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CONVERGENCE OF SEQUENCES OF ITERATES OF RANDOM-VALUED VECTOR FUNCTIONS

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Abstract. Given a probability space $(\Omega, \mathcal{A}, P)$ and a closed subset X of a Banach lattice, we consider functions $f : X \times \Omega \to X$ and their iterates $f^n : X \times \Omega^{\mathbb{N}} \to X$ defined by $f^1(x, \omega) = f(x, \omega_1)$, $f^{n+1}(x, \omega) = f(f^n(x, \omega), \omega_{n+1})$, and obtain theorems on the convergence (a.s. and in L^1) of the sequence $(f^n(x, \cdot))$.

It is well known that iteration processes play an important role in mathematics and they are especially important in solving equations. However, it may happen that instead of the exact value of a function at a point we know only some parameters of this value. In [1] iterates of such functions were defined and simple results on the behaviour of the iterates were obtained for scalar-valued functions. It is the aim of the present paper to consider such functions with values in Banach lattices. The basic theorem on the convergence of iterates is obtained in [1] (see also [10; Chapter 12]) by using a submartingale convergence theorem. It is well known (see e.g. [5]) that for martingales with values in a Banach space the convergence theorem holds only if the space has the Radon–Nikodym property. Hence beside a direct use of submartingale convergence theorems we also apply some other martingale methods to get the convergence of the sequence of iterates for an arbitrary AL-space. Basic notions and facts connected with lattices and used in this paper may be found in [4] and [14].

Fix a probability space $(\Omega, \mathcal{A}, P)$, a separable Banach lattice E and its closed subset X. Let $\mathcal{B}$ denote the σ-algebra of all Borel subsets of X. We say that $f : X \times \Omega \to X$ is a *random-valued vector function* if it is measurable with respect to the product σ-algebra $\mathcal{B} \otimes \mathcal{A}$. The iterates of f are defined by

$$f^1(x, \omega_1, \omega_2, \ldots) = f(x, \omega_1),$$
$$f^{n+1}(x, \omega_1, \omega_2, \ldots) = f(f^n(x, \omega_1, \omega_2, \ldots), \omega_{n+1}),$$

for $x \in X$ and $(\omega_1, \omega_2, \ldots) \in \Omega^\infty := \Omega^{\mathbb{N}}$. Note that $f^n : X \times \Omega^\infty \to X$ is

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a random-valued function on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$. More exactly, the nth iterate f^n is $\mathcal{B}\otimes\mathcal{A}_n$-measurable, where $\mathcal{A}_n$ denotes the σ-algebra of all sets of the form

$$\{(\omega_1,\omega_2,\ldots)\in\Omega^\infty : (\omega_1,\omega_2,\ldots,\omega_n)\in A\}$$

with A in the product σ-algebra $\mathcal{A}^n$.

In what follows, $f: X\times\Omega\to X$ is a fixed random-valued function such that

$$E\|f^n(x,\cdot)\|<\infty \quad \text{for } x\in X \text{ and } n\in\mathbb{N}. \tag{1}$$

We also assume that the mean $m: X\to E$ defined by

$$m(x)=Ef(x,\cdot)$$

is continuous. Moreover we assume that $x_0\in X$ is fixed and the sequence $(f^n(x_0,\cdot))$ is L^1-bounded. Concerning this assumption consult the Remark, Proposition 1, and Example below. It is easy to check that then

$$E(f^{n+1}(x,\cdot)\,|\,\mathcal{A}_n)=m\circ f^n(x,\cdot) \tag{2}$$

for $x\in X$ and $n\in\mathbb{N}$.

Our first theorem shows that the limit of $(f^n(x_0,\cdot))$ is a fixed point of m.

Theorem 1. *Assume that E does not contain isomorphic copies of c_0 and either*

$$m(x)\ge x \quad \textit{for } x\in X \tag{3}$$

or

$$m(x)\le x \quad \textit{for } x\in X. \tag{4}$$

If the sequence $(f^n(x_0,\cdot))$ converges in measure to an integrable $\xi:\Omega^\infty\to E$, then $m\circ\xi=\xi$.

Proof. Applying Fatou's lemma to a subsequence of $(\|m(f^n(x_0,\omega))\|)$ we get integrability of $m\circ\xi$. Assume (3) and put $g=m\circ\xi-\xi$, $g_n=m\circ f^n(x_0,\cdot)-f^n(x_0,\cdot)$ and (pointwise) $h_n=\inf\{g_n,g\}$ for $n\in\mathbb{N}$. Then the sequence (h_n) converges to g in measure, $h_n\le g_n$ and $h_n\le g$ for $n\in\mathbb{N}$. Moreover, the sequence $\big(Ef^n(x_0,\cdot)\big)$ is bounded and (in view of (2) and (3)) increasing, whence, according to the theorem of Tzafriri ([18], see also [12; Theorem 1.c.4]), convergent. Consequently,

$$0\le Eg=\lim_{n\to\infty}Eh_n\le\lim_{n\to\infty}Eg_n=\lim_{n\to\infty}E\big(f^{n+1}(x_0,\cdot)-f^n(x_0,\cdot)\big)=0. \ \blacksquare$$

In the next theorem, which is our main result, we assume additionally that the Banach lattice considered is an *AL-space*, i.e. $\|x+y\|=\|x\|+\|y\|$ for all $x,y\ge 0$ in E (cf. [14], [16]).

