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AN EXISTENCE RESULT FOR BALANCE LAWS WITH MULTIFUNCTIONS: A MODEL FROM THE THEORY OF GRANULAR MEDIA

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Abstract. We study an example of the balance law with a multifunction source term, coming from the theory of granular media. We prove the existence of "weak entropy solutions" to this system, using the vanishing viscosity method and compensated compactness. Because of the occurrence of a multifunction we give a new definition of the weak entropy solutions.

1. Introduction. We consider an avalanche running down a slope. Mathematically, our problem can be written in the following form: find the height $h:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$, density $\varrho:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$, and velocity $v:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ of the avalanche satisfying the following system including a differential inclusion:

$$\text{(SH)}\qquad \begin{gathered}\frac{\partial}{\partial t}(\varrho h)+\frac{\partial}{\partial x}(\varrho h v)=0,\\ \frac{\partial}{\partial t}(\varrho h v)+\frac{\partial}{\partial x}\left(\varrho h v^2+\frac{1}{2}\beta\varrho h^2\right)\in \varrho h g,\end{gathered}$$

where $\beta:=\beta(x)$ is a given function and $g:=g(x,v)$ is a given multifunction. The first equation in (SH) describes the conservation of mass whereas the differential inclusion describes the balance of linear momentum.

A physical motivation for the use of a differential inclusion in (SH) instead of a differential equation can be found in [9]. In experiments one can observe a rich family of different static states. The main idea is to include related static solutions using differential inclusions. This phenomenon is typical only for granular matter.

Obviously, these two balance laws do not determine the evolution of the three variables (ϱ,h,v) uniquely. We need an additional constitutive relation. One possible approach is to assume that ϱ is a function of h and v. In this case we obtain a system of two differential inclusions for the two

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independent variables (h, v). In particular, constitutive relations of the form $\varrho = \varrho(h)$ were investigated by R. Balean [1], [2], and by the author [9] ([1]).

The paper has two basic purposes. First, we want to present the main ideas of the general theory. Second, we want to avoid technical problems as much as possible. For this reason we set the following constitutive relation: $\varrho = h^{-1/2}$, which turns out to be the simplest case in calculations. Particularly, we can avoid using difficult results of [13] and [12]. Moreover, we assume that the constant β and the multifunction $g(v)$ are defined by

$$\beta = k\cos(\gamma),$$
$$g(v) = \begin{cases} \sin(\gamma) + [-\cos(\gamma), +\cos(\gamma)] & \text{for } v = 0,\\ \sin(\gamma) - \frac{v}{|v|}\cos(\gamma) & \text{for } v \neq 0,\end{cases}$$

where $-\pi/2 < \gamma < \pi/2$ is the angle between the gravitational force and a constant slope ground and k is a positive constant. The general Savage–Hutter model for the one-dimensional snow avalanche [15] includes the case where $\gamma = \gamma(x)$, i.e. the case of a curved slope ground ([2]). Next, we introduce $H = \varrho h$. Our system (SH) takes the form

$$\frac{\partial}{\partial t}(H) + \frac{\partial}{\partial x}(Hv) = 0,$$
$$\frac{\partial}{\partial t}(Hv) + \frac{\partial}{\partial x}\left(Hv^2 + \frac{1}{2}\beta H^3\right) \in Hg(v).$$

It turns out that the above system for the evolution of (H, vH) is difficult to handle ([3]). In order to simplify our investigations we consider a new system whose classical solutions with $H > 0$ coincide with those of (SH). However, it is well known that under nonlinear transformations weak solutions can change considerably (for example the speed of shock waves can change: look at the Rankine–Hugoniot jump condition). The new system is

$$\frac{\partial}{\partial t}H + \frac{\partial}{\partial x}(Hv) = 0,$$
$$\frac{\partial}{\partial t}v + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2 + \frac{3\beta}{4}H^2\right) \in g(v).$$

Next, we introduce $u = (u_1, u_2) = \left(\frac{\sqrt{2}}{\sqrt{3\beta}}H, v\right)$, restating our system in the

([1]) More precisely, the constitutive relations in the cited references were $\varrho = \varrho_0$ and $\varrho = h^{\alpha}$, $\alpha \geq 0$, respectively.

([2]) See [1] and [9].

([3]) Actually, the problem is to obtain some additional technical estimates for the viscous regularized problem (see [9]).

following way:

$$\text{(CL)} \qquad \frac{\partial}{\partial t}u + \frac{\partial}{\partial x}F(u) \in \widetilde{G}(u),$$

where

$$F(u) = \begin{pmatrix} u_1u_2 \\ u_2^2/2 + u_1^2/2 \end{pmatrix}, \qquad \widetilde{G}(u) = \begin{pmatrix} 0 \\ g(u_2) \end{pmatrix}.$$

We will define a proper class of solutions that we are looking for. This class contains the "weak entropy solutions" defined below.

