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A CONVOLUTION PROPERTY OF THE CANTOR–LEBESGUE MEASURE, II

BY

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Abstract. For $1 \leq p, q \leq \infty$, we prove that the convolution operator generated by the Cantor–Lebesgue measure on the circle $\mathbb{T}$ is a contraction whenever it is bounded from $L^p(\mathbb{T})$ to $L^q(\mathbb{T})$. We also give a condition on p which is necessary if this operator maps $L^p(\mathbb{T})$ into $L^2(\mathbb{T})$.

Let $\mathbb{T}$ be the circle group $\mathbb{R}/\mathbb{Z}$ and, for $1 \leq p \leq \infty$, write L^p for the Lebesgue space formed using normalized Lebesgue measure on $\mathbb{T}$. Let λ be the usual Cantor–Lebesgue measure on $\mathbb{T}$. We are interested in determining the L^p-L^q mapping properties of the convolution operator defined by λ: we would like to know the indices $p, q \in [1, \infty]$ for which there is an inequality

$$\|\lambda * f\|_{L^q} \leq C(p,q)\|f\|_{L^p} \tag{1}$$

for $f \in L^p$. Since (1) is trivial if $q \leq p$, our interest is in the case $p < q$. The following results are in [O].

Lemma 1. *Suppose $1 \leq p < q \leq \infty$. If the inequality*

$$\left(\frac{1}{3}\left[\left(\frac{a+b}{2}\right)^q + \left(\frac{b+c}{2}\right)^q + \left(\frac{a+c}{2}\right)^q\right]\right)^{1/q} \leq \left(\frac{a^p+b^p+c^p}{3}\right)^{1/p} \tag{2}$$

holds for all $a, b, c \geq 0$, then (1) holds with $C(p,q) = 1$.

Lemma 2. *Inequality (2) holds for $q = 2$ when $p \geq 2/(1+3^{-1/2}) \approx 1.2679$.*

It follows from duality and interpolation that if $1 < p < \infty$ then there is q satisfying $p < q < \infty$ and such that (1) holds with $C(p,q) = 1$. Similar results for more general measures are in [BJJ] and [R], while [C] establishes the "L^p-improving" property for a larger class of singular measures using a quite different method.

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The known cases of (1) are all applications of Lemma 1 and so satisfy $C(p,q)=1$. The main result of this note is that if convolution with λ maps L^p into L^q, then it does so as a contraction:

THEOREM 3. *If* (1) *holds for* $p,q\in[1,\infty]$ *then* (1) *holds with* $C(p,q)=1$.

A more difficult and interesting problem is to determine exactly the indices for which (1) holds. Here we focus on the case $q=2$. In addition to the information above, there are the following results.

PROPOSITION 4 ([B], [BJJ]). *Inequality* (2) *holds for* $q=2$ *exactly when* $p\geq \log 4/\log 3\approx 1.2619$.

PROPOSITION 5. *If* (1) *holds then*

$$\frac{1}{p}+\left(1-\frac{\log 2}{\log 3}\right)\left(1-\frac{1}{q}\right)\leq 1.$$

Proposition 5 is checked by testing (1) on the indicator functions of small intervals. It shows in particular that if (1) holds with $q=2$ then $p\geq 2(1+\log 2/\log 3)^{-1}\approx 1.2263$, providing a necessary condition to pair with the sufficient condition provided by Lemma 1 and Proposition 4. The second result of this note narrows the gap between these two conditions.

PROPOSITION 6. *Suppose* (1) *holds with* $q=2$. *Then the following inequality holds whenever* $0<a<b<1$ *and* $2b<1+a$:

$$\left(\frac{2^a}{6a^a(b-a)^{2(b-a)}(1+a-2b)^{(1+a-2b)}}\right)^{1/2}\leq\left(\frac{2^b}{3b^b(1-b)^{(1-b)}}\right)^{1/p}. \tag{3}$$

Numerical calculations indicate that (3) fails when $p=1.244$, $b=.0770$, and $a=.0105$. This rules out the possibility that the condition provided by Proposition 5 is sufficient as well as necessary (but leaves open the interesting possibility that the sufficient condition supplied by Beckner's Proposition 4 is necessary). In the remainder of this note we give the proofs of Theorem 3 and Proposition 6.

