq M} |f(y)|P^0((x^{-1}y)^{a^{-1}})a^{-Q}\,dy \\ &\leq C \int_{|y|\leq M} |f(y)|(1 + a^{-1}|x^{-1}y|)^{-Q}a^{-Q}\,dy \\ &= C \int_{|y|\leq M} |f(y)|(a + |x^{-1}y|)^{-Q}\,dy.\end{aligned}$$

We note that

$$(a + |x^{-1}y|)^{-Q} \leq \min\{a^{-Q}, |x^{-1}y|^{-Q}\}.$$

The boundedness for large a is automatic. The boundedness for large x follows from Lemma 4.4. ∎

The other implication of Theorem 1.4 is proved as follows.

We see from Theorem 1.1 that F_0^1 is supported in K. Specifically, if $x \notin K$, then

$$F_0^1(x) = \lim_{a\to 0^+} \frac{F(x,a)}{\ln a} = 0$$

from our boundedness assumption. Let

$$H = -C_0^{-1}P^0(F_0^1),$$

where C_0 is as in Theorem 1.3. From Lemma 4.5, there is a compact set $K' \subset N$ such that $M = F - H$ is uniformly bounded on $G \setminus (K' \times (0,1])$.

On the other hand, it follows from Theorem 1.3 and Corollary 4.2 that $M_0^1 = 0$. Hence, from Corollary 2.11 with $\alpha_i = 0$ and $n = 0$, $\widetilde{M}(t)$ is bounded independently of t for $t \in (-\infty, 0]$ in the sense of distributions. Hence, for all $\varphi \in C_c^\infty(G)$, the function

$$M_\varphi(g) = \langle \varphi, M(\cdot g)\rangle$$

is a uniformly bounded $\mathcal{L}^0$-harmonic function on G, which, from the results of [3], must therefore be constant. This constant is 0 since the first boundary function of M is zero. ∎

Proof of Theorem 1.6. Let F^γ be as in (4.1). It follows from Theorem 1.1 and formula (4.5) that as a distribution

$$\lim_{a \to 0^+} a^{-\gamma}(F^\gamma(\cdot, a) - \|P^\gamma\|_1 \delta_e) = G_0^\gamma.$$

Hence the limit statement in Theorem 1.6 is valid in the sense of distributions on $N \setminus \mathcal{U}(\varepsilon)$, where for $\varepsilon > 0$,

$$\mathcal{U}(\varepsilon) = \{x \in N \mid |x| < \varepsilon\}.$$

It follows easily from Lemma 1.2 and (1.4) that the L^2 norms of $a^{-\gamma}F^\gamma(\cdot, a)$ on $N \setminus \mathcal{U}(\varepsilon)$ are bounded independently of a. The convergence of our limit in the sense of measures follows from the weak compactness of the unit ball in L^2, proving Theorem 1.6. ∎

In order to prove Theorem 1.7, we will need the following lemma.

LEMMA 4.6. *Let $f \in C_c^\infty(N)$ be supported in a compact set $K \subset N$. Then for every $x \in N$, there are constants $C_x > 0$ and $\alpha > 1$ such that for all $y \in N$,*

$$|f(xy) - f(x)| \le C_x |y|^\alpha.$$

Proof. Since f is compactly supported, it suffices to prove the lemma for $|y| < 1$. We identify N with $\mathcal{N}$ via the exponential mapping and let $\|\cdot\|$ be a norm on $\mathcal{N}$. Since f is compactly supported and differentiable, there is a constant C_x such that

$$|f(xy) - f(x)| \le C_x \|y\|$$

for all $|y| < 1$.

For $a \in \mathbb{R}^+$, $\mathrm{Ad}(a)$ is linear and its matrix elements are polynomials in $\ln a$ times a^{β_i}. Hence, for any norm, there are constants $C > 0$ and $1 < \alpha < \min\{\mathrm{Re}\,\beta_i\}$ such that

$$\|x^a\| \le C a^\alpha \|x\|$$

for all $a \in (0,1]$. For $y \in \mathcal{N}$, we may write $y = x_0^{|y|}$, where $|x_0| = 1$. It follows that for $|y| < 1$,

$$\|y\| \leq C_0 |y|^{\alpha},$$

proving the lemma. ∎

Proof of Theorem 1.7. We note that from Lemma 4.6, Lemma 1.2, (1.4), and the compact support of f, the following functions have L^1 norm bounded independently of a for $a \in (0,1]$:

$$y \mapsto (f(xy) - f(x)) a^{-Q-\gamma} P^{\gamma}(y^{a^{-1}}).$$

Furthermore, if $f(x\cdot)$ is supported in the compact set $K \subset N$, then

$$\begin{aligned} a^{-\gamma}(P^{\gamma}(f)(x,a) - \|P^{\gamma}\|_1 f(x)) = & \int_K (f(xy) - f(x)) a^{-Q-\gamma} P^{\gamma}(y^{a^{-1}})\, dy \\ & - \int_{N\setminus K} f(x) a^{-Q-\gamma} P^{\gamma}(y^{a^{-1}})\, dy. \end{aligned}$$

Let ϱ_γ be the almost homogeneous norm defined in (1.6). From the dominated convergence theorem and Theorem 1.6 the above expression converges to

$$\begin{aligned} \int_K (f(xy) - f(x)) \varrho_\gamma(y)^{-Q-\gamma}\, dy - \int_{N\setminus K} f(x) a^{-Q-\gamma} \varrho_\gamma(y)^{-Q-\gamma}\, dy \\ = \int_N (f(xy) - f(x)) \varrho_\gamma(y)^{-Q-\gamma}\, dy. \end{aligned}$$

proving our theorem. ∎

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