NOTATION. Let Ω be a subset of a Euclidean space, ω a positive function on Ω, and X a subset of a Banach space. We denote by $\mathcal{C}^0_\omega(\Omega; X)$ the set of continuous functions $u : \Omega \to X$ with $\|u\|_{\mathcal{C}^0_\omega} = \sup_{x\in\Omega} \|\omega(x)u(x)\| < \infty$. Similarly by $\mathcal{C}_{\mathrm{b}}$, $\mathcal{C}_0$, $\mathcal{C}_{\mathrm{c}}$ we denote respectively the sets of: continuous bounded functions, continuous functions vanishing at infinity and continuous functions with compact support. By $\mathcal{C}^r_{\mathrm{b}}(\Omega; X)$ we denote the Banach space of r-times differentiable functions with the standard norm. By $\mathcal{M}$ we denote the set of bounded Radon measures.

DEFINITION 1.1. Suppose that $\eta = \eta(u_1, u_2)$, $q = q(u_1, u_2)$ are scalar $\mathcal{C}^1$-functions satisfying

$$\nabla_{(u_1,u_2)}\eta(u_1, u_2) \cdot \nabla_{(u_1,u_2)}F(u_1, u_2) = \nabla_{(u_1,u_2)}q(u_1, u_2).$$

Such functions (η, q) are called *entropy-flux pairs*. If η is convex, then (η, q) is called a *convex entropy-flux pair.*

DEFINITION 1.2. We call $u \in \mathbb{L}^\infty([0, T) \times \mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ a *weak entropy solution* to the system (CL) with the initial data $u^0 \in \mathcal{C}^0_{\mathrm{b}}(\mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ if there exists $G \in \mathbb{L}^\infty([0, T) \times \mathbb{R}; \mathbb{R}^2)$ with the following properties:

(i) $G(t, x) \in \widetilde{G}(u(t, x))$ for a.a. $(t, x) \in [0, T) \times \mathbb{R}$.

(ii) u is a weak solution, i.e.

$$\int\limits_{[0,T)\times\mathbb{R}} \bigg[u(t, x) \cdot \frac{\partial}{\partial t}\psi(t, x) + F(u(t, x)) \cdot \frac{\partial}{\partial x}\psi(t, x) + G(t, x) \cdot \psi(t, x)\bigg]\, dt\, dx = -\int\limits_{\mathbb{R}} u^0(x) \cdot \psi(0, x)\, dx$$

for all test functions $\psi \in \mathcal{C}^1_{\mathrm{c}}([0, T) \times \mathbb{R}; \mathbb{R}^2)$.

(iii) The entropy inequality

$$\int\limits_{[0,T)\times\mathbb{R}} \bigg[\eta(u(t, x))\, \frac{\partial}{\partial t}\phi(t, x) + q(u(t, x))\, \frac{\partial}{\partial x}\phi(t, x) + \nabla_u\eta(u(t, x)) \cdot G(t, x)\phi(t, x)\bigg]\, dt\, dx \ge -\int\limits_{\mathbb{R}} \eta(u^0(x))\phi(0, x)\, dx$$

holds for all nonnegative test functions $\phi \in \mathcal{C}_{\mathrm{c}}^1([0,T)\times\mathbb{R};\mathbb{R})$ and all convex entropy-flux pairs (η, q).

REMARK. The above definition is nonstandard because of the differential inclusion that occurs in (i).

Let us present the main result:

THEOREM 1.1. *Assume that the initial data satisfies*

$$u^0 = (u_1^0, u_2^0) \in \mathcal{C}_{\mathrm{b}}^3(\mathbb{R};\mathbb{R}^2), \qquad \inf_{x\in\mathbb{R}} u_1^0(x) \ge 0, \qquad (u_1^0 - \overline{u}, u_2^0) \in \mathcal{C}_{\omega}^0(\mathbb{R};\mathbb{R}^2)$$

with $\omega(x) = 1 + |x|$ for some positive constant $\overline{u}$. Then the problem (CL) *has a weak entropy solution in the sense of Definition* 1.2 *for all positive T, with* $\inf_{x\in\mathbb{R}} u_1(t,x) \ge 0$ *for a.a. $t \in [0,T)$.*

REMARK. The strong assumption on the initial data appears only for technical reasons.

2. Existence of solutions and a priori estimates for the viscous perturbed problem. We define the viscous system by adding a second order term and replacing $\widetilde{G}$ by a smooth bounded function G_ε. More precisely, we consider the problem

$$\text{(VP)} \qquad \frac{\partial}{\partial t}u + \frac{\partial}{\partial x}F(u) = G_\varepsilon(u) + \varepsilon\,\frac{\partial^2}{\partial x^2}u,$$

with initial data

$$u^0 = (u_1^0, u_2^0).$$

Here $G_\varepsilon(u)$ is constructed with the help of Friedrichs mollifiers φ_ε ($\operatorname{supp}\varphi_\varepsilon = [-\varepsilon,\varepsilon]$) and given by the formula

$$[G_\varepsilon(u)]^T = \left(0, \sin(\gamma) - \frac{u_2}{|u_2|} * \varphi_\varepsilon(u_2)\cos(\gamma)\right).$$

The function $u_2/|u_2|$ is well defined for a.a. u_2 in $\mathbb{R}$.