Proof of Theorem 3. We will show that if (1) holds for $C(p,q)\in[1,\infty)$ then (1) also holds when $C(p,q)$ is replaced by $\sqrt{C(p,q)}$. It is convenient to replace λ with its translate by $1/2$. We will need the facts that then the Fourier transform of λ is given by

$$\widehat{\lambda}(n)=\prod_{j=0}^{\infty}\cos(2\pi 3^{-j}n)$$

and that when f is 1-periodic and continuous on $\mathbb{R}$ we have, for integral M,

$$\lim_{M\to\infty}\int_0^1 f(\theta)f(M\theta)\,d\theta=\Big(\int_0^1 f(\theta)\,d\theta\Big)^2. \tag{4}$$

Fix a trigonometric polynomial

$$t(\theta) = \sum_{n=-L}^{L} \widehat{t}(n) e^{2\pi i n\theta}. \tag{5}$$

Then, for positive integers N,

$$\lambda * t(\theta)\lambda * t(3^N\theta)$$
$$= \sum_{n_1,n_2} \prod_{j=0}^{\infty} (\cos(2\pi 3^{-j} n_1)\cos(2\pi 3^{-j} n_2))\widehat{t}(n_1)\widehat{t}(n_2)e^{2\pi i(n_1+3^N n_2)\theta}.$$

Also, $t(\theta)t(3^N\theta) = \sum_{n_1,n_2} \widehat{t}(n_1)\widehat{t}(n_2)e^{2\pi i(n_1+3^N n_2)\theta}$ and so

$$\lambda*(t(\cdot)t(3^N\cdot))(\theta) = \sum_{n_1,n_2} \prod_{j=0}^{\infty} \cos(2\pi 3^{-j}(n_1+3^N n_2))\widehat{t}(n_1)\widehat{t}(n_2)e^{2\pi i(n_1+3^N n_2)\theta}.$$

Now

$$\prod_{j=0}^{\infty} \cos(2\pi 3^{-j}(n_1 + 3^N n_2))$$
$$= \prod_{j=0}^{N} \cos(2\pi 3^{-j} n_1) \prod_{j=N+1}^{\infty} \cos(2\pi[3^{-j} n_1 + 3^{N-j} n_2])$$
$$= \prod_{j=0}^{N} \cos(2\pi 3^{-j} n_1)$$
$$\times \prod_{j=N+1}^{\infty} [\cos(2\pi 3^{-j} n_1)\cos(2\pi 3^{N-j} n_2) - \sin(2\pi 3^{-j} n_1)\sin(2\pi 3^{N-j} n_2)].$$

For $M \geq N+1$,

$$\prod_{j=N+1}^{M} [\cos(2\pi 3^{-j} n_1)\cos(2\pi 3^{N-j} n_2) - \sin(2\pi 3^{-j} n_1)\sin(2\pi 3^{N-j} n_2)]$$
$$= \prod_{j=N+1}^{M} \cos(2\pi 3^{-j} n_1)\cos(2\pi 3^{N-j} n_2) + e$$

where the error term $e = e(n_1, n_2, N, M)$ satisfies

$$|e| \leq \prod_{j=N+1}^{M} [1 + |\sin(2\pi 3^{-j} n_1)|] - 1 = O(3^{-N} L)$$

since $|n_1| \leq L$. Then

$$\Big|\prod_{j=0}^{\infty}\cos(2\pi 3^{-j}(n_1+3^N n_2)) - \prod_{j=0}^{\infty}(\cos(2\pi 3^{-j}n_1)\cos(2\pi 3^{-j}n_2))\Big| = O(3^{-N}L)$$

and it follows that

$$|\lambda * t(\theta)\lambda * t(3^N\theta) - \lambda * (t(\cdot)t(3^N\cdot))(\theta)| \leq C(t)\cdot 3^{-N},$$

where $C(t)$ is a positive constant depending on the trigonometric polynomial t.

Thus

$$\big|\|\lambda * t(\theta)\ \lambda * t(3^N\theta)\|_{L^q} - \|\lambda * \big(t(\cdot)t(3^N\cdot)\big)(\theta)\|_{L^q}\big| \to 0$$

as $N \to \infty$. Since

$$\|\lambda * t(\theta)\lambda * t(3^N\theta)\|_{L^q} \to \|\lambda * t\|_{L^q}^2$$

by (4), and also

$$\|\lambda * (t(\cdot)t(3^N\cdot))(\theta)\|_{L^q} \leq C(p,q)\|t(\theta)t(3^N\theta)\|_{L^p} \to C(p,q)\|t\|_{L^p}^2,$$

it follows that

$$\|\lambda * t\|_{L^q} \leq \sqrt{C(p,q)}\,\|t\|_{L^p}$$

as desired. Thus the proof of Theorem 3 is complete.