Theorem 2. *Let E be an AL-space. Assume that either* (3) *or* (4) *holds. If m is a contraction, then the sequence $\big(f^n(x_0,\cdot)\big)$ converges, both a.s. and in L^1, to the unique fixed point of m.*

Proof. Assume (3) and put $X_n = f^n(x_0, \cdot)$ for $n \in \mathbb{N}$. Since $(X_n, \mathcal{A}_n)$ is an L^1-bounded submartingale with values in an AL-space, we have

$$\sum_{n=1}^{N} E\|E(X_{n+1} \mid \mathcal{A}_n) - X_n\| = \Big\| \sum_{n=1}^{N} E(E(X_{n+1} \mid \mathcal{A}_n) - X_n) \Big\| = \|E(X_{N+1} - X_1)\| \leq 2 \sup_{n \in \mathbb{N}} E\|X_n\|$$

for every $N \in \mathbb{N}$. Hence

$$\sum_{n=1}^{\infty} E\|E(X_{n+1} \mid \mathcal{A}_n) - X_n\| \leq 2 \sup_{n \in \mathbb{N}} E\|X_n\| < \infty, \tag{5}$$

which jointly with (2) shows that

$$\lim_{n \to \infty} E\|m \circ X_n - X_n\| = 0. \tag{6}$$

On the other hand, if L denotes the Lipschitz constant of m, then

$$E\|X_p - X_q\| \leq \frac{1}{1-L} \left(E\|m \circ X_p - X_p\| + E\|m \circ X_q - X_q\| \right)$$

for all positive integers p, q. From this and (6) we infer that (X_n) converges in L^1 to a $\xi : \Omega^\infty \to E$. According to Theorem 1 (see also [14; Example 7, p. 92]) we have $m \circ \xi = \xi$. In particular, m has a fixed point, and being a contraction, it has at most one fixed point. Consequently, (X_n) converges in L^1 to the unique fixed point of m. Hence, applying (5) and [11; Theorem 1.3] (cf. also [2]), we obtain the a.s. convergence of (X_n) as well. ■

The following shows a possible realization of the assumptions of Theorems 1 and 2 in the simplest non-deterministic (vector) case, viz. $\Omega = \{\omega_1, \omega_2\}$.

Example. Let p_1, p_2 be positive reals with $p_1 + p_2 = 1$ and $h_1, h_2 : [0, \infty) \to [0, \infty)$ be continuous functions such that

$$p_1 h_1(t) + p_2 h_2(t) \leq t \quad \text{for every } t \geq 0.$$

Given a finite separable measure μ put $E = L^1(\mu)$, consider the subset X of E of all positive elements of E and define $f : X \times \{\omega_1, \omega_2\} \to X$ by

$$f(x, \omega_i) = h_i \circ x.$$

Then

$$m(x) = p_1 h_1 \circ x + p_2 h_2 \circ x \leq x \quad \text{and} \quad E\|f^n(x, \cdot)\| \leq \|x\|$$

for $x \in X$ and $n \in \mathbb{N}$. Moreover, m is continuous. Hence all the assumptions of Theorem 1 are satisfied. If additionally $p_1 h_1 + p_2 h_2$ is a contraction, then so is m (with zero as its only fixed point) and all the assumptions of Theorem 2 hold.

Of course, the convergence in L^1 implies the uniform integrability of the sequence. Concerning the uniform integrability of $\big(f^n(x_0,\cdot)\big)$ note the following simple fact.

PROPOSITION 1. *If there exists an integrable $\Phi:\Omega\to[0,\infty)$ such that*

$$\|f(x,\omega)\|\le\Phi(\omega)\qquad \textit{for } x\in X \textit{ and } \omega\in\Omega,$$

then the sequence $\big(f^n(x,\cdot)\big)$ is L^1-bounded and uniformly integrable for every $x\in X$.