THEOREM 2.1. *Assume that the initial data satisfies*

$$u^0 = (u_1^0, u_2^0) \in \mathcal{C}_{\mathrm{b}}^3(\mathbb{R};\mathbb{R}^2), \qquad (u_1^0 - \overline{u}, u_2^0) \in \mathcal{C}_{\omega}^0(\mathbb{R};\mathbb{R}^2),$$

where $\omega(x) = 1+|x|$, for some positive constant $\overline{u}$. Then the problem (VP) *has a classical, global-in-time solution, i.e. $u \in \mathcal{C}^0([0,T);\mathcal{C}_{\mathrm{b}}^2(\mathbb{R};\mathbb{R}^2))$, $\frac{\partial}{\partial t}u \in \mathcal{C}_{\mathrm{b}}^0([0,T)\times\mathbb{R};\mathbb{R}^2)$, where T is arbitrary. Moreover*:

(i) *if* $\inf_{x\in\mathbb{R}} u_1^0(x) \ge 0$, *then* $\inf_{x\in\mathbb{R}} u_1(t,x) \ge 0$;
(ii) $\|u(t)\|_{\mathbb{L}^\infty(\mathbb{R})} \le (|\sin(\gamma)| + |\cos(\gamma)|)t + \sqrt{2}\,\|u^0\|_{\mathbb{L}^\infty(\mathbb{R})}$ *for all $t \in [0,T)$.*

Proof. STEP 1 (proof of local-in-time existence). Using the Gaussian kernel

$$\Theta(t,x) = \frac{1}{\sqrt{4\pi\varepsilon t}}\exp\left(\frac{-x^2}{4\varepsilon t}\right)$$

of the heat equation we restate (VP) in the integral form

$$u(t,x) = \int_{\mathbb{R}} \Theta(t,x-y)u^0(y)\,dy + \int_{\mathbb{R}}\int_0^t \frac{\partial}{\partial x}\Theta(t-\tau,x-y)F(u(\tau,y))\,d\tau\,dy + \int_{\mathbb{R}}\int_0^t \Theta(t-\tau,x-y)G_\varepsilon(u(\tau,y))\,d\tau\,dy. \tag{2.1}$$

Note that

$$\left|\frac{\partial}{\partial x}\Theta(t,x)\right| \le \frac{c}{\varepsilon t}\exp\left(\frac{-x^2}{2\varepsilon t}\right),$$

$$\int_{\mathbb{R}}\int_0^t \left|\frac{\partial}{\partial x}\Theta(t-\tau,x-y)\right| d\tau\,dy \le c\sqrt{\frac{t}{\varepsilon}}.$$

Define the bounded set

$$\mathcal{U} = \operatorname{conv}\{u^0(x) \mid x \in \mathbb{R}\},$$

and the compact set

$$\mathcal{V} = \{y \in \mathbb{R}^2 \mid \operatorname{dist}(y,\mathcal{U}) \le 1\}.$$

Note that G_ε is globally Lipschitz, with Lipschitz constant depending on ε, but F is only locally Lipschitz.

To solve (2.1) we define the sequence $u^{(1)}, u^{(2)}, \dots$ by

$$\begin{aligned} u^{(1)}(t,x) &= \int_{\mathbb{R}} \Theta(t,x-y)u^0(y)\,dy, \\ u^{(n+1)}(t,x) &= \int_{\mathbb{R}} \Theta(t,x-y)u^0(y)\,dy \\ &\quad + \int_{\mathbb{R}}\int_0^t \frac{\partial}{\partial x}\Theta(t-\tau,x-y)F(u^{(n)}(\tau,y))\,d\tau\,dy \\ &\quad + \int_{\mathbb{R}}\int_0^t \Theta(t-\tau,x-y)G_\varepsilon(u^{(n)}(\tau,y))\,d\tau\,dy. \end{aligned}$$

Obviously $u^{(1)}(t,x) \in \mathcal{U}$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. It is also easy to observe that there exists a sufficiently small time bound T_{loc}, which depends only on the $\mathbb{L}^\infty$-norm u^0, such that

$$\sum_{n=1}^{\infty} \|u^{(n+1)} - u^{(n)}\|_{\mathcal{C}^0([0,T_{\mathrm{loc}})\times\mathbb{R})} \le 1.$$

Then $\lim_{n\to\infty} u^{(n)}(t,x) = u(t,x)$ for all $(t,x) \in [0,T_{\mathrm{loc}}) \times \mathbb{R}$, where $u(t,x)$ is a solution to (2.1), and thus a "mild solution" to (VP). Moreover $u(t,x) \in \mathcal{V}$ for all $(t,x) \in [0,T_{\mathrm{loc}}) \times \mathbb{R}$.

Note that $F, G_\varepsilon \in \mathcal{C}^3_{\mathrm{b}}(\mathcal{V};\mathbb{R}^2)$. Then regularity of the solution (i.e. $u \in \mathcal{C}^0((0,T_{\mathrm{loc}});\mathcal{C}^2_{\mathrm{b}}(\mathbb{R};\mathbb{R}^2)))$ will be obtained by examining difference quotients of first and second order; see also [11]).