Proof of Proposition 6. If (1) holds it is easy to see that convolution with λ yields a bounded operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$. If $q = 2$ it follows that

$$\langle \lambda * \widetilde{\lambda}, \chi_E * \chi_{-E}\rangle \leq C(p)|E|^{2/p} \tag{6}$$

for Borel $E \subseteq \mathbb{R}$. To discretize (6) define

$$C_N = \Big\{\sum_{j=0}^{N-1}\varepsilon_j 3^j : j \in \{0,2\}\Big\}.$$

With "$*$" now representing the usual convolution on the group of integers and "$|\cdot|$" standing for cardinality, (6) implies that

$$\frac{1}{12^N}\langle \chi_{C_N} * \chi_{-C_N}, \chi_F * \chi_{-F}\rangle \leq C(p)\Big(\frac{|F|}{3^N}\Big)^{2/p} \tag{7}$$

whenever $F \subseteq \mathbb{Z}$. We will establish (3) by applying (7) to certain sets $F_{N,k}$.

Fix a positive integer N. For $J \subseteq \{0,1,\ldots,N-1\}$ put

$$F_J = \Big\{\sum_{j\in J}\varepsilon_j 3^j : j \in \{-2,2\}\Big\}$$

so that

$$\chi_{F_J} = \mathop{*}_{j\in J}(\delta_{-2\cdot 3^j} + \delta_{2\cdot 3^j}).$$

For $1 \le k \le N-1$ define

$$F_{N,k} = \bigcup \{F_J : J \subseteq \{0,1,\ldots,N-1\},\ |J| = k\}.$$

Note that if J_1 and J_2 are disjoint then

$$\chi_{F_{J_1}} * \chi_{F_{J_2}} = \chi_{F_{J_1 \cup J_2}},$$

and that, in general,

$$\chi_{F_{J_1}} * \chi_{F_{J_2}} = \mathop{*}_{j \in J_1 \cap J_2} (\delta_{-4\cdot 3^j} + 2\delta_0 + \delta_{4\cdot 3^j}) * \chi_{F_{(J_1 \cup J_2)\setminus(J_1 \cap J_2)}}.$$

It follows that

$$\chi_{F_{J_1}} * \chi_{F_{J_2}} \ge 2^{|J_1 \cap J_2|} \chi_{F_{(J_1 \cup J_2)\setminus(J_1 \cap J_2)}}.$$

Thus, since F_{J_1} and F_{J_2} are disjoint if $J_1 \ne J_2$,

$$\begin{aligned}\langle \chi_{C_N} * \chi_{-C_N}, \chi_{F_{N,k}} * \chi_{F_{N,k}} \rangle \\ &= \sum_{|J_1|=|J_2|=k} \langle \chi_{C_N} * \chi_{-C_N}, \chi_{F_{J_1}} * \chi_{F_{J_2}} \rangle \\ &\ge \sum_{|J_1|=|J_2|=k} 2^{|J_1 \cap J_2|} \langle \chi_{C_N} * \chi_{-C_N}, \chi_{F_{(J_1 \cup J_2)\setminus(J_1 \cap J_2)}} \rangle.\end{aligned}$$

Now

$$\langle \chi_{C_N} * \chi_{-C_N}, \chi_{F_J} \rangle = \sum_{f \in F_J} |f + C_N \cap C_N| = 2^{|J|} 2^{N-|J|} = 2^N$$

so

$$\begin{aligned}\langle \chi_{C_N} * \chi_{-C_N}, \chi_{F_{N,k}} * \chi_{F_{N,k}} \rangle &\ge 2^N \sum_{|J_1|=|J_2|=k} 2^{|J_1 \cap J_2|} \\ &= 2^N \binom{N}{k} \sum_{l=0}^{k} 2^l \binom{k}{l} \binom{N-k}{k-l}.\end{aligned}$$

Thus (7) implies that for $l = 0, \ldots, k$ there is the inequality

$$\frac{2^l}{6^N} \binom{N}{k} \binom{k}{l} \binom{N-k}{k-l} \le C(p) \left(\frac{\binom{N}{k} 2^k}{3^N} \right)^{2/p}. \tag{8}$$

By continuity, it is enough to establish (3) when a and b are rational. With such a and b fixed, N will now stand for a positive integer such that both aN and bN are integers. Take $k = bN$ and $l = aN$ in (8), estimate the binomial coefficients using Stirling's formula, take Nth roots of both sides of the resulting inequality, and then let $N \to \infty$. This gives

$$\frac{2^a}{6a^a (b-a)^{2(b-a)} (1+a-2b)^{(1+a-2b)}} \le \left(\frac{2^b}{3 \cdot b^b (1-b)^{(1-b)}} \right)^{2/p},$$

the conclusion of Proposition 6.

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