Proof. Clearly $\|f^n(x,\omega)\|\le\Phi(\omega_n)$ for $x\in X$ and $\omega\in\Omega^\infty$. In particular $(f^n(x,\cdot))$ is L^1-bounded for $x\in X$. Moreover, if $x\in X$ and $n\in\mathbb{N}$ are fixed, then for every $A\in\mathcal{A}^\infty$ with $P^\infty(A)<N^{-1}\int_{\{\Phi>N\}}\Phi\,dP$ we have

$$\begin{aligned}\int_A\|f^n(x,\omega)\|\,dP^\infty(\omega)&\le\int_A\Phi(\omega_n)\,dP^\infty(\omega)\\&\le\int_{\{\omega\in\Omega^\infty\,:\,\Phi(\omega_n)>N\}}\Phi(\omega_n)\,dP^\infty(\omega)+NP^\infty(A)\\&\le 2\int_{\{\Phi>N\}}\Phi\,dP.\ \blacksquare\end{aligned}$$

In the case where the function f considered has the form

$$f(x,\omega)=x\Phi(\omega)\qquad\text{for } x\in X \text{ and } \omega\in\Omega, \tag{7}$$

we have the following observation.

PROPOSITION 2. *If f has the form* (7) *with $\Phi:\Omega\to\mathbb{R}$ integrable, $(f^n(x_0,\cdot))$ is uniformly integrable and $x_0\neq 0$, then either $E|\Phi|<1$ or $|\Phi|=1$ a.s.*

Proof. Clearly

$$f^n(x_0,\omega)=x_0\prod_{k=1}^n\Phi(\omega_k)$$

on Ω^∞, whence

$$E\|f^n(x_0,\cdot)\|=\|x_0\|(E|\Phi|)^n$$

for every $n\in\mathbb{N}$. Consequently, $E|\Phi|\le 1$. Assume $E|\Phi|=1$ and define a probability measure μ on $\mathcal{A}$ by

$$\mu(A)=\int_A|\Phi|\,dP$$

and a sequence (μ_n) of probability measures on $\mathcal{A}^\infty$ by

$$\mu_n(A)=\int_A\Big|\prod_{k=1}^n\Phi(\omega_k)\Big|\,dP^\infty(\omega).$$

If $N \in \mathbb{N}$, $A \in \mathcal{A}_N$ and $n \geq N$, then $\mu_n(A) = \mu^\infty(A)$. Hence the sequence (μ_n) is pointwise convergent on $\bigcup_{n=1}^\infty \mathcal{A}_n$ to μ^∞. Applying the uniform integrability of $(f^n(x_0,\cdot))$ we get

$$\lim_{P^\infty(A)\to 0} \sup_{n\in\mathbb{N}} \mu_n(A) = 0.$$

This allows us to check that the union of every increasing sequence of sets of the family

$$\{A \in \mathcal{A}^\infty : \lim_{n\to\infty} \mu_n(A) = \mu^\infty(A)\} \tag{8}$$

belongs to this family. According to the Dynkin lemma ([8], see also [3; Theorem 1.3.2]), the family (8) coincides with $\mathcal{A}^\infty$. In particular, μ^∞ is absolutely continuous with respect to P^∞. Hence, by the theorem of Kakutani [13; Proposition III.2.6], $E\sqrt{|\Phi|} \geq 1$. But $E\sqrt{|\Phi|} \leq \sqrt{E|\Phi|} \leq 1$, and so $E\sqrt{|\Phi|} = 1 = E|\Phi|$. Consequently, $|\Phi| = 1$ a.s. ∎

Now we proceed to the case where E has the Radon–Nikodym property. Since such a lattice does not contain isomorphic copies of c_0 (see [6]), our Theorem 1 and the theorem of Heinich [9] (cf. also [7] and [15]) imply what follows.

THEOREM 3. *Assume that E has the Radon–Nikodym property. If either f is lattice bounded from below and* (3) *holds, or f is lattice bounded from above and* (4) *holds, then the sequence $(f^n(x_0,\cdot))$ converges a.s. to an integrable $\xi : \Omega^\infty \to E$ and $m \circ \xi = \xi$.*

Note that [16; Proposition 3 and Theorem 1] and [14; Example 7, p. 92] imply the following.

REMARK. Assume that E is an AL-space and f satisfies (1). If either f is lattice bounded from above and (3) holds, or f is lattice bounded from below and (4) holds, then the sequence $(f^n(x_0,\cdot))$ is L_1-bounded for any $x_0 \in X$.

We finish with some special cases of E.

THEOREM 4. *Assume that $E = l_1$ or E is finite-dimensional. If* (3) *or* (4) *holds, then the sequence $(f^n(x_0,\cdot))$ converges a.s. to an integrable $\xi : \Omega^\infty \to E$ and $m \circ \xi = \xi$.*

Proof. Assume (2) and let

$$f^n(x_0,\cdot) = M_n + A_n, \quad n \in \mathbb{N},$$

be the Doob decomposition [17]. Since $\big(f^n(x_0,\cdot)\big)$ is L_1-bounded, it is easy to check that $\sup_{n\in\mathbb{N}} E\|M_n^-\| < \infty$. Applying the theorem of J. Szulga and W. A. Woyczyński [17; Theorem 4.1] we obtain the desired limit. ∎

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