Step 2 ($\mathbb{L}^\infty$-bound of solutions). Let us observe that system (VP) is equivalent to the system of two equations

$$\begin{aligned}\frac{\partial}{\partial t}(u_1-u_2)-\frac{1}{2}\frac{\partial}{\partial x}(u_1-u_2)^2&=\varepsilon\frac{\partial^2}{\partial x^2}(u_1-u_2)-g_\varepsilon(u_2),\\ \frac{\partial}{\partial t}(u_1+u_2)+\frac{1}{2}\frac{\partial}{\partial x}(u_1+u_2)^2&=\varepsilon\frac{\partial^2}{\partial x^2}(u_1+u_2)+g_\varepsilon(u_2),\end{aligned}$$

where $g_\varepsilon(u_2)=\sin(\gamma)-\frac{u_2}{|u_2|}*\varphi_\varepsilon(u_2)\cos(\gamma)$. Note that $\|g_\varepsilon(u_2)\|_{\mathbb{L}^\infty}\le|\sin(\gamma)|+|\cos(\gamma)|$. From the maximum principle for a scalar parabolic equation we obtain

$$\|u(t)\|_{\mathbb{L}^\infty(\mathbb{R})}\le(|\sin(\gamma)|+|\cos(\gamma)|)t+\sqrt{2}\,\|u^0\|_{\mathbb{L}^\infty(\mathbb{R})}$$

for all $t\in[0,T_{\mathrm{loc}})$.

Step 3 (global-in-time existence). Direct application of results obtained in Steps 1 and 2 yields the global-in-time existence.

Step 4 (minimum estimate for u_1). Simple calculation as in Step 1 shows that $(u_1(t)-\overline{u},u_2(t))$ is bounded in the space $\mathcal{C}^0_\omega(\mathbb{R};\mathbb{R}^2)$ (with $\omega(x)=1+|x|$) for all $t\in[0,T_{\mathrm{loc}})$ (see the estimation of the x-derivative of the heat kernel).

Claim. *If $u_2\in\mathcal{C}^1_{\mathrm{b}}([0,T)\times\mathbb{R})$, $u_1\in\mathcal{C}^0([0,T);\mathcal{C}^2_{\mathrm{b}}(\mathbb{R}))$, $\frac{\partial}{\partial t}u_1\in\mathcal{C}^0_{\mathrm{b}}([0,T)\times\mathbb{R})$, $u_1^0\ge0$, $u_1(t)-\overline{u}$ is bounded in the space $\mathcal{C}^0_\omega(\mathbb{R})$ ($\omega(x)=1+|x|$) for all $t\in[0,T)$, and $u_1(t,x)$ is a classical solution to the equation*

$$\frac{\partial}{\partial t}u_1+\frac{\partial}{\partial x}(u_1u_2)=\varepsilon\frac{\partial^2}{\partial x^2}u_1$$

then $u_1(t,x)\ge0$.

Proof of Claim. Our proof follows in part the idea of [8, Proposition 4.1]. First, we multiply the equation by $\phi=\min\{0,u_1\}$ and integrate over $(-r,r)\times(0,t)$ to get

$$\begin{aligned}\int_{-r}^{r}\frac{1}{2}\phi^2(x,t)\,dx+\varepsilon\int_0^t\int_{-r}^{r}|\phi_x(x,s)|^2\,dx\,ds+\varepsilon\int_0^t[\phi_x(x,s)\phi(x,s)]_{-r}^{r}\,ds\\ =\int_0^t\int_{-r}^{r}u_2(x,s)\phi(x,s)\phi_x(x,s)\,dx\,ds+\int_0^t[u_2(x,s)\phi^2(x,s)]_{-r}^{r}\,ds,\end{aligned}$$

Observe that the third term of the left hand side and the second term of the right hand side tend to zero as r tends to infinity. Moreover

$$\int_0^t\int_{-r}^r u_2(x,s)\phi(x,s)\phi_x(x,s)\,dx\,ds \le \varepsilon\int_0^t\int_{-r}^r |\phi_x(x,s)|^2\,dx\,ds + \frac{\|u_2\|_\infty}{4\varepsilon}\int_0^t\int_{-r}^r \phi^2(x,s)\,dx\,ds.$$

Hence

$$\int_{-\infty}^{\infty} \phi^2(x,t)\,dx \le \frac{\|u_2\|_\infty}{4\varepsilon}\int_0^t\int_{-\infty}^{\infty}\phi^2(x,s)\,dx\,ds$$

and it follows from Gronwall's inequality that $\phi(x,t)=0$. ∎

In order to pass to the limit $\varepsilon\to 0^+$ we need the following estimate.

LEMMA 2.2. *Assume that $u(t,x)$ is a classical solution to the equation*

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}F(u) = G_\varepsilon(u) + \varepsilon\,\frac{\partial^2}{\partial x^2}u.$$

Moreover, assume that $u, F(u), G_\varepsilon(u) \in \mathbb{L}^\infty([0,T)\times\mathbb{R};\mathbb{R}^2)$. Then $\sqrt{\varepsilon}\frac{\partial}{\partial x}u$ is bounded independently of ε, $0<\varepsilon<1$, in the space $\mathbb{L}^2(\Omega)$ for any open bounded $\Omega\subset[0,T)\times\mathbb{R}$.

Proof. Let (η,q) be a convex entropy-flux pair with η strictly convex. Then we have $D^2\eta\big(\frac{\partial}{\partial x}u,\frac{\partial}{\partial x}u\big) > K\big|\frac{\partial}{\partial x}u\big|^2$ for some $K>0$. Multiplying (VP) by $\nabla\eta$ we obtain

$$\frac{\partial}{\partial t}\eta + \frac{\partial}{\partial x}q = \varepsilon\bigg(\frac{\partial^2}{\partial x^2}\eta - D^2\eta\bigg(\frac{\partial}{\partial x}u,\frac{\partial}{\partial x}u\bigg)\bigg) + \nabla\eta G_\varepsilon.$$

Next we multiply the above equation by a nonnegative test function $\theta\in C_0^\infty([0,T)\times\mathbb{R})$ with $\theta_{|\Omega}=1$, and integrate over $[0,T)\times\mathbb{R}$. This yields

$$\varepsilon K\int_{[0,T)\times\mathbb{R}}\theta\bigg|\frac{\partial}{\partial x}u\bigg|^2 dt\,dx \le \int_{\mathbb{R}}\theta\,\eta\,dx + \int_{[0,T)\times\mathbb{R}}\bigg\{\bigg|\frac{\partial^2}{\partial x^2}\theta\eta\bigg| + \frac{\partial}{\partial x}\theta q + \frac{\partial}{\partial t}\theta\eta + \theta\nabla\eta G_\varepsilon\bigg\}\,dt\,dx,$$

where the right hand side and consequently the left hand side are bounded uniformly w.r.t. ε. ∎

3. Existence of weak entropy solutions to the original problem. In this section we pass to the limit $\varepsilon\to0^+$ in the viscous perturbed problem (VP) and also in the regularization G_ε of our multifunction $\widetilde{G}$. First, we recall two basic theorems.

THEOREM (Convergence in the sense of Young measures). *Let $\{u_k\}$ be a sequence in $\mathbb{L}^\infty(\Omega;K)$, where $\Omega\subset[0,T)\times\mathbb{R}$ and K is a compact set in $\mathbb{R}^2$. Then there exists a subsequence $\{u_{k_j}\}\subset\{u_k\}$ and for a.a. $(t,x)\in\Omega$ a Borel probability measure $\mu_{(t,x)}$ on K such that for each $F\in\mathcal{C}(K;\mathbb{R}^2)$ we have*

$$\lim_{k_j\to\infty}\int_\Omega F(u_{k_j}(t,x))\varphi(t,x)\,dt\,dx=\int_\Omega \overline{F(t,x)}\varphi(t,x)\,dt\,dx \qquad \forall\varphi\in\mathbb{L}^1(\Omega;\mathbb{R}^2),$$

where

$$\overline{F(t,x)}=\int_K F(y)\,d\mu_{(t,x)}(y) \quad \textit{for a.a. } (t,x)\in\Omega.$$

For a proof see [7] or [3].

THEOREM (Div-Curl lemma). *Let (η_k^1,q_k^1) and $(-q_k^2,\eta_k^2)$ be bounded sequences in $\mathbb{L}^2(\Omega;\mathbb{R}^2)$ such that $\frac{\partial}{\partial t}\eta_k^i+\frac{\partial}{\partial x}q_k^i$ (for $i=1,2$) lies in a compact subset of $\mathbb{W}^{-1,2}(\Omega;\mathbb{R})$. Suppose further that $(\eta_k^1,q_k^1)\rightharpoonup(\eta^1,q^1)$ and $(-q_k^2,\eta_k^2)\rightharpoonup(-q^2,\eta^2)$ in $\mathbb{L}^2(\Omega;\mathbb{R}^2)$. Then*

$$\int_\Omega (q_k^2\eta_k^1-\eta_k^2q_k^1)\varphi\,dt\,dx\to\int_\Omega (q^2\eta^1-\eta^2q^1)\varphi\,dt\,dx \qquad \forall\varphi\in\mathcal{C}_c^\infty(\Omega;\mathbb{R}).$$

Proof. Observe that

$$\mathrm{curl}_{(t,x)}(-q_k^2,\eta_k^2)=\frac{\partial}{\partial t}\eta_k^2+\frac{\partial}{\partial x}q_k^2,\qquad \mathrm{div}_{(t,x)}(\eta_k^1,q_k^1)=\frac{\partial}{\partial t}\eta_k^1+\frac{\partial}{\partial x}q_k^1.$$

Thus the above statement follows from [7, Theorem 4, p. 53] or [14, Lemma 3.3, p. 158].

Proof of the existence of a weak entropy solution. 1. First we write the viscous perturbed problem in the weak form

$$\int_{[0,T)\times\mathbb{R}}[u^\varepsilon\psi_t+F(u^\varepsilon)\psi_x+G_\varepsilon(u^\varepsilon)\psi]\,dt\,dx=\varepsilon\int_{[0,T)\times\mathbb{R}}u_x^\varepsilon\psi_x\,dt\,dx+\int_{\mathbb{R}}u_0\psi(0,x)\,dx,$$

for any test function $\psi\in\mathcal{C}_c^1([0,T)\times\mathbb{R};\mathbb{R}^2)$. The functions ψ_t,ψ_x,ψ are elements of the space $\mathcal{C}_c^0([0,T)\times\mathbb{R};\mathbb{R}^2)$. Hence, for the sequence of approximate problems (for example setting $\varepsilon=1/k$, $k\in\mathbb{N}$) we can pass to the limit in the sense of Young measures. Using the fact that $k^{-1}u_x\to0$ strongly in $\mathbb{L}^2_{\mathrm{loc}}([0,T)\times\mathbb{R};\mathbb{R}^2)$ we deduce that the first term on the right hand side tends to zero as $k\to\infty$. This yields

$$\int_{[0,T)\times\mathbb{R}}[\overline{u}(t,x)\psi_t(t,x)+\overline{F}(t,x)\psi_x(t,x)+\overline{G}(t,x)\psi(t,x)]\,dt\,dx=\int_{\mathbb{R}}u_0\psi(0,x)\,dx,$$

where

$$\overline{u}(t,x)=\int_K(y_1,y_2)\,d\mu_{(t,x)}(y_1,y_2),\qquad \overline{F}(t,x)=\int_K F(y_1,y_2)\,d\mu_{(t,x)}(y_1,y_2)$$

for a.a. $(t,x) \in [0,T) \times \mathbb{R}$ and $G_\varepsilon(u^\varepsilon) \rightharpoonup \overline{G}$ in $\mathbb{L}^p_{\mathrm{loc}}([0,T) \times \mathbb{R};\mathbb{R}^2)$. In the last equation $\mu_{(t,x)}$ denotes the Young measure generated by a subsequence $\{(u_1,u_2)^{k_j}\}$. From now on we restrict our considerations to the subsequence k_j. To simplify the notation we denote it by k.

If the Young measures $\mu_{(t,x)}$ are the Dirac measures (i.e. $\mu_{(t,x)} = \delta_{\overline{u}(t,x)}$) for a.e. (t,x) in $[0,T) \times \mathbb{R}$ then

$$F(\overline{u}(t,x)) = \int_K F(y_1,y_2)\, d\delta_{\overline{u}(t,x)}(y_1,y_2) = \overline{F}(t,x)$$

a.e. in $[0,T) \times \mathbb{R}$. It turns out that $\overline{G}$ cannot be determined by means of the standard "Young measure characterization" because of the differential inclusion in the original problem. This raises the question about the properties of $\overline{G}$.

2. In order to prove that $\mu_{(t,x)}$ is a Dirac measure we apply the Div-Curl lemma to a family consisting of only two entropy-flux pairs. More precisely, we choose

$$\begin{aligned}(\eta^1,q^1) &= (2(u_1+u_2),(u_1+u_2)^2),\\ (\eta^2,q^2) &= (3(u_1+u_2)^2, 2(u_1+u_2)^3).\end{aligned}$$

Next, we multiply the equation (VP) by $\nabla_v \eta^i(u)$ $(i=1,2)$, which yields

$$\frac{\partial}{\partial t} u^k \nabla_u \eta^i(u^k) + \frac{\partial}{\partial x} F(u^k)\nabla_u\eta^i(u^k) = G_k(u^k)\nabla_u\eta^i(u^k) + \frac{1}{k}\frac{\partial^2}{\partial x^2} u^k \nabla_u\eta^i(u^k),$$

and by a simple calculation

$$\begin{aligned}\frac{\partial}{\partial t}\eta^i(u^k) &+ \frac{\partial}{\partial x} q^i(u^k)\\ &= G_k(u^k)\nabla_u\eta^i(u^k) + \frac{1}{k}\bigg(\frac{\partial^2}{\partial x^2}\eta^i(u^k) + (u^k)_x^T\nabla_u^2\eta^i(u^k)u^k_x\bigg).\end{aligned} \tag{3.1}$$

We want to prove that the right hand side and thus the left hand side of (3.1) lies in a compact subset of the space $\mathbb{W}^{-1,2}(\Omega;\mathbb{R}^2)$ for all open bounded sets $\Omega \subset [0,T) \times \mathbb{R}$. A simple computation yields $\|\nabla_u\eta^i(u^k)\|_{\mathbb{L}^\infty(\Omega;\mathbb{R}^2)} \le C$, $\|G_k(u^k)\|_{\mathbb{L}^\infty(\Omega;\mathbb{R}^2)} \le C$, and $\|\nabla^2_u\eta^i(u^k)\|_{\mathbb{L}^\infty(\Omega;\mathbb{R}^2)} \le C$, where C is independent of k and Ω. This yields

(i) $\{G_k(u^k)\nabla_u\eta^i(u^k)\}$ is bounded in $\mathbb{L}^\infty(\Omega;\mathbb{R}^2)$ and thus precompact in $\mathbb{W}^{-1,2}(\Omega;\mathbb{R}^2)$,

(ii) $\{k^{-1}(u^k_x)^T\nabla^2_u\eta^1(u^k)u^k_x\} = 0$ (since η^1 is a linear function),

(iii) $\{k^{-1}(u^k_x)^T\nabla^2_u\eta^2(u^k)u^k_x\}$ is bounded in $\mathbb{L}^1(\Omega;\mathbb{R}^2)$ and hence in $\mathcal{M}(\Omega;\mathbb{R}^2)$,

(iv) $k^{-1}\frac{\partial}{\partial x}\eta^i(u^k) = k^{-1/2}\nabla_u\eta^i(u^k)k^{-1/2}u^k_x \to 0$ in $\mathbb{L}^2(\Omega;\mathbb{R}^2)$; consequently, $\{k^{-1}\frac{\partial^2}{\partial x^2}\eta^i(u^k)\}$ lies in a compact subset of $\mathbb{W}^{-1,2}(\Omega;\mathbb{R}^2)$,

(v) the left hand side of (3.1) is bounded in $\mathbb{W}^{-1,\infty}(\Omega;\mathbb{R}^2)$ and therefore in $\mathbb{W}^{-1,p}(\Omega;\mathbb{R}^2)$ for all $1<p<\infty$.

We will need the following

THEOREM (Murat's lemma). *Let f_k be a bounded sequence in $\mathbb{W}^{-1,p}(\Omega)$ for some $p>2$. Suppose that $f_k=g_k+h_k$, where the sequence g_k is precompact in $\mathbb{W}^{-1,2}(\Omega)$ and the sequence h_k is bounded in $\mathcal{M}(\Omega)$. Then the sequence f_k is precompact in $\mathbb{W}^{-1,2}(\Omega)$.*

For a proof see [7, Corollary 1, pp. 7–8].

Hence, by Murat's lemma and (i)–(v) the left hand side of (3.1) lies in a compact subset of $\mathbb{W}^{-1,2}(\Omega;\mathbb{R}^2)$.

3. Next we apply the Div-Curl lemma to the sequence $(\eta^i(u^k),q^i(u^k))$ for $i=1,2$ and use the equality

$$\eta^1(y_1+y_2)q^2(y_1+y_2)-q^1(y_1+y_2)\eta^2(y_1+y_2)=(y_1+y_2)^4.$$

By the Young measure characterization of the weak limit we have

$$\begin{aligned}\int\limits_{[0,T)\times\mathbb{R}}\Big[4\int\limits_K(y_1+y_2)\,d\mu_{(t,x)}(y_1,y_2)\int\limits_K(y_1+y_2)^3\,d\mu_{(t,x)}(y_1,y_2)\\
-3\int\limits_K(y_1+y_2)^2\,d\mu_{(t,x)}(y_1,y_2)\int\limits_K(y_1+y_2)^2\,d\mu_{(t,x)}(y_1,y_2)\Big]\varphi\,dt\,dx\\
=\int\limits_{[0,T)\times\mathbb{R}}\int\limits_K(y_1+y_2)^4\,d\mu_{(t,x)}(y_1,y_2)\varphi\,dt\,dx\end{aligned}$$

for all $\varphi\in\mathcal{C}_0^\infty(\Omega;\mathbb{R})$. We recall the following algebraic inequality:

$$a^4-4a^3b+6a^2b^2-4ab^3+b^4=(a-b)^4\ge0.$$

We set $a=y_1+y_2$ and $b=\int_K(y_1+y_2)\,d\mu_{(t,x)}(y_1,y_2)$, and integrate the above inequality with respect to the measure $d\mu_{(t,x)}(y_1,y_2)$. This yields

$$\begin{aligned}\int\limits_K(y_1+y_2)^4\,d\mu_{(t,x)}(y_1,y_2)\\
-4\int\limits_K(y_1+y_2)^3\,d\mu_{(t,x)}(y_1,y_2)\int\limits_K(y_1+y_2)\,d\mu_{(t,x)}(y_1,y_2)\\
+6\int\limits_K(y_1+y_2)^2\,d\mu_{(t,x)}(y_1,y_2)\Big[\int\limits_K(y_1+y_2)\,d\mu_{(t,x)}(y_1,y_2)\Big]^2\\
-3\Big[\int\limits_K(y_1+y_2)\,d\mu_{(t,x)}(y_1,y_2)\Big]^4\ge0.\end{aligned}$$

Combining the above results we deduce that

$$\int_{[0,T)\times\mathbb{R}} \Big[\int_K (y_1+y_2)^2\, d\mu_{(t,x)}(y_1,y_2) - \Big(\int_K (y_1+y_2)\, d\mu_{(t,x)}(y_1,y_2)\Big)^2\Big]^2 \varphi\, dt\, dx \le 0$$

for all $\varphi \in \mathcal{C}_0^\infty(\Omega;\mathbb{R})$ such that $\varphi \ge 0$. This implies

$$\int_K (y_1+y_2)^2\, d\mu_{(t,x)}(y_1,y_2) = \Big(\int_K (y_1+y_2)\, d\mu_{(t,x)}(y_1,y_2)\Big)^2.$$

Let us now consider another two entropy-flux pairs:

$$(\eta^3,q^3) = (2(u_1-u_2),(u_1-u_2)^2),$$
$$(\eta^4,q^4) = (3(u_1-u_2)^2,2(u_1-u_2)^3).$$

As before, we can prove that

$$\int_K (y_1-y_2)^2\, d\mu_{(t,x)}(y_1,y_2) = \Big(\int_K (y_1-y_2)\, d\mu_{(t,x)}(y_1,y_2)\Big)^2$$

and

$$\int_K [(y_1+y_2)^2+(y_1-y_2)^2]\, d\mu_{(t,x)}(y_1,y_2) = \Big(\int_K (y_1+y_2)\, d\mu_{(t,x)}(y_1,y_2)\Big)^2 + \Big(\int_K (y_1-y_2)\, d\mu_{(t,x)}(y_1,y_2)\Big)^2.$$

We apply Lemma 2.27 (generalized Jensen inequality) from [14, pp. 155–156] to the strictly convex function $(y_1+y_2)^2+(y_1-y_2)^2$ to obtain

$$\mu_{(t,x)}(y_1,y_2) = \delta_{\overline{u}(t,x)} \quad \text{for a.a. } (t,x) \in [0,T)\times\mathbb{R}.$$

4. In order to prove that $\overline{u}(t,x) = \int_K (y_1+y_2)d\delta_{\overline{u}(t,x)}(y_1,y_2)$ is a weak entropy solution to the original problem we have to show that the limit function $G(t,x)$ satisfies

$$G(t,x) \in \widetilde{G}(u(t,x)) \quad \text{for a.a. } (t,x) \in [0,T)\times\mathbb{R}.$$

First, we prove that $u^k \to u$ in the strong topology of $\mathbb{L}^p(\Omega;\mathbb{R}^2)$ for $1 \le p < \infty$ and an arbitrary open bounded $\Omega \subset [0,T)\times\mathbb{R}$. By the Radon–Riesz theorem the following implication holds: $[f^k \rightharpoonup f$ weakly in $\mathbb{L}^p$ and $\|f^k\|_{\mathbb{L}^p} \to \|f\|_{\mathbb{L}^p}] \Rightarrow f^k \to f$ strongly in $\mathbb{L}^p$ (for any $p \in (1,\infty)$). Consequently, it suffices to prove that $\|u^k\|_{\mathbb{L}^p} \to \|u\|_{\mathbb{L}^p}$. However, by the convergence in the sense of Young measures, and by the fact that the limit measure is the Dirac delta, we have

$$\int_\Omega |u^k|^p\, dt\, dx \to \int_\Omega\int_K |u|^p\, d\delta_{(t,x)}\, dt\, dx = \int_\Omega |u|^p\, dt\, dx.$$

Note that the characteristic function of Ω belongs to $\mathbb{L}^1(\mathbb{R}^+\times\mathbb{R})$.

Define

$$\operatorname{sign}(u)=\begin{cases}-1 & \text{for } u<0,\\ [-1,1] & \text{for } u=0,\\ 1 & \text{for } u>0.\end{cases}$$

Next, we prove the following lemma.

LEMMA 3.1. *Assume that* $u_2^k \to u_2$ *in the strong topology of* $\mathbb{L}^p(\Omega;\mathbb{R})$ *for some* $p\in(1,\infty)$, *Let* $\operatorname{sign}_k = \operatorname{sign} * \varphi_{1/k}$. *Then there exist a function* $S\in\mathbb{L}^p(\Omega;\mathbb{R})$ *and a subsequence* k_j *such that*

(i) $\operatorname{sign}_{k_j}(u_2^{k_j}) \rightharpoonup S$ *in the weak topology of* $\mathbb{L}^p(\Omega;\mathbb{R})$,

(ii) $S\in\operatorname{sign}(u_2)$ *a.e. in* Ω.

Proof. By assumption $|\operatorname{sign}_k|\le 1$. Consequently, $\operatorname{sign}_{k_j}(u_2^{k_j})\rightharpoonup S$ and $|S(t,x)|\le 1$ a.e. in Ω. It remains to show that $S\in\operatorname{sign}(u_2)$ a.e. in Ω. Therefore, we divide Ω into three sets

$$\begin{aligned}\Omega_- &= \{(t,x)\in\Omega \mid u_2(t,x)<0\},\\ \Omega_0 &= \{(t,x)\in\Omega \mid u_2(t,x)=0\},\\ \Omega_+ &= \{(t,x)\in\Omega \mid u_2(t,x)>0\}.\end{aligned}$$

It is obvious that $S\in\operatorname{sign}(u_2)$ a.e. in Ω_0. Without loss of generality, we can assume that $u_2^{k_j}\to u_2$ a.e. in Ω. Thus $\operatorname{sign}_{k_j}(u_2^{k_j})\to 1$ for a.a. $(t,x)\in\Omega_+$ and $\operatorname{sign}_{k_j}(u_2^{k_j})\to -1$ for a.a. $(t,x)\in\Omega_-$. This completes the proof of the lemma. ∎

By applying Lemma 3.1 we end the proof of the existence of a weak entropy solution in the sense of Definition 1.2 to our system (CL). Items (i) and (ii) of Definition 1.2 are obviously satisfied. Note that $[G_k(u^k)]^T = (0,\sin(\gamma)-\operatorname{sign}_k(u_2^k)\cos(\gamma))$.

Item (iii) follows from the fact that $G_k(u^k)\rightharpoonup G$ in $\mathbb{L}^p_{\mathrm{loc}}([0,T)\times\mathbb{R};\mathbb{R}^2)$ and $\nabla_u\eta(u^k)\to\nabla_u\eta(u)$ in $\mathbb{L}^p_{\mathrm{loc}}([0,T)\times\mathbb{R};\mathbb{R}^2)$. Then $G_k(u^k)\cdot\nabla_u\eta(u^k)\rightharpoonup G\cdot\nabla_u\eta(u)$ in $\mathbb{L}^p_{\mathrm{loc}}([0,T)\times\mathbb{R};\mathbb{R})$ for all $p\in[1,\infty)$